SOLID GEOMETRY

BY

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PREFACE TO THE SECOND EDITION.

It was with a feeling of great discouragement that I began the preparation of another Edition of this work, deprived, as I was, of the valuable assistance of my friend Mr. Wolstenholme, in working with whom I had had so much pleasure while writing the First Edition. Mr. Wolstenholme, who is now Professor of Mathematics in the Royal Engineering College at Cooper's Hill, thought that there would be great difficulty in carrying on this work satisfactorily by correspondence, even if the important duties in which he is engaged did not fully occupy his time; I was, therefore, reluctantly obliged to undertake the whole labour of remodelling our original work.

As we contemplated making additions, and many alterations both in form and substance, my friend desired that his name might not appear in the Second Edition, and I have been compelled to alter the title of the work, and to take the responsibility of the changes which have been introduced.

The problems which appeared in the former Edition were for the most part original, and a large proportion of them were due to Mr. Wolstenholme; in this department, therefore, a most important one in my opinion, I have not lost the advantage of his valuable assistance.

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I feel bound to say a few words with respect to my persistence in retaining the word 'Conicoid' to represent the locus for the equation of the second degree. It was natural that the distinguished analyst, who has done so much towards the investigation of the properties of surfaces of higher degrees than the second, should seek a term for that of the second degree, which would connect it with those of higher degrees. But I cannot help thinking it unfortunate that the term

'quadric' should have been selected, which had already a different meaning. I quote the words of the author of the well-known treatise on Higher Algebra: "It is convenient "to have a word to denote the function itself without being "obliged to speak of the equation got by putting the function "$=0$. The term 'quantic' denotes, after Mr. Cayley, a "homogeneous function in general, using the words 'quadric,' "cubic,' 'quartic,' 'n-ic,' to denote quantics of the 2nd, 3rd, "4th, nth, degrees." Now, 'quadric,' as used in the other sense, is not even the equation found, but it takes two steps and becomes the locus of the equation.

I consider that the surface of the second degree at present, whatever may be the case in some future development, stands on a platform of its own, on account of the services which it has rendered to all departments of Mathematical Science, and well deserves a distinctive name instead of being recognised only by its number, a mode of designation which, I am informed, a convict feels so acutely. Man might be always called a biped, because besides himself there exists a quadruped, an octopus, and a centipede, but, on account of his superiority, it is more complimentary to call him by some special name.

The useful word 'conic' being well-established, the term 'conicoid' seems to suggest all that can be required, when it is employed to designate the locus of the equation of the second degree in three dimensions, at least so long as the analogous words spheroid, ellipsoid, and hyperboloid are in use, at all events it is not open to the great objection of being equally applicable to plane curves, as is the term 'quadric;' cubics and quartics being actually so employed in Salmon's *Higher Plane Curves*, Chapters V. and VI.

To the many excellent mathematicians, whose talent is shewn in the composition of the yearly College papers and the papers set for the Mathematical Tripos examination, I am indebted in the highest degree both for the problems which I have added to the collection, and also for the hints derived from them in the treatment of the subject itself.

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CAMBRIDGE,
October, 1875.

PREFACE TO THE THIRD EDITION.

I HAVE reprinted the principal part of the Preface to the Second Edition because it contains the expression of my feelings of regret when I was deprived of the co-operation of Dr. Wolstenholme in the preparation of that Edition.

I have especially retained my plea for the use of the term 'conicoid,' and the statement of my objections to the term 'quadric,' which weigh so much with me, that they have overbalanced my desire to do honour to the distinguished sponsor, to whom I owe so much, by accepting the name which he selected as most suitable.

In the present Edition the problems placed at the end of each chapter have been selected and arranged with great care in groups, each of which illustrates most of the points of the chapter to which they are attached.

In an Appendix, now nearly ready for the Press, I shall give hints sufficient for the solution of the problems.

The use of the 'solidus,' described by Prof. Stokes in the Preface to his *Physical Papers*, has enabled me to introduce a great deal of matter not contained in the last Edition without increasing the bulk of the volume. It certainly will not have been employed in vain if it induce any student to go through the work himself, in order to use another notation, instead of only reading the book.

I have in the body of the work given references for many of the mathematical papers which I have had occasion to use; but I find that I have omitted to attach the names of the authors to so many theorems that I think it best to supply

the omission by giving here a list of articles which contain theorems due to authors not mentioned in the text.

Cayley, Art. 790.

Chasles, Arts. 513, 771 and Cor., 773, 776.

Hart, Arts. 783, 784.

Joachimsthal, Art. 692.

Legendre, Art. 945.

Mac Cullagh, Arts. 301, 326.

Monge, Arts. 608, 698 &c., 736, 850 &c., 868.

M. Roberts, Arts. 767–770, 774 Cor. 2, 782, 788 and Cors., 791.

W. Thomson, Art. 706.

The arrangement of lines of a cubic surface, called a double sixer, Art. 541, was found out by Schläfli.

I am afraid that there still remain many omissions, but I should like especially to acknowledge how much I owe to Dr. Salmon for whatever knowledge I possess of many departments of the subject on which I have been engaged.

This is the proper place to express my thanks for the great assistance which I have received from Mr. Chree, Mr. Berry, and Mr. Richmond, of King's College, who have kindly not only corrected the proof sheets, which is a very tedious business, but helped me materially by testing the correctness of a great many of the problems. I wish also to thank Mr. Stearn, of King's College, for his help in the earlier portions of the work, and for his kind superintendence of the printing during my absence from Cambridge.

CAMBRIDGE,
March, 1886.

Page 149, line 28, *for* ξ and η, *read* ξ^2 and η^2.
„ 351, line 2, *for* k, *read* kX radius of AB.
Index, conoidal surface *misplaced*.
„ Pendlebury, *for* 28β, *read* 280.

PROBLEMS.

ERRATA *majora*.

PAGE
113, XX. (8), *insert* $+abc=0$ *after* cxy.
128, XXIII. (9), line 5, *insert* $-$ *before* x.
179, XXIX. (1), line 12, *for* $\sqrt{\frac{3}{2}}$, *read* $\sqrt{3}$.
line 13, *for* 1, 0, 1, *read* 1, 0, -1.
180, XXIX. (9), line 3, *insert* $+a''\sqrt{a}/(a+b)$ *after* $y\sqrt{b}$.
225, XXXVI. (9), *add* for the same height of the luminous point.
XXXVII. (7), line 2, *dele* double.
226, XXXVIII. (8), *add* a is the intersection of tangent planes at B, C, D.
236, XL. (3), *dele* of revolution.
(7), *insert* $+a'$ *after* $-C'z$).
301, XLIX. (6), *for* $4\pi\{1-c/\sqrt{(a^2+c^2)}\}$, *read* $4\pi a/\sqrt{(a^2+c^2)}$.
303, LI. (7), line 3, *for* the portion, *read* any portion.
line 4, *for* π, *read* 2π.
line 5, *add* estimated symmetrically with respect to the portion.
(9), line 4, *for* ; also &c., *read* along circular parts of their intersection.
328, LV. (3), *add* and the central circular sections.
(5), *for* conoidal surface, *read* right conoid.
329, LVI. (2), line 6, *for* tangent...at P, *read* generator of the scroll through P.
354, LVII. (7), *for* $\frac{dq}{dp}$, *read* $\left(\frac{dq}{dp}\right)^2$.
LVIII. (3), *add* if p, q be measured along fixed generating lines.
(4), line 6, *for* conicoid, *read* helicoidal surface.
356, LX. (2), line 5, *insert* $-$ *before* $\rho_r\sigma_p\tau_q$.
(6), *for* epicycloid, *read* hypocycloid.
(7), line 6, *insert* $+i\{\phi(p+iu)-\phi(p-iu)\}$ *after* $f(p-iu)$.
372, LXII. (1), line 6, *for* m^2, *read* m.
389, LXVI. (5), *for* $n-2$, *read* $2(n-2)$.

ERRATA *minora*.

89, XV. (4), *for* BC, *read* PC.
101, XVIII. (14), line 1, *for* (11), *read* (14).
126, XXI. (10), *for* b^2-m^2, *read* $(b^2-m^2)^2$.
181, XXXI. (7), line 4, *for* $a'^2b'^2c'^2$, *read* $27a'^2b'^2c'^2$.
224, XXXV. (6), *for* ax, *read* az.
248, XLII. (8), line 5, *add* and abc *after* $a'b'c$.
249, XLIII. (10), *for* pair, *read* pairs.
276, XLVI. (7), *for* β, *read* γ.
302, L. (4), *for* p/z, *read* p/x.
329, LVI. (4), *for* ϕ, *read* ψ.
354, LVIII. (1), *for* square, *read* rectangular.
356, LX. (7), line 2, *for* a, *read* a.
lines 9, 10, 11, *for* index ^{2}, *read* ^{3}.
line 10, *omit* $+$ *before* sin sp.
line 11, *for* sin sq, *read* sin sp.
389, LXVI. (3), line 6, *for* a'^2, *read* a^2.
402, LXVII. (6), *for* $(5a^{-1}+b^{-1})$, *read* $(5a+b)$.

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ERRATA *majora*.

PAGE

53, line 22, *for* Art. 114. *read* Art. 116.

77, line 9, *for* $b\ (gz - hy)^2 = (z - h)^2$, *read* $b\ (g - \beta h)^2 = 1$.

90, Ex. (10), line 3, *for* or a plane, *read* or a cone.

106, line 13, *for* axes, *read* semi-axes.

113, Ex. (10), lines 4, 5, *insert* $= 0$ in both lines.

205, last line, *for* element of the line of striction, *read* shortest distance.

274, line 11, *for* two, *read* three.

302, line 3, *for* $2\ \sqrt{(x^2 + y^2)}$, *read* $\sqrt{(x^2 + y^2)}$.

337, line 12 from below, *after* P of, *insert* the normal sections touching.

340, line 2 from below, *for* P_{21}, *read* P_{12}.

ERRATA *minora*.

16, line 2 from below, *for* A, *read* A_y.

30, line 1, *for* $(b = nx - lz)/m +$, *read* $(b + nx - lz)/m =$.

30, line 4, *for* ml, *read* mz.

67, line 5, *for* $a - br$, *read* $a - br^2$.

76, line 12 from below, *for* equations, *read* equation.

89, Ex. (10), *for* b^{-9}, *read* b^{-2}.

99, Ex. (10), *for* $(\theta_2\phi_3)$, *read* $(\theta_3\phi_3)$.

101, line 13, *for* (11), *read* (14).

106, line 7 from below, *for* action, *read* section.

110, line 4 from below, *for* equations, *read* equation.

110, line 8 from below, *for* are, *read* is.

225, line 11, *for* $b' \sin'\theta$, *read* $b' \sin^2\theta$.

237, line 4 from below, *for* $\theta\lambda^2\mu$, *read* $\theta\lambda^2\mu$.

238, line 11, *for* touches, *read* touch.

239, line 17, *for* $\lambda b + \mu a'$, *read* $\lambda b + \mu b'$.

325, line 13 from below, *for* $\log \frac{1}{2}$ an $\frac{1}{2}\theta$, *read* $\log \frac{1}{2} \tan \frac{1}{2}\theta$.

344, line 5 from below, *for* helix, *read* helicoid.

GEOMETRY OF THREE DIMENSIONS.

CHAPTER I.

ON COORDINATE SYSTEMS.

1. BEFORE entering upon the application of Algebra to the investigation of Theorems, and to the solution of Problems, in Solid Geometry, we shall premise on the part of the student a complete knowledge of all the ordinary processes adopted in the case of Plane Geometry.

By this means we shall avoid the necessity of entering upon the subject of the interpretation of the affection denoted by the sign $(-)$ prefixed to a symbol; since it is known that, if $+a$ denote a line of length a measured in any direction from a point in a line straight or curved, $-a$ may be interpreted to denote a line of length a measured in the opposite direction from any other point in the line, without this hypothesis involving any infringement of the laws of combination of these signs, assumed as the fundamental laws of Symbolical Algebra.

2. Our first object will be to explain how the position of a point in space can be represented by algebraical symbols, and with this view we shall describe three of the different coordinate systems which it has been found convenient to adopt; reserving the consideration of other systems for future chapters, when the student shall have acquired some familiarity with the subject. And it will be found that each of the systems has its peculiar advantage, according to the nature of the theorem or problem which is the subject of examination.

Coordinate System of Three Planes.

3. In the coordinate system of three planes, three planes xOy, yOz, zOx are fixed upon as planes of reference, which may be either perpendicular to one another, or inclined at angles which are known.

The three lines in which they intersect are called *coordinate axes*, and the point in which they meet the *origin of coordinates*.

The position of a point in space is then completely determined, when its distance from each of the planes, estimated parallel to the

coordinate axes, and the direction in which those distances are measured, are given.

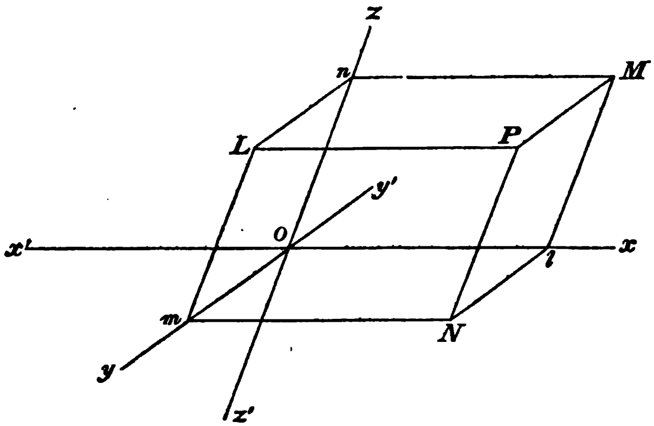
The absolute distance and the direction of measurement are included in the term *algebraical distance*.

Thus $+a$ and $-a$ are the algebraical distances of two points whose absolute distances from the plane yOz are each a, and which are measured, the first in the direction Ox, the second in the direction xO from that plane.

These algebraical distances are called the *coordinates* of a point in this system, and are usually denoted by the letters x, y, and z.

The point, of which these are coordinates, is described as the point (x, y, z).

Produce xO, yO, zO backwards to x', y', z'; then, if a, b, c



are absolute lengths, (a, b, c) denotes a point in the compartment $xyzO$, $(-a, b, c)$ in $x'yzO$, $(a, -b, c)$ in $xy'zO$, $(a, b, -c)$ in $xyz'O$, $(a, -b, -c)$ in $xy'z'O$, $(-a, b, -c)$ in $x'yz'O$, $(-a, -b, c)$ in $x'y'zO$, $(-a, -b, -c)$ in $x'y'z'O$.

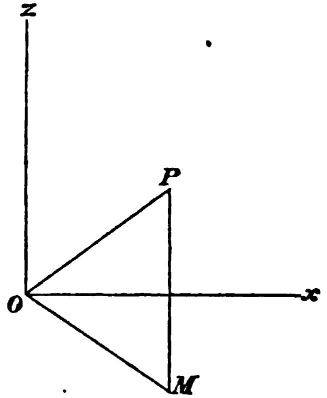
4. If a parallelepiped be constructed, whose faces are parallel to the coordinate planes, the point $P(a, b, c)$ being the other extremity of the diagonal drawn from the origin, the edges LP, MP, NP will be the coordinates of the point P supposed in the compartment $xyzO$.

Also, it is obvious that $x = a$ for every point in the plane face $PNlM$, or that $x = a$ is the equation of that plane, as $y = b$ and $z = c$ are the equations of the planes $PLmN$ and $PMnL$ indefinitely extended in every direction.

Thus, the point P may be considered as the intersection of the three planes, whose equations are $x = a$, $y = b$, $z = c$. The points l, O may be denoted by $(a, 0, 0)$ and $(0, 0, 0)$ and the points L and M by $(0, b, c)$ and $(a, 0, c)$.

Polar Coordinate System.

5. In the system of Polar Coordinates, a plane zOx is chosen, and in this plane a straight line Oz is drawn from a fixed point O.



The position of a point P in space is completely determined when its distance from the fixed point O is given, the angle through which OP has revolved from Oz in a plane zOP passing through Oz, and the angle through which this plane has revolved into its position from the fixed plane of reference zOx. These coordinates are usually denoted by the symbols r, θ, and ϕ, and the point P by (r, θ, ϕ).

Thus, if the longitude of a place be l, the latitude λ, and the radius of the earth a, we may take the first meridian for the plane zOx, the axis of the earth for the line Oz, and the position of the place will be expressed by $(a, \frac{1}{2}\pi - \lambda, l)$. The position of Greenwich, latitude λ', is given by $(a, \frac{1}{2}\pi - \lambda', 0)$.

Cylindrical Coordinates.

6. In some cases it is convenient to define the position of the foot of the ordinate z by the polar coordinates r, θ instead of x, y; r, θ, z are then called *cylindrical coordinates.*

I.

(1) Construct the positions of points which are represented by the equations $x^2 - y^2 = z^2$, $x + y = 4a$, $x - y = a$.

(2) $x^2 + y^2 = 2z^2$, $x + y = 2z$, $xy = a^2$.

(3) Shew that, for every point in OP, P being (a, b, c), $x/a = y/b = z/c$.

(4) Shew that, for every point in the plane $LMlm$ in the preceding figure, $x/a + y/b = 1$.

(5) Draw a figure, for every point of which $x^2 + y^2 = a^2$ and $z = 0$.

CHAPTER II.

GENERAL DESCRIPTION OF LOCI OF EQUATIONS. SURFACES. CURVES.

Locus of an Equation.

7. If an equation $\phi(x, y, z) = 0$ be given, in which the variables are the coordinates of any point, the number of solutions of this equation is generally infinite, *i.e.* the number of points whose coordinates satisfy the equation is infinite; we shall proceed to shew what is the general nature of the distribution of these points.

We shall prove, in the first place, that no algebraical equation can be satisfied by every point of *any solid* figure, but, in the most general case, only by every point in some surface or surfaces.

8. If an equation involve only one of the coordinates as x, we know that such an equation, $\phi(x) = 0$, has a finite or an infinite number of roots $a, b, c, \ldots$ separated by definite intervals, and is equivalent to the equations $x = a$, $x = b$, ... each of which, as $x = a$, is satisfied by every point in a plane parallel to the plane yOz at an algebraical distance a from that plane. Hence, all the points, whose coordinates satisfy the equation $\phi(x) = 0$, lie in a series of planes parallel to yOz at algebraical distances $a, b, c, \ldots$ from it.

If the given equation involve two only of the variables, as $\phi(y, z) = 0$, on the plane yz let the curve be constructed, every point of which satisfies this equation, and let a straight line be drawn parallel to Ox through any point in this curve, then every point in this line is such that its coordinates, as well as those of the point through which it is drawn, satisfy the given equation, and the same is true of all points in the curve, but of no other points. Hence, all the points which satisfy the proposed equation lie in a surface generated by a straight line parallel to Ox, which passes successively through every point of the curve traced on the plane yz; such a surface is called a *cylindrical* surface, and the curve is called the *trace* on the plane yz, being one of an infinite number of curves called *guiding curves* to the cylindrical surface; in this case the trace on the plane yz is the *projection* by lines parallel to Ox, of any one of the guiding curves. The number of guiding curves is infinite, since, if any curve be traced upon the cylindrical surface so as to cross every generating line, a line, moving parallel to Ox so as to traverse every portion of such a curve traced in space, would generate the entire cylindrical surface, that curve serving to guide the direction of motion of the generating line.

9. We may notice here that if the equation $\phi(y, z) = 0$ be equivalent to a series of equations of such forms as

$$(y-b)^2 + (z-c)^2 = 0, \quad (my - nz)^2 + (z-c)^2 = 0,$$

the trace on yz is reduced to a series of points, and the locus of the equation $\phi(y, z) = 0$ becomes a series of straight lines parallel to Ox and passing through those points.

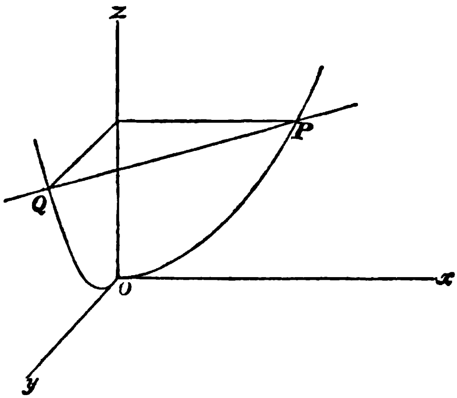
In such cases the locus appears to be different in character from that of the general case, since it is a series of lines instead of being a surface. But it may be seen that this is only in appearance, since each of the equations, whose locus is called a point, represents a closed curve of infinitely small dimensions, and the lines are cylinders whose breadths are infinitely small, and the locus of the equation $\phi(y, z) = 0$ is, as in the general case, a series of surfaces. A similar interpretation may be given in every case.

10. We shall now proceed to the general case, in which all the coordinates are involved, $\phi(x, y, z) = 0$, and examine the position of all the points which satisfy the equation.

We shall first find the position of those points which are at an algebraical distance f from the plane of yz, which is the same thing as finding those points of the locus which lie in a plane whose equation is $x = f$.

Such points are contained in the cylindrical surface whose equation is $\phi(f, y, z) = 0$. The trace of this surface on the plane $x = f$ is the line which contains all the points of the locus which lie on that plane; and, if the series of lines be supposed traced corresponding to different positions of the plane $x = f$, for values of f varying from $-\infty$ to $+\infty$, we shall evidently obtain a surface which will contain all the points which satisfy the equation $\phi(x, y, z) = 0$.

11. As an illustration of tracing surfaces, we will take the case of the surface whose equation is $(x + y)^2 = az$.



If $x = 0$, $y^2 = az$, therefore the trace upon the plane of yz is a parabola whose axis is Oz and vertex O.

Similarly the trace on zx is an equal parabola having the same axis and vertex.

If $z = h$, $(x + y)^2 = ah$, the latter is the equation of two planes parallel to Oz, equally inclined to the planes yz, zx, therefore the trace on the plane $z = h$ is two straight lines equally inclined to the planes of yz, zx.

Hence, the surface may be generated by straight lines such as PQ, which move parallel to the line $x + y = 0$ in the plane xy, constantly passing through the traces OP, OQ on the planes zx, yz, and inclined to these planes at equal angles.

The surface is therefore cylindrical.

Locus of the Polar Equation.

12. We shall examine in order the loci of equations in polar coordinates which involve one or more of these coordinates.

i. If the equation be $F(r) = 0$, this is equivalent to a series of equations $r = a$, $r = b$, ... any one of which being satisfied the original equation is satisfied; $r = a$ is satisfied by all points at a distance a from the origin, measured in any direction; therefore the locus of $F(r) = 0$ is a series of concentric spheres, whose centre is the origin.

ii. If the equation be $F(\theta) = 0$, it is equivalent to $\theta = \alpha$, $\theta = \beta$, ..., any one $\theta = \alpha$ is satisfied by every point of lines through O inclined to Oz at angles equal to α; therefore the locus $F(\theta) = 0$ is a series of conical surfaces, whose common axis is Oz, common vertex O, and vertical angles 2α, 2β, ...

iii. If the equation be $F(\phi) = 0$, it is equivalent to $\phi = \alpha$, $\phi = \beta$, ..., any one $\phi = \alpha$ is satisfied by every point in a plane through Oz inclined at an angle α to the fixed plane zOx; therefore the locus of $F(\phi) = 0$ is a series of planes through Oz inclined to zOx at angles α, β, ...

iv. If the equation involve only r and θ, as $F(r, \theta) = 0$, since for all values of ϕ the same relation exists between r and θ, the locus of the equation is the surface generated by the revolution of a curve traced on a plane passing through Oz, as this plane revolves about Oz as an axis.

v. If the equation involve only θ and ϕ, as $F(\theta, \phi) = 0$, for every value of ϕ, there is a series of values of θ, corresponding to which if straight lines be drawn through O, every point in these lines will be such that its coordinates will satisfy the equation; as ϕ changes, or the plane through Oz revolves, these lines assume new positions relative to Oz, and generate, during the revolution of the plane, conical surfaces, a conical surface being defined to be a surface generated by a straight line moving in any manner with the restriction that it passes through a fixed point.

vi. If the only coordinates involved be r, ϕ, as in $F(r, \phi) = 0$, for each position of the plane through Oz inclined at any angle ϕ to the plane zOx, there is a series of values of r which are constant for all values of θ, *i.e.* there is a series of concentric circles in the plane, the coordinates of each point in which satisfy the equation.

The locus of the equation is therefore a surface generated by circles having their centres in O, and varying in magnitude as their planes revolve about the line Oz through which they pass.

vii. If the equations involve all the coordinates, as $F(r, \theta, \phi) = 0$, let any value, as β, be given to ϕ, then corresponding to this value there is a plane through Oz, and if the locus of $F(r, \theta, \beta) = 0$ be traced on this plane, and such curves be drawn upon all planes corresponding to values of ϕ from $-\infty$ to $+\infty$, the surface which contains all these curves will be the locus of the equation.

Curves.

13. Curves in space may be considered as the limits of polygons whose sides are indefinitely small, and the peculiarity of such curves is, that the plane which contains two consecutive sides of the polygon of which the curve is the limit, does not generally contain the next side; so that the plane in which the curvature is taking place at any point changes as the point changes.

This property is sufficiently represented by calling such curves *tortuous curves*, the old name of *curves of double curvature* was misleading, since there are not two curvatures.*

Equations of Curves.

14. Through every curve there can be drawn an infinite number of surfaces, the intersections of any two of which will include every point of the curve. At the same time we must observe that two surfaces, each of which contains a given curve, may not be sufficient to determine the position of the curve definitely, because they may intersect in other points which are not connected with the given curve.

Thus, if we take the case of a circle, it is true that it lies entirely in the intersection of a certain sphere and cylinder, but the sphere and cylinder are not sufficient to determine the circle without ambiguity, because they also intersect in another circle. It is possible, however, in this case to find two surfaces which do define the circle completely, as, for example, a plane and either a sphere or cylinder.

15. If $\phi(x, y, z) = 0$ and $\psi(x, y, z) = 0$ be equations of two surfaces, these surfaces, by their intersection, determine a certain

* Thomson and Tait's Natural Philosophy, Vol. I. Part I. Art. 7.

curve, and if another equation $\chi(x, y, z) = 0$ be derived from these equations by any algebraical process, this third equation will be satisfied by every point in the curve determined by the intersection of the first two surfaces, and we may employ this equation and either of the first two to obtain properties of the curve, although the new equations which we employ may, and generally would, represent surfaces which intersect in other points than those of the curve originally proposed.

For example, the equations

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \text{ and } x^2 + y^2 + z^2 = b^2$$

represent two surfaces, the first of which is called an ellipsoid, the second is a sphere, now, dividing the second equation by b^2, and, subtracting the first from it, we obtain the equation

$$(b^{-2} - a^{-2})\, x^2 = (c^{-2} - b^{-2})\, z^2,$$

which, if $a > b > c$, represents two planes and shews that the curve of intersection is composed of two circles, which are the intersection of the sphere and the two planes.

When we eliminate x from the two general equations of the curve, the result being of the form $f(y, z) = 0$, the curve on the plane yz, whose coordinates satisfy this equation, is the projection of the given curve by lines parallel to Ox, the given curve being one of the guiding curves mentioned in Art. 8.

16. It is often convenient in practice to consider a curve as the intersection of two cylindrical surfaces, whose generating lines are parallel to two of the coordinate axes. In this way of considering curves, the equations of the surfaces are of the form $\phi(x, z) = 0$, $\psi(y, z) = 0$.

As a simple example of this method the straight line joining the points n, N in the figure on page 2 is determined by the two plane surfaces whose equations are

$$x/a + z/c = 1 \quad\text{and}\quad y/b + z/c = 1.$$

II.

(1) Trace the surfaces represented by the equations

(i) $x^2 + y^2 = ax$. (ii) $z^2 = ax + by$. (iii) $x^2 + y^2 + z^2 = 2ax + 2by + 2cz$.

(iv) $x^2 + y^2 = az$. (v) $xz^2 = c^2y$. (vi) $xy = az$.

(vii) $c^2y^2 = z^2(a^2 - x^2)$ (viii) $(x + y)^2 = c(z - x)$.

(2) Describe the three surfaces

(i) $r = a \sin\theta$, (ii) $r = a \cos\phi$, and (iii) $4\theta = 2\pi + \pi \sin 4\phi$.

CHAPTER III.

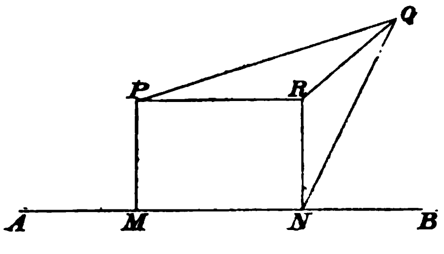
PROJECTIONS OF LINES AND AREAS. DIRECTION-COSINES AND DIRECTION-RATIOS.

17. Def. The *geometrical projection* of a straight line of limited length upon any other straight line given in position is the distance intercepted between the feet of the perpendiculars let fall from the extremities of the limited line upon the straight line on which it is to be projected.

18. *The geometrical projection of a straight line of limited length on a given straight line is equal to the given length multiplied by the cosine of the acute angle contained between the lines.*

Let PQ be the line of limited length, AB the indefinite line upon which it is to be projected.

Let QRN be a plane through Q perpendicular to AB meeting it in N, PR parallel to AB meeting QRN in R.



Therefore PR being parallel to AB is perpendicular to the plane QRN, and therefore to RN and QR, and QN is perpendicular to AB; hence, if PM be drawn perpendicular to AB, MN is the projection of PQ, and QPR is the acute angle contained between PQ and AB, and since $PRNM$ is a rectangle,

$$MN = PR = PQ \cos QPR.$$

If PQ intersect AB, the proposition is obviously true.

19. Def. The *algebraical projection* of a line PQ upon an indefinite line AB given in position is the projection estimated in a given direction, as AB.

If α be the angle through which PQ may be supposed to have revolved from PR, drawn in the positive direction AB, the algebraical projection of $PQ = PQ \cos \alpha$.

When N lies in the opposite direction with reference to M, α is obtuse, and $PQ \cos\alpha$ is negative.

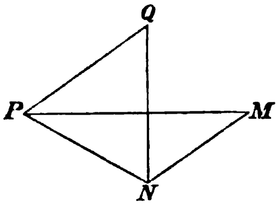
The algebraical projection of a limited straight line upon a line given in position measures the distance traversed in the direction of the latter line in passing from one extremity of the former to the other.

This consideration shews that, if all the sides of a closed polygon taken in order be projected on any straight line given in position, the sum of the algebraical projections of these sides will be zero; since, in passing round the perimeter of the polygon from any point, the whole distance advanced in any direction is zero.

Hence, the algebraical projection of any side AB of a closed polygon is the sum of the algebraical projections of the remaining sides commencing from A and terminating in B.

Note. In future, when the term projection is used with reference to straight lines, the algebraical projection is to be understood.

20. Let PQ be any line, PM, MN, NQ three straight lines drawn in any given directions so as to terminate in Q, and l, m, n the cosines of the angles which PQ makes with these directions.



Then PQ will be the sum of the projections of PM, MN, and NQ on PQ; therefore $PQ = l \, . \, PM + m \, . \, MN + n \, . \, NQ$.

Direction-Cosines.

21. The direction of a straight line in space is determined when the angles which it makes with the coordinate axes are known.

Def. When the coordinate axes are perpendicular, the cosines of the inclinations to the three axes are called *direction-cosines.*

22. *To find the relation between the direction-cosines of a straight line.*

If l, m, n be the direction-cosines of PQ, and PM, MN, NQ be parallel to the coordinate axes,

$$PM = PQ \, . \, l, \quad MN = PQ \, . \, m, \quad NQ = PQ \, . \, n.$$

Join PN, then, since QN is perpendicular to NM, MP, and therefore to the plane PMN, PNQ is a right angle;

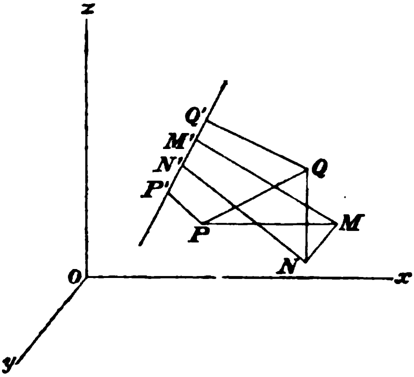
$$\text{hence } PQ^2 = PN^2 + NQ^2 = PM^2 + MN^2 + NQ^2;$$

$$\therefore \; 1 = l^2 + m^2 + n^2,$$

which is the relation required. Hence the three angles of inclination cannot all be assumed arbitrarily.

23. *To find the angle between two straight lines in terms of their direction-cosines.*

Let PQ, $P'Q'$ be two straight lines whose direction-cosines are (l, m, n) and (l', m', n') respectively.



Let PM, MN, NQ be drawn parallel to the axes, connecting any two points P, Q, and PP', QQ' perpendicular to $P'Q'$, and let θ be the angle between PQ and $P'Q'$.

Then $P'Q'$, the projection of PQ on $P'Q'$, will be equal to the sum of the projections of PM, MN, NQ on $P'Q'$, namely, $P'M'$, $M'N'$, $N'Q'$;

$$\text{therefore } PQ\cos\theta = PM \,.\, l' + MN \,.\, m' + NQ \,.\, n',$$

$$\text{and since } PM = PQ.l,\ MN = PQ.m,\ NQ = PQ.n,$$

$$\therefore \cos\theta = ll' + mm' + nn';$$

$$\text{hence, } \sin^2\theta = (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn')^2$$

$$= (mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2.$$

Aliter. Since the angle θ between the two given lines is equal to the angle between two lines OR, OR' parallel to them through the origin, let R and R' be at unit distance from O, their coordinates will be l, m, n and l', m', n', therefore since $2\,OR\sin\frac{1}{2}\theta = RR'$, assuming the formula for the distance between two points given in Art. 39,

$$4\sin^2\tfrac{1}{2}\theta = (l - l')^2 + (m - m')^2 + (n - n')^2 = 2 - 2(ll' + mm' + nn'),$$

$$\therefore \cos\theta = ll' + mm' + nn'.$$

When $\theta = \frac{1}{2}\pi$, $ll' + mm' + nn' = 0$, the condition that the two straight lines may be at right angles. The student should obtain this condition without assuming the general case.

24. *To find the direction-cosines of a straight line perpendicular to two straight lines whose direction-cosines are given.*

Let l, m, n, and l', m', n' be the given direction-cosines, and λ, μ, ν the required cosines of the perpendicular; then, from the condition of perpendicularity,

$$l\lambda + m\mu + n\nu = 0, \text{ and } l'\lambda + m'\mu + n'\nu = 0,$$

$$\text{whence } (ln' - l'n)\lambda + (mn' - m'n)\mu = 0,$$

and, by symmetry, $\dfrac{\lambda}{mn'-m'n} = \dfrac{\mu}{nl'-n'l} = \dfrac{\nu}{lm'-l'm} = \pm\dfrac{1}{\sin\theta}$, Art. 23,

if θ be the angle between the lines.

25. *To find the direction-cosines of two straight lines which lie in the plane containing two straight lines, whose direction-cosines are given, and bisect the angles between them.*

Let AP, AQ be the two given lines, whose direction-cosines are l, m, n and l', m', n'.

Take $AP = AQ = r$, join PQ and bisect it in R, AR is one of the bisecting lines, let its direction-cosines be λ, μ, ν, and if 2θ be the angle between AP and AQ, $AR = r\cos\theta$.

If AP, AQ, and AR be projected upon the axis Ox, the projection of R will bisect that of PQ; $\therefore\ 2r\cos\theta.\lambda = lr + l'r$, whence λ, and similarly μ and ν can be found, θ being known by the equation $\cos 2\theta = ll' + mm' + nn'$, Art. 23.

Produce QA to Q' so that $AQ' = r$, and bisect PQ' in R', AR' is the other bisector, and since the direction-cosines of AQ' are $-l'$, $-m'$, $-n'$ and $AR' = 2r\sin\theta$, if λ', μ', ν' be the direction-cosines of AR'; $\therefore\ 2r\sin\theta.\lambda = lr + (-l')r$, and similarly for μ' and ν'.

26. *To find the angle between the two straight lines whose direction-cosines are given by two homogeneous equations of the first and second degrees respectively.*

Let the given equations be

$$al^2 + bm^2 + cn^2 + 2a'mn + 2b'nl + 2c'lm = 0, \text{ and } \alpha l + \beta m + \gamma n = 0.$$

That there are two lines may be seen by eliminating n from the two equations, whereby we obtain the equation giving two values of $l : m$,

$$\gamma^2(al^2 + bm^2 + 2c'lm) - 2\gamma(\alpha l + \beta m)(b'l + a'm) + c(\alpha l + \beta m)^2 = 0,$$

$$\text{or } vl^2 + 2w'lm + um^2 = 0,$$

$$\text{where } v = a\gamma^2 - 2b'\gamma\alpha + c\alpha^2,$$

$$w' = c'\gamma^2 - (a'\alpha + b'\beta)\gamma + c\alpha\beta,$$

$$u = c\beta^2 - 2a'\beta\gamma + b\gamma^2.$$

Now, let l_1, m_1, n_1, and l_2, m_2, n_2 be the direction-cosines of the two straight lines, then, $l_1 : m_1$ and $l_2 : m_2$ being roots of the equation,

$$\frac{l_1l_2}{u} = \frac{m_1m_2}{v} = \frac{l_1m_2 + l_2m_1}{-2w'} = \left\{\frac{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}{4(w'^2 - uv)}\right\}^{\frac{1}{2}}.$$

Now, it can be shewn, by collecting the coefficients of the different powers of γ, that

$$w'^2 - uv = \gamma^2(A\alpha^2 + B\beta^2 + C\gamma^2 + 2A'\beta\gamma + 2B'\gamma\alpha + 2C'\alpha\beta),$$

$$\text{where } A = a'^2 - bc \text{ and } A' = aa' - b'c',$$

and similar expressions.

We have, therefore, from symmetry,

$$\frac{l_1 l_2}{u} = \frac{m_1 m_2}{v} = \frac{n_1 n_2}{w} = \frac{m_1 n_2 - m_2 n_1}{2aP} = \frac{n_1 l_2 - n_2 l_1}{2\beta P} = \frac{l_1 m_2 - l_2 m_1}{2\gamma P},$$

where P^2 is written for the symmetrical expression $Aa^2 + \ldots + 2A'\beta\gamma + \ldots$.

Therefore, if ϕ be the angle between the lines,

$$\frac{\cos\phi}{u+v+w} = \frac{\sin\phi}{2P(a^2+\beta^2+\gamma^2)^{\frac{1}{2}}}.$$

COR. The conditions that two such equations may represent two perpendicular or two parallel directions are

$$u + v + w = 0, \text{ and } P = 0, \text{ respectively.}$$

The condition of perpendicularity may be written

$$(a+b+c)(a^2+\beta^2+\gamma^2) - f(a, \beta, \gamma) = 0,$$

if $f(l, m, n) = 0$ be the equation of the second degree. The condition of parallelism may be expressed by the determinant

$$\begin{vmatrix} a, & c', & b', & a \\ c', & b, & a', & \beta \\ b', & a', & c, & \gamma \\ a, & \beta, & \gamma, & 0 \end{vmatrix} = 0.$$

Direction-Ratios.

27. DEF. If the coordinate axes be not perpendicular to each other, the direction of a line PQ is fully determined, when the ratios of PM, MN, NQ to PQ are given, PM, MN, NQ being parallel to the axes. These ratios are called *direction-ratios.*

28. *To find the relation between the direction-ratios of a straight line.*

In the figure on page 11, let the angles yOz, zOx, xOy be λ, μ, ν, and let α, β, γ be the angles between PQ and the axes, l, m, n the direction-ratios of PQ.

Projecting the line PQ and the bent line $PMNQ$ terminated in the same points on Ox,

$$PQ\cos\alpha = PM + MN\cos\nu + NQ\cos\mu,$$

$$\left.\begin{aligned} \therefore\ \cos\alpha &= l + m\cos\nu + n\cos\mu; \\ \text{similarly}\ \cos\beta &= l\cos\nu + m + n\cos\lambda, \\ \text{and}\ \cos\gamma &= l\cos\mu + m\cos\lambda + n. \end{aligned}\right\}$$

Also, projecting $PMNQ$ on PQ,

$$PM\cos\alpha + MN\cos\beta + NQ\cos\gamma = PQ,$$

$$\therefore\ l\cos\alpha + m\cos\beta + n\cos\gamma = 1,$$

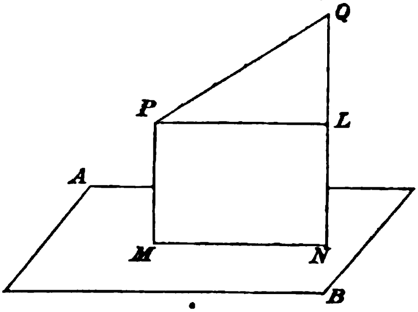
$$\therefore\ 1 = l^2 + m^2 + n^2 + 2mn\cos\lambda + 2nl\cos\mu + 2lm\cos\nu,$$

which is the relation required.

Projection of a Straight Line on a Plane.

29. DEF. The orthogonal projection of a line of limited length on a plane is the line intercepted between the perpendiculars drawn from the extremities of the limited line upon the plane.

30. *The orthogonal projection of a line upon a plane is the length of the line multiplied by the cosine of the angle of inclination of the line to the plane.*



Let PQ be the given line, AB the plane, PM, QN perpendiculars upon the plane.

Since PM, QN are perpendicular to the plane AB, PM is parallel to QN, and the plane $MPQN$ is perpendicular to the plane AB; join MN, and draw PL parallel to MN;

$$\therefore \angle PLQ = \angle MNQ = \text{a right angle};$$

$$\therefore MN = PL = PQ \cos QPL,$$

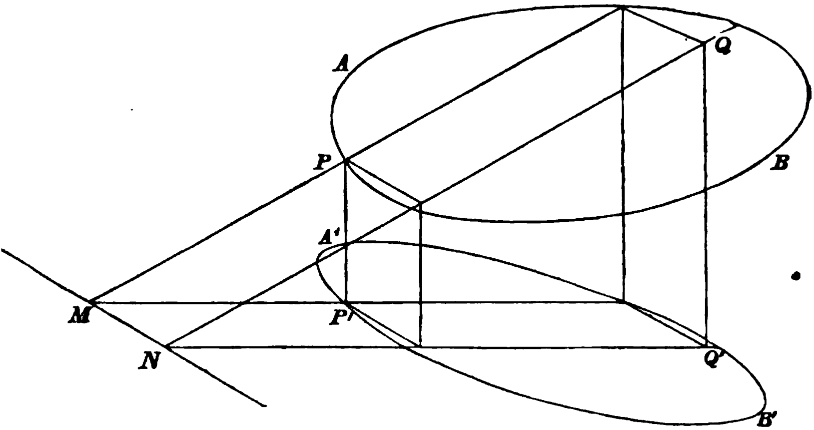
therefore, since MN is the projection of PQ on AB, and $\angle QPL$ is the inclination of PQ to the plane, the proposition is proved.

Projection of a Plane Area upon a Plane.

31. DEF. The orthogonal projection of a closed plane area upon a fixed plane, is the area on that plane included within the line which is the locus of the feet of perpendiculars drawn from every point in the boundary of the plane area.

32. *The orthogonal projection of any plane area on a given plane is the area multiplied by the cosine of the inclination of the plane of the area to the given plane.*

Let APB be any closed curve described upon a given plane, and $A'P'B'$ the *orthogonal projection* upon any other fixed plane, which is the locus of the feet of the perpendiculars drawn to the second plane from every point of the curve APB.



The areas APB, $A'P'B'$ may have inscribed in them any number of parallelograms, such as PQ, $P'Q'$, whose sides are in planes PMP', QNQ' drawn perpendicular to the line of intersection of the given planes, and parallel to that line, and these parallelograms are in the ratio of 1 : cosine of the inclination of the planes; therefore the sums of the parallelograms are in the same ratio.

Hence, proceeding to the limit when the breadths of these parallelograms are indefinitely diminished, the area of the projection of $APB =$ area of $APB \times$ cosine of the inclination of the planes.

When the inclination of the plane area A to the plane on which it is projected is given by the angle θ between the normals drawn in a definite direction with respect to the two planes, the expression $A \cos\theta$ is called the *algebraical projection*, which will be negative if θ be an obtuse angle.

33. *To find the area of the projection of a triangle upon the coordinate planes, the coordinates of the angular points being given.*

Let (a, b, c), (a', b', c'), (a'', b'', c'') be the angular points of the triangle, and let the projections of these points on the plane of yz be P, Q, R. Take a case in which $b'' > b' > b$, and $c'' > c' > c$, and let the motion along the sides of the triangle in the order PQR be in the direction of the motion of the hands of a watch. Draw QU, RV parallel to Oy, meeting a line through P parallel to Oz in U, V, and join QV.

Then $$\Delta PQR = \Delta PVQ + \Delta RVQ - \Delta RVP;$$

$$\therefore\ 2\Delta PQR = QU.PV - RV.PU = (b'-b)(c''-c) - (b''-b)(c'-c)$$
$$= b(c'-c'') + b'(c''-c) + b''(c-c'),$$

which may be written

$$2A_x = \begin{vmatrix} b, & c, & 1 \\ b', & c', & 1 \\ b'', & c'', & 1 \end{vmatrix}.$$

If PQR had been the direction opposite to the motion of the hands of a watch, or Q on the other side of PR,

$$\Delta PQR = \Delta RVP - \Delta RVQ - \Delta PVQ,$$

$$\text{and } 2\Delta PQR = RV.PU - QU.PV,$$

or the above determinant with a negative sign.

$$\text{Similarly, } 2A_y = \begin{vmatrix} c, & a, & 1 \\ c', & a', & 1 \\ c'', & a'', & 1 \end{vmatrix} \text{ and } 2A_z = \begin{vmatrix} a, & b, & 1 \\ a', & b', & 1 \\ a'', & b'', & 1 \end{vmatrix}.$$

34. *If the faces of any closed polyhedron be projected on any plane, the sum of the algebraical projections of the faces on any fixed plane will be zero.*

One side of the fixed plane being selected as that to which the normal is drawn, the angle between this normal and the normal, drawn outwards, at any point of the closed polyhedron, is quite definite; and the projection of any face will be positive or negative according as this angle is acute or obtuse. Now any straight line whatever (produced indefinitely both ways) will meet the polyhedron in 0, 2, 4, ... or some *even* number of points, since passing from outside to inside, or from inside to outside, necessitates crossing a face once. Draw a straight line parallel to the normal to the plane of projection meeting the polyhedron in points P_1, P_2, P_3, ..., P_{2m}, and round it an indefinitely small cylinder whose transverse section is a, then the projections of the sections of this cylinder made by the faces of the polyhedron which it meets will be alternately $+a$ and $-a$, and since the number of them is even, their sum will always be zero. This being true for every straight line perpendicular to the plane of projection, will be true for the total projection of the polyhedron; and will also be true when the number of faces is indefinitely increased, and the areas of some, or all of them, diminished indefinitely; that is, the sum of the algebraical projections of all the elements of a closed surface on any fixed plane is zero.

35. *To find the area of any plane surface in terms of the areas of the projections upon any rectangular coordinate planes.*

Let l, m, n be the direction-cosines of a normal to the plane on which the given area A lies, A_x, A_y, A_z the areas of the projections upon the coordinate planes of yz, zx, xy.

Then, since l is the cosine of the angle between Ox and the normal to the plane, which is the same as the angle between the plane of A and the plane of yz, $A_x = Al$,

$$\text{and similarly, } A_y = Am, \text{ and } A_z = An;$$

$$\therefore A^2 = A^2 (l^2 + m^2 + n^2) = A_x^2 + A_y^2 + A_z^2.$$

36. *To find the plane upon which the sum of the algebraical projections of any number of given plane areas is a maximum.*

Let A, A', A''... be any number of plane areas, l, m, n, l', m', n'... the direction-cosines of the normals to their planes, λ, μ, ν those of the normal to a plane upon which they are projected; and let A_x, A_y, A_z and A'_x, A'_y, A'_z...be the areas of the projections of the given areas upon the coordinate planes.

Then since $l\lambda + m\mu + n\nu$ is the cosine of the angle between the plane of A, and the plane upon which it is projected, the projection of A is

$$A(l\lambda + m\mu + n\nu) = A_x\lambda + A_y\mu + A_z\nu;$$

therefore the sum of the projections of all the areas upon the plane (λ, μ, ν) is $\lambda\Sigma(A_x) + \mu\Sigma(A_y) + \nu\Sigma(A_z)$ which is to be a maximum by the variation of λ, μ, ν, subject to the condition $\lambda^2 + \mu^2 + \nu^2 = 1$;

$$\therefore \Sigma(A_x)\,d\lambda + \Sigma(A_y)\,d\mu + \Sigma(A_z)\,d\nu = 0, \quad \text{and} \quad \lambda d\lambda + \mu d\mu + \nu d\nu = 0,$$

must be true for an infinite number of values of $d\lambda : d\mu : d\nu$;

$$\therefore \frac{\lambda}{\Sigma(A_x)} = \frac{\mu}{\Sigma(A_y)} = \frac{\nu}{\Sigma(A_z)} = \frac{1}{\sqrt{[\{\Sigma(A_x)\}^2 + \{\Sigma(A_y)\}^2 + \{\Sigma(A_z)\}^2]}},$$

which determine the direction of the plane of projection, in order that the sum of the projections of the areas may be a maximum.

III.

(1) Two straight lines are drawn in the planes of xy and yz, making angles a, γ with the axes of x, z respectively; the direction-cosines of the straight line perpendicular to the two are proportional to $\tan a$, -1, $\tan\gamma$.

(2) If two straight lines be inclined at an angle of 60°, and their direction-cosines be l, m, n, l', m', n', there will be a straight line whose direction-cosines are $l-l'$, $m-m'$, and $n-n'$, and this straight line will be inclined at angles of 60° and 120° to the former straight lines.

(3) If the angles which a straight line through the origin forms with the coordinate planes be an arithmetical progression, whose difference is 45°, the line must lie in one of the coordinate planes.

If it form angles a, $2a$, $3a$ with the coordinate axes, it must lie in one of the coordinate planes.

(4) The angle between two faces of a regular tetrahedron is $\sec^{-1}3$.

(5) Find the angle between the two straight lines, whose direction-cosines are given by $l^2 + m^2 = n^2$ and $l + m + n = 0$.

(6) Shew, by projecting upon the base, that the area of the surface of a right cone is πal, a being the radius of the base, and l the length of a slant side.

(7) Shew *a priori* that the rational equation connecting the direction-cosines of a straight line can only involve even powers of those quantities.

(8) Three circles whose areas are in the ratio 3 : 4 : 5 lie in three perpendicular planes, shew that the plane on which the sum of the projection is greatest is inclined at an angle 45° to the plane of one of the circles.

IV.

(1) The sum of the acute angles which any straight line makes with rectangular coordinate axes can never be less than $\frac{3}{2}\sec^{-1}(-3)$.

(2) The direction-cosines of a straight line perpendicular to the two whose direction-cosines are proportional to l, m, n and $m+n, n+l, l+m$, are proportional to $m-n, n-l, l-m$.

(3) The straight lines whose direction-cosines are given by the equations

$$al + bm + cn = 0, \quad \alpha l^2 + \beta m^2 + \gamma n^2 = 0,$$

will be perpendicular, if $a^2(\beta+\gamma) + b^2(\gamma+\alpha) + c^2(\alpha+\beta) = 0$,

and parallel, if $a^2/\alpha + b^2/\beta + c^2/\gamma = 0$.

(4) The straight lines whose direction-cosines are given by the equations

$$al + bm + cn = 0, \quad \alpha/l + \beta/m + \gamma/n = 0,$$

will be perpendicular, if $\alpha/a + \beta/b + \gamma/c = 0$,

and parallel, if $\sqrt{(a\alpha)} \pm \sqrt{(b\beta)} \pm \sqrt{(c\gamma)} = 0$.

(5) The direction-cosines of a line making equal angles with three straight lines whose direction-cosines are (l, m, n), (l', m', n'), (l'', m'', n''), are proportional to

$$m(n'-n'') + m'(n''-n) + m''(n-n'),$$
$$n(l'-l'') + n'(l''-l) + n''(l-l'),$$
$$l(m'-m'') + l'(m''-m) + l''(m-m').$$

If the given lines be mutually at right angles, the direction-cosines will be $(l+l'+l'')/\sqrt{3}$, $(m+m'+m'')/\sqrt{3}$, $(n+n'+n'')/\sqrt{3}$.

(6) If the direction-cosines of two straight lines be given by the equations

$$amn + bnl + clm = 0, \quad \alpha l + \beta m + \gamma n = 0,$$

prove that the tangent of the angle between the lines will be

$$\frac{\{(\alpha^2+\beta^2+\gamma^2)(a^2\alpha^2+\ldots-2bc\beta\gamma-\ldots)\}^{\frac{1}{2}}}{a\beta\gamma + b\gamma\alpha + c\alpha\beta}.$$

(7) Find the direction-cosines of the two straight lines which are equally inclined to the axis of z, and are perpendicular to each other and to the line which makes equal angles with the coordinate axes.

(8) If a plane mirror be equally inclined to each of the three coordinate planes, and λ, μ, ν be the direction-cosines of a ray incident on it, shew that those of the reflected ray will be

$$\tfrac{1}{3}(2\mu+2\nu-\lambda), \quad \tfrac{1}{3}(2\nu+2\lambda-\mu), \quad \text{and} \quad \tfrac{1}{3}(2\lambda+2\mu-\nu),$$

(9) If $\delta\theta$ be the small angle between two lines, whose direction-cosines are respectively l, m, n and $l+\delta l, m+\delta m, n+\delta n$, prove that

$$(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2.$$

(10) Determine the plane and the area of the maximum projection of the hexagon formed by the six edges of a cube that do not meet a given diagonal.

(11) If A, B, C, D be four points in a plane, A', B', C', D' their projections on any other plane, the volumes of the tetrahedrons $ABCD'$ $A'B'C'D$ will be equal.

(12) If l, m, n be the cosines of the angles which a straight line makes with three oblique coordinate axes, and λ, μ, ν be the angles between the axes,

$$\begin{vmatrix} l, & 1, & \cos\nu, & \cos\mu \\ m, & \cos\nu, & 1, & \cos\lambda \\ n, & \cos\mu, & \cos\lambda, & 1 \\ 1, & l, & m, & n \end{vmatrix} = 0.$$

CHAPTER IV.

DIVISION OF LINES IN A GIVEN RATIO. DISTANCES OF POINTS. EQUATIONS OF A STRAIGHT LINE.

37. *To find the coordinates of a point which divides the straight line joining two given points in a given ratio.*

Let the given points be $P(x, y, z)$, and $P'(x', y', z')$, and let $Q(\xi, \eta, \zeta)$ divide PP' in a given ratio, so that $PQ : QP' :: \lambda' : \lambda$. If M, N, M' be the feet of the ordinates of P, Q, P' parallel to Oz, and mQm' parallel to MNM' meet MP in m, and $M'P'$ in m',

$$Pm : m'P' :: PQ : QP' :: \lambda' : \lambda;$$

$$\therefore \lambda(\zeta - z) = \lambda'(z' - \zeta'), \text{ and } \zeta = (\lambda z + \lambda' z')/(\lambda + \lambda');$$

similarly for ξ and η.

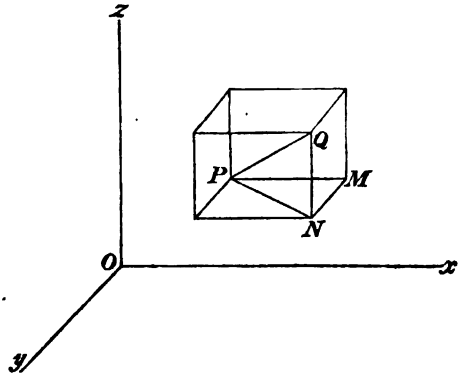
When Q lies in PP' produced in the direction of P', PQ and QP', being measured in opposite directions, are affected with opposite signs and λ is negative. In like manner, when Q is in PP' produced in the direction of P, λ' is negative. In all cases, due regard being paid to the signs of λ and λ', we have

$$PQ/\lambda' = QP'/\lambda = PP'/(\lambda + \lambda').$$

Distance between two points.

38. *To find the distance between two points whose coordinates are given, referred to rectangular axes.*

Let (x, y, z), (x', y', z') be two points P, Q whose coordinates are given referred to a rectangular system; and let a parallelepiped be constructed whose diagonal is PQ, and whose edges PM, MN, NQ are parallel to the coordinate axes Ox, Oy, Oz; and join PN.



Then, since QN is perpendicular to the plane PMN, and therefore to PN, $PQ^2 = PN^2 + QN^2$, but $PN^2 = PM^2 + MN^2$;

$$\therefore\ PQ^2 = PM^2 + MN^2 + NQ^2.$$

PM is the difference of the algebraical distances of Q and P from the plane yOz, and similarly for MN, NQ;

$$\therefore\ PQ^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2.$$

If α, β, γ be the inclinations of PQ to the axes of coordinates,

$$x' - x = PQ\cos\alpha,\quad y' - y = PQ\cos\beta,\quad z' - z = PQ\cos\gamma;$$

$$\therefore\ 1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma.$$

The double sign, which appears in the value of PQ, may be interpreted in a manner similar to that adopted in the case of the radius vector in polar coordinates in Plane Geometry.

If the angles α, β, γ define the direction of measurement of the distance PQ of Q from P, the opposite direction is defined by $\pi + \alpha$, $\pi + \beta$, $\pi + \gamma$, and therefore these angles with an algebraical distance $-PQ$ equally determine the position of the point Q with reference to P.

The distance of (x', y', z') from the origin is $\sqrt{(x'^2 + y'^2 + z'^2)}$.

39. *To find the distance between two points referred to oblique axes.*

Let λ, μ, ν be the angles between the axes, (x, y, z), (x', y', z') two points P and Q, and let α, β, γ be the angles which PQ makes with the axes.

Construct a parallelepiped whose diagonal is PQ, and whose edges PM, MN, NQ are parallel to Ox, Oy, Oz.

Now, the projections on PM of the line PQ, and of the bent line $PMNQ$ terminated in the same points, are equal;

$$\left.\begin{aligned} \text{hence,}\quad PQ\cos\alpha &= PM + MN\cos\nu + NQ\cos\mu,\\ \text{similarly,}\quad PQ\cos\beta &= MN + NQ\cos\lambda + PM\cos\nu,\\ \text{and}\quad PQ\cos\gamma &= NQ + PM\cos\mu + MN\cos\lambda, \end{aligned}\right\}\qquad (1).$$

Also PQ is the projection of $PMNQ$ on PQ;

$$\therefore\ PQ = PM\cos\alpha + MN\cos\beta + NQ\cos\gamma. \qquad (2).$$

Therefore multiplying the equations (1) by PM, MN, NQ we have by (2), $PQ^2 = PM^2 + MN^2 + NQ^2 + 2MN.NQ\cos\lambda + 2NQ.PM\cos\mu + 2PM.MN\cos\nu$, and PM is the difference of the distances of Q and P from yOz, measured parallel to Ox, and therefore $= x' - x$, and similarly $MN = y' - y$, and $NQ = z' - z$; $\therefore\ PQ^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2 + 2(y' - y)(z' - z)\cos\lambda + 2(z' - z)(x' - x)\cos\mu + 2(x' - x)(y' - y)\cos\nu$, whence PQ is determined as required.

Cor. If l, m, n be the direction-ratios of PQ,

$$PM = l.PQ,\quad MN = m.PQ,\quad NQ = n.PQ;$$

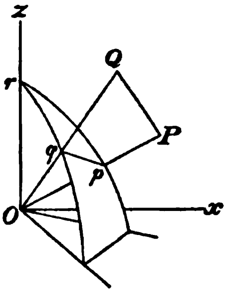
$$\therefore\ 1 = l^2 + m^2 + n^2 + 2mn\cos\lambda + 2nl\cos\mu + 2lm\cos\nu,$$

which is the equation connecting the direction-ratios of any line referred to oblique axes.

40. *To find the distance between two points whose polar coordinates are given.*

Let (r, θ, ϕ) and (r', θ', ϕ') be the given points P and Q.

Join OP, OQ, QP, and let a spherical surface, whose centre is O and radius unity, intersect OP, OQ, and OZ in p, q, and r.



Then, $rp = \theta$, $rq = \theta'$ and $\angle qrp = \phi' - \phi$.

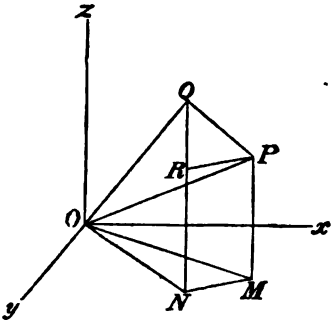
$$PQ^2 = OP^2 + OQ^2 - 2\,OP.OQ\cos pq$$
$$= r^2 + r'^2 - 2rr'\cos pq.$$

But $\cos pq = \cos pr \cos qr$
$$+ \sin pr \sin qr \cos prq$$
$$= \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi' - \phi);$$
$$\therefore PQ^2 = r^2 + r'^2 - 2rr'\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi' - \phi)\},$$

whence the distance PQ is determined in terms of the polar coordinates of P and Q.

41. The distance may be determined without Spherical Trigonometry as follows:

Draw PM, QN perpendicular to the plane of xy, join MN, OM and ON, and draw PR perpendicular to QN;



$$\therefore PQ^2 = QR^2 + PR^2$$
$$= QR^2 + MN^2,$$
$$QR = r'\cos\theta' - r\cos\theta,$$

and $MN^2 = OM^2 + ON^2$
$$- 2\,OM.ON\cos MON$$
$$= r^2\sin^2\theta + r'^2\sin^2\theta' - 2rr'\sin\theta\sin\theta'\cos(\phi' - \phi);$$
$$\therefore PQ^2 = r^2 + r'^2 - 2rr'\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi' - \phi)\},$$

which gives the required distance.

The Straight Line.

42. The general equations of the straight line which will be employed are of two forms: one form is symmetrical, and the equations are deduced from the consideration that the position of a straight line is completely determined, when one point in the line is given, and the direction in which the straight line is drawn. The symmetry of this form gives great advantages, and in all questions of a general nature the general symmetrical equations will be

almost exclusively employed. The other form is unsymmetrical, and the equations are deduced from the consideration that a straight line is the intersection of two planes, and is completely determined when the equations of the two planes are given. These equations in their simplest forms are the equations of planes parallel to two of the coordinate axes, and are the same as the equations of the projections of the straight line parallel to these axes upon two of the coordinate planes. It will be seen that, in cases in which the elimination of the constants is an essential part of the solution of a problem, the unsymmetrical equations may be used with advantage.

43. *To find the symmetrical equations of a straight line.*

Let A be a fixed point (a, b, c) of a straight line, P any other point (x, y, z), and let l, m, n the direction-cosines of AP, let $AP = r$. Then the projection of AP on the axis of x is $x - a$, and it is also lr, hence $x - a = lr$, similarly, $y - b = mr$, and $z - c = nr$. The equations of the straight line are therefore

$$(x-a)/l = (y-b)/m = (z-c)/n, \text{ or } (x-a)/L = (y-b)/M = (z-c)/N,$$

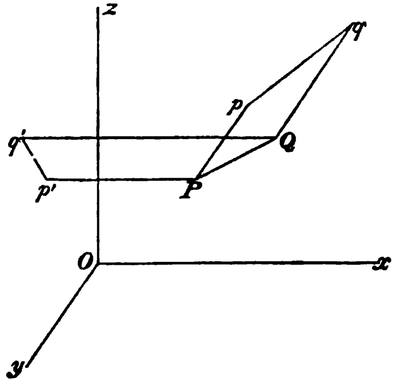
L, M, N being any quantities proportional to l, m, n.

It should be carefully remembered that, when the former equations are used, each member of the equations is equal to the distance r of the current point (x, y, z) from the fixed point (a, b, c).

The equations of a straight line will be of the same form if the axes be oblique, the same interpretation being given to r, and l, m, n being the direction-ratios. The projections employed in the above proof will then be the intercepts on the axes made by planes through A and P parallel to the coordinate planes.

44. *To find the non-symmetrical equations of a straight line.*

If a straight line PQ be projected by straight lines parallel to the axes Oy, Ox, whether rectangular or oblique, on the two coordinate planes zx, yz, each projection will be a straight line, as pq, $p'q'$, in these planes respectively.



Hence, the coordinates x, z of any point (x, y, z) in PQ being the same as those of the projection of the point in pq, satisfy an equation of the form $x = pz + h$, and the coordinates y, z similarly an equation of the form $y = qz + k$; and, consequently, the equations of the line may be written $x = pz + h$, $y = qz + k$.

45. *On the number of independent constants employed in the equations of a straight line.*

It may be noticed that the latter system of equations involves only four constants, whilst the symmetrical system involves six.

Of the three l, m, n, however, we know that they are connected by the relation $l^2 + m^2 + n^2 = 1$, Art. 22, or an equivalent relation, Art. 28, if the axes be oblique, which renders them equivalent to only two independent constants; and, if we take L, M, N, since they are only required to be proportional to l, m, n, one of these may be assumed arbitrarily, and they are still equivalent to two constants only.

Also, of the three a, b, c, one may be assumed at pleasure; for since the straight line cannot be parallel to all the coordinate planes, let it not be parallel to that of yz; then at whatever distance a from yz we take a parallel plane, the straight line will meet this plane, and we may take the point where they meet for the point (a, b, c), that is, we may give to a any value we please, and the three a, b, c are consequently equivalent to two independent constants only.

46. *To find the equations of a straight line parallel to a coordinate plane.*

If a straight line be parallel to a coordinate plane, as that of yz, every point in it will be at a constant distance from this plane, and we have the equation $x = h$, therefore the equations will be of the form $x = h$, $y = qz + k$.

For the symmetrical form, since the line will be perpendicular to the axis of x, $l = 0$, and therefore $L = 0$, and the equations of the line assume the form $(x - a)/0 = (y - b)/M = (z - c)/N$, which form implies that $x = a$ for every point in the line at a finite distance, since the members are not infinite for such values.

47. *To find the equations of a straight line parallel to one of the coordinate axes.*

If the straight line be parallel to one of the coordinate axes, it will be parallel to the two coordinate planes passing through that axis, and consequently any point in it will be at an invariable distance from each of these planes. Hence, if a straight line be parallel to the axis of z, the distances of any point in it from the planes yz, xz will be constant, a fact expressed by the equations

$x=h$, $y=k$, which will, therefore, be the equations of the line, and, as before, the symmetrical form is $(x-a)/0=(y-b)/0=(z-c)/N$.

48. *To find the angle between two straight lines whose equations are given.*

If the equations of a straight line be given in the form

$$(x-a)/L=(y-b)/M=(z-c)/N,$$

then, if l, m, n be its direction-cosines,

$$\frac{l}{L}=\frac{m}{M}=\frac{n}{N}=\frac{\pm\sqrt{(l^2+m^2+n^2)}}{\sqrt{(L^2+M^2+N^2)}}=\frac{\pm 1}{\sqrt{(L^2+M^2+N^2)}},$$

If the equations be given in the form $x=pz+h$, $y=qz+k$, since these may be written $(x-h)/p=(y-k)/q=z/1$, the direction-cosines of the line will be

$$\frac{\pm p}{\sqrt{(p^2+q^2+1)}},\quad \frac{\pm q}{\sqrt{(p^2+q^2+1)}},\quad \frac{\pm 1}{\sqrt{(p^2+q^2+1)}}$$

in which the ambiguities have the same sign.

Hence, if the equations of two straight lines be $(x-a)/L=(y-b)/M=(z-c)/N$, and $(x-a')/L'=(y-b')/M'=(z-c')/N'$, the cosine of the angle between them will be

$$\frac{LL'+MM'+NN'}{\sqrt{(L^2+M^2+N^2)}\sqrt{(L'^2+M'^2+N'^2)}},\quad \text{Art. 23.}$$

And, if the equations be

$$x=pz+h,\quad y=qz+k,$$
$$x=p'z+h',\quad y=q'z+k',$$

the cosine of the angle between them will be

$$\frac{pp'+qq'+1}{\sqrt{(p^2+q^2+1)}\sqrt{(p'^2+q'^2+1)}}.$$

49. *To find the conditions that two straight lines whose equations are given may be parallel.*

If the two straight lines in the last Article be parallel, they will have the same direction-cosines, and, since L, M, N and also L', M', N' are respectively proportional to these direction-cosines, $L/L'=M/M'=N/N'$ (1) will be the conditions of parallelism.

These conditions follow directly from the consideration that parallel lines pass through the same point at infinity.

They may also be derived from the general value of the cosine of the angle between them, which will then be unity;

$$\therefore\ (L^2+M^2+N^2)(L'^2+M'^2+N'^2)-(LL'+MM'+NN')^2=0,$$
$$\text{or } (LM'-L'M)^2+(MN'-M'N)^2+(NL'-N'L)^2=0,$$

which is equivalent to the conditions (1).

With the unsymmetrical form, if the straight lines be parallel, their projections will also be parallel, $\therefore p=p'$, and $q=q'$.

50. *To find the condition that two straight lines, whose equations are given, may be perpendicular.*

If the straight lines be perpendicular, the cosine of the angle between them will vanish, and the condition that this may be the case is $LL'+MM'+NN'=0$, or $pp'+qq'+1=0$, according to the systems of equations given.

51. *To find the condition that two straight lines, whose equations are given, may intersect.*

Let the equations of the two straight lines be

$$(x-a)/L=(y-b)/M=(z-c)/N=R,$$
$$\text{and } (x-a')/L'=(y-b')/M'=(z-c)/N'=R'.$$

Then, if the lines intersect, these equations must be simultaneously satisfied by the coordinates of the point in which they intersect.

$$\text{Hence, } a'-a+L'R'-LR=0,$$
$$b'-b+M'R'-MR=0,$$
$$c'-c+N'R'-NR=0,$$

and eliminating R and R', we obtain the required condition

$$\begin{vmatrix} a'-a, & L', & L \\ b'-b, & M', & M \\ c'-c, & N', & N \end{vmatrix}=0.$$

With the equations in the unsymmetrical form the condition is found by eliminating x, y, and z, to be $(h'-h)/(p'-p)=(k'-k)/(q'-q)$.

Straight line under given conditions.

52. *To find the equations of a straight line passing through a given point.*

If (a, b, c) be the given point, these will be in the symmetrical form $(x-a)/L=(y-b)/M=(z-c)/N$; in the unsymmetrical form $x-a=p(z-c)$, $y-b=q(z-c)$.

53. *To find the equations of a straight line passing through two given points.*

Let (a, b, c) and (a', b', c') be the two given points, the equations of the straight line are

$$(x-a)/(a'-a)=(y-b)/(b'-b)=(z-c)/(c'-c).$$

If one point be the origin, the equations are $x/a'=y/b'=z/c'$.

54. *To find the equations of a straight line passing through a given point and parallel to a given straight line.*

If the given point be (a, b, c) and L, M, N be proportional to the direction-cosines of the given line, the required equations will be $(x-a)/L = (y-b)/M = (z-c)/N$.

55. *To find the equations of a straight line passing through a given point, and perpendicular to and intersecting a given straight line.*

Let (a, b, c) be the given point, and the equations of the given straight line be $(x-a')/l = (y-b')/m = (z-c')/n$; then $(x-a)/L = (y-b)/M = (z-c)/N$ will be the required equations of the straight line, where the ratios $L:M:N$ are to be determined from the equations

$$Ll + Mm + Nn = 0, \quad \begin{vmatrix} a'-a, & l, & L \\ b'-b, & m, & M \\ c'-c, & n, & N \end{vmatrix} = 0. \quad \text{Arts. 50, 51.}$$

56. *To find the equations of a straight line passing through a given point and intersecting two given straight lines.*

Let (a, b, c) be the given point, and let the equations of the two given straight lines be

$(x-a')/L' = (y-b')/M' = (z-c')/N'$, and $(x-a'')/L'' = (y-b'')/M'' = (z-c'')/N''$;

and let the equations of the straight line satisfying the required conditions be $(x-a)/L = (y-b)/M = (z-c)/N$.

By the conditions of intersection given in Art. 51 L, M, N satisfy the equations $LP' + MQ' + NR' = 0$, $LP'' + MQ'' + NR'' = 0$, where P', Q', R', &c. are the first minors of the two corresponding determinants, whence the equations of the straight line become

$$\frac{x-a}{Q'R''-Q''R'} = \frac{y-b}{R'P''-R''P'} = \frac{z-c}{P'Q''-P''Q'}.$$

57. *To find the equations of a straight line passing through a given point, parallel to a given plane, and intersecting a given straight line.*

Let (a, b, c) be the given point, l, m, n the direction-cosines of a normal to the plane, which will therefore be perpendicular to the straight line whose equations are required, and let the equations of the given straight line be $(x-a')/l' = (y-b')/m' = (z-c')/n'$.

The required equations will then be $(x-a)/L = (y-b)/M = (z-c)/N$, where $L:M:N$ are determined by the equations

$$Ll + Mm + Nn = 0, \text{ and } \begin{vmatrix} a'-a, & l', & L \\ b'-b, & m', & M \\ c'-c, & n', & N \end{vmatrix} = 0.$$

58. *To find the distance from a given point to a given straight line.*

Let $(x-a)/l = (y-b)/m = (z-c)/n$ be the equations of the given straight line, B being the point (a, b, c); and let A be the given point (x', y', z'), AP the perpendicular from A on the straight line; then the projections of BA on the axes of x, y, z are respectively $x'-a$, $y'-b$, $z'-c$; and the projections of these on the given line are $l(x'-a)$, $m(y'-b)$, $n(z'-c)$, but the sum of these projections is the projection of BA on the straight line, or

$$BP = l(x'-a) + m(y'-b) + n(z'-c);$$

hence, $AP^2 = BA^2 - BP^2$

$$= (x'-a)^2 + (y'-b)^2 + (z'-c)^2 - \{l(x'-a) + m(y'-b) + n(z'-c)\}^2,$$

giving the required distance, which may be written

$$\sqrt{[\{n(y'-b)-m(z'-c)\}^2 + \{l(z'-c)-n(x'-a)\}^2 + \{m(x'-a)-l(y'-b)\}^2]}.$$

59. *To find the equation of a circular cylinder, the equations of whose axis and the radius of a circular section of which are given.*

The circular cylinder being the locus of a point whose distance from the axis is constant and equal to the given radius r, if $(x-a)/l = (y-b)/m = (z-c)/n$ be the equations of the axis, the equation of the surface will be, by the preceding article, $(x-a)^2 + (y-b)^2 + (z-c)^2 - \{l(x-a) + m(y-b) + n(z-c)\}^2 = r^2$.

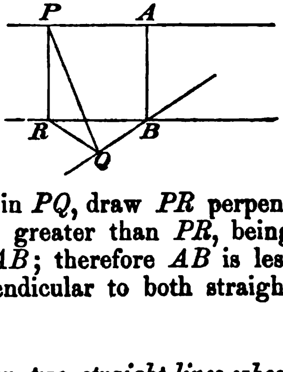
60. *To find the equation of a circular cone, whose vertex, vertical angle, and the equations of whose axis are given.*

If V be the vertex, P any point of the cone, PQ perpendicular on the axis, and 2α the vertical angle, $VQ^2 = VP^2 \cos^2\alpha$; therefore, if (a, b, c) be the vertex, the equations of the axis being as before, the equation of the cone will be

$$\{l(x-a) + m(y-b) + n(z-c)\}^2 = \cos^2\alpha\,\{(x-a)^2 + (y-b)^2 + (z-c)^2\}.$$

61. *To shew that the shortest distance between two straight lines which do not intersect is perpendicular to both.*

Let AP, BQ be the two straight lines, and let a plane be drawn through BQ parallel to AP, and BR be the orthogonal projection of AP upon this plane, B being the projection of A; therefore AB will be perpendicular to both straight lines, for it meets two parallel lines AP, BR, to one of which, BR, it is perpendicular, and it is also perpendicular to BQ, since it is drawn perpendicular to the plane QBR.



Let P, Q be any points in AP, BQ, join PQ, draw PR perpendicular to BR, and join QR; then PQ is greater than PR, being opposite to the greater angle, and $PR = AB$; therefore AB is less than PQ, or the distance which is perpendicular to both straight lines is less than any other distance.

62. *To find the shortest distance between two straight lines whose equations are given.*

Let the equations of the two straight lines be

$(x-a)/l = (y-b)/m = (z-c)/n$, and $(x-a')/l' = (y-b')/m' = (z-c')/n'$,

and let λ, μ, ν be the direction-cosines of the straight line perpendicular to each, then by Art. 24

$$\frac{\lambda}{mn'-m'n} = \frac{\mu}{nl'-n'l} = \frac{\nu}{lm'-l'm} = \frac{1}{\sin\theta},$$

θ being the angle between the lines.

Now, if we suppose P, Q, in the last figure, to be the points (a, b, c), (a', b', c'), the projection of PQ on AB, which is AB itself, will be $\lambda(a-a')+\mu(b-b')+\nu(c-c')$, hence

$$AB = \frac{(a-a')(mn'-m'n)+(b-b')(nl'-n'l)+(c-c')(lm'-l'm)}{\{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2\}^{\frac{1}{2}}}.$$

Observe that the numerator is the determinant which vanishes when the two lines intersect.
$$\begin{vmatrix} a-a', & l, & l' \\ b-b', & m, & m' \\ c-c', & n, & n' \end{vmatrix}$$

63. *To find equations of the line on which lies the shortest distance between two straight lines whose equations are given.*

Taking the equations of the last article, if (ξ, η, ζ) be any point of the line considered, the equations of the line will be

$$\frac{x-\xi}{mn'-m'n} = \frac{y-\eta}{nl'-n'l} = \frac{z-\zeta}{lm'-l'm}.$$

Hence, by Art. 51, since it meets each of the two given lines,

$$\text{we have} \quad \begin{vmatrix} \xi-a, & \eta-b, & \zeta-c \\ l, & m, & n \\ mn'-m'n, & nl'-n'l, & lm'-l'm \end{vmatrix} = 0,$$

$$\text{and} \quad \begin{vmatrix} \xi-a', & \eta-b', & \zeta-c' \\ l', & m', & n' \\ mn'-m'n, & nl'-n'l, & lm'-l'm \end{vmatrix} = 0,$$

and, since (ξ, η, ζ) is any point on the line these are equations of the line.

If l, m, n and l', m', n' be direction-cosines, since

$$m(lm'-l'm)-n(nl'-n'l) = l(mm'+nn')-l'(m^2+n^2) = l(ll'+mm'+nn')-l',$$

these equations may be written

$$(l\cos\theta-l')(x-a)+(m\cos\theta-m')(y-b)+(n\cos\theta-n')(z-c)=0,$$
$$(l'\cos\theta-l)(x-a')+(m'\cos\theta-m)(y-b')+(n'\cos\theta-n)(z-c')=0,$$

where θ is the angle between the given lines.

This form of the equations may be obtained directly by projections. Suppose, in the figure of page 27, PQ perpendicular to QB, P the point (a, b, c), and S any point (x, y, z) in BA. Project PS on PA and QB, then

$$PA = l(x-a)+m(y-b)+n(z-c), \text{ and } QB = l'(x-a)+m'(y-b)+n'(z-c),$$

also $QB = PA\cos\theta$, which gives the first of the equations, and similarly for the second.

64. A very simple form, in which the equations of two straight lines can be presented, will be obtained by taking the middle point of the shortest distance between them for the origin, the line in which it lies for one of the axes, suppose that of z, and the two planes equally inclined to the two straight lines for those of zx, zy.

If 2α be the angle between the two straight lines, $2c$ the shortest distance between them, their equations will then become

$$y = x\tan\alpha, \quad z = c, \quad \text{and} \quad y = -x\tan\alpha, \quad z = -c.$$

V.

(1) The straight line given by the equations

$$x + 2y + 3z = 0, \quad 3x + 2y + z = 0,$$

makes equal angles with the axes of x and z, and an angle $\frac{1}{2}\sec^{-1}3$ with the axis of y.

(2) Prove that the equations $\frac{x^2-1}{x-1} = \frac{y^2-1}{y-1} = \frac{z^2-1}{z-1}$ represent seven straight lines which all pass through the same point.

(3) Find the direction-cosines of the straight line determined by equations $lx + my + nz = mx + ny + lz = nx + ly + mz$.

(4) The angle between the two straight lines given by the equations $x = y$ and $xy + yz + zx = 0$, is $\sec^{-1} 3$.

(5) Find the equations of the straight line passing through the points (b, c, a) (c, a, b) and shew that it is perpendicular to the line passing through the origin and through the middle point of the line joining the two points, and also to each of the straight lines whose equations are

$$x = y = z, \quad x/a = y/b = z/c.$$

(6) Find the shortest distance between the axis of z and the straight line $(x-a)/l = (y-b)/m = z/n$, and find its equations.

(7) Find the shortest distance between an edge of a cube and a diagonal which does not meet it.

(8) Prove that the equations of any straight line intersecting the two straight lines $y = mx$, $z = c$; $y = -mx$, $z = -c$; may be written in the form

$$\frac{mx - \lambda' \cos\theta}{m \sin\theta} = \frac{y - m\lambda \sin\theta}{\cos\theta} = \frac{\lambda z}{c}.$$

(9) The equations of two straight lines are $y = x\tan a$, $z = c$, and $y = -x\tan a$, $z = -c$; shew that the distance between two points on these straight lines whose distances from the axis of z are a, b respectively is $\sqrt{(4c^2 + a^2 + b^2 \mp 2ab\cos 2a)}$.

(10) Interpret the equation $(x^2 + y^2 + z^2)(l^2 + m^2 + n^2) = (lx + my + nz)^2$, and give a geometrical illustration.

(11) The locus of the middle points of all straight lines terminated by two fixed straight lines is a plane bisecting the shortest distance between the fixed straight lines.

VI.

(1) Find the equations of the straight line which passes through the origin and intersects at right angles the straight line whose equations are

$$(m+n)x + (n+l)y + (l+m)z = a, \quad (m-n)x + (n-l)y + (l-m)z = a;$$

and obtain the coordinates of the point of intersection.

(2) The equations $\frac{x^3+1}{x+1} = \frac{y^3+1}{y+1} = \frac{z^3+1}{z+1}$ denote thirteen straight lines. Shew that four are the four diagonals of a cube, and construct for the rest.

(3) The straight lines determined by the equations

$$l(b-c)yz + m(c-a)zx + n(a-b)xy = 0, \text{ and } lx + my + nz = 0$$

are at right angles to each other.

(4) Shew that the equations $(a+mz-ny)/l=(b-nx-lz)/m+(c+ly-mx)/n$ are reducible to the form $(x+nb-mc)/l=(y+lc-na)/m=(z+ma-lb)/n$.

(5) The equations of a straight line are given in the form

$$(a-ny+ml)/\lambda=(b-lz+nx)/\mu=(c-mx+ly)/\nu,$$

obtain them in the form

$$\{(l\lambda+m\mu+n\nu)x-\mu c+\nu b\}/l=\{(l\lambda+m\mu+n\nu)y-\nu a+\lambda c\}/m=\{(l\lambda+m\mu+n\nu)z-\lambda b+\mu a\}/n.$$

(6) ABC, $A'B'C'$ are two straight lines, BB' the shortest distance between them, C, C' any two points on the two lines, such that CA' is perpendicular to $A'B'C'$ and $C'A$ to ABC; prove that $AB.BC=A'B'.B'C'$.

(7) When a ray of light is reflected from a plane mirror, the shortest distance between the incident ray and any straight line on the mirror is equal to that between the reflected ray and the same straight line.

(8) The cosine of the angle between the two straight lines whose equations are $lx+my+nz=0$, $ax^2+by^2+cz^2=0$,

$$\text{is } \frac{l^2(b+c)+m^2(c+a)+n^2(a+b)}{\sqrt{\{l^4(b-c)^2+\ldots+2m^2n^2(a-b)(a-c)+\ldots\}}}.$$

(9) If the straight lines, whose directions are determined by the equations $al^2+bm^2+cn^2=0$ or $Al^2+Bm^2+Cn^2=0$, combined with $ul+vm+wn=0$, form a harmonic pencil, prove that $u^2(bC+cB)+v^2(cA+aC)+w^2(aB+bA)=0$.

(10) The locus of the middle points of all straight lines of constant length terminated by two fixed straight lines, is an ellipse whose centre bisects the shortest distance between the fixed lines, and whose axes are equally inclined to them.

(11) If the axes of coordinates be inclined at angles α, β, γ, shew that the equations of the four straight lines, each point of which is equidistant from the three coordinate planes, will be $x^2/\sin^2\alpha=y^2/\sin^2\beta=z^2/\sin^2\gamma$.

(12) If a system of straight lines be represented by $y=\lambda x+\mu$, $z=\lambda' x+\mu'$, where λ, μ, λ', μ' are given functions of a single parameter, what will be the condition that any two consecutive lines of the system intersect?

CHAPTER V.

GENERAL EQUATION OF THE FIRST DEGREE. EQUATION OF A PLANE.

65. *The locus of the general equation of the first degree is a plane.* The general equation of the first degree is

$$Ax + By + Cz + D = 0.$$

Let (a, b, c), (a', b', c') be any two points P, P' in the locus of this equation, so that

$$Aa + Bb + Cc + D = 0 \text{ and } Aa' + Bb' + Cc' + D = 0. \quad (1)$$

The equations of the line joining P, P' are

$$(x - a)/l = (y - b)/m = (z - c)/n = r,$$

$$\text{where } (a' - a)/l = (b' - b)/m = (c' - c)/n. \quad (2).$$

The straight line PP' meets the locus at every point for which

$$A(a + lr) + B(b + mr) + C(c + nr) + D = 0, \quad (3)$$

but $Aa + Bb + Cc + D = 0$ and $A(a' - a) + B(b' - b) + C(c' - c) = 0$ by (1), $\therefore Al + Bm + Cn = 0$ by (2); hence (3) is true for all values of r; that is the straight line joining any two points of the locus lies wholly in the locus which is therefore a plane.

66. The student will readily deduce the following special positions of the plane.

i. If $D = 0$, the plane passes through the origin.

ii. If $A = 0$, the plane is parallel to the axis of x.

iii. If A and $B = 0$, the plane is parallel to the plane of xy.

iv. If A, B and $D = 0$, the plane is that of xy.

v. If A, B and $C = 0$, while D remains finite, the plane is at an infinite distance. For, the point in which the axis of x meets the plane is given by the equations $y = 0$, $z = 0$, $Ax + D = 0$; hence, the distance from the origin being $-D/A$, if A be indefinitely diminished, while D is finite, the plane cuts the axis of x at an infinite distance from the origin, and the same being true for each axis, it follows that the plane is at an infinite distance from the origin.

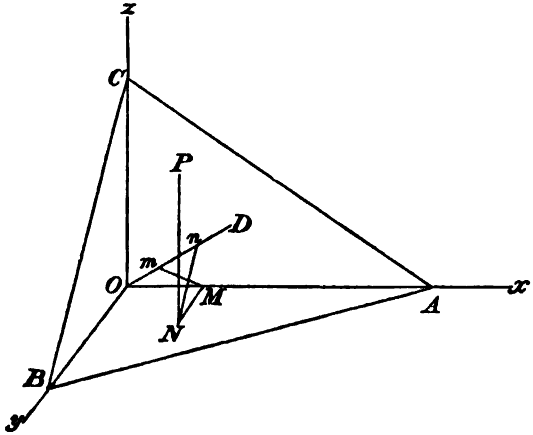
67. It is important to observe that the existence of three arbitrary constants in the general equation of the first degree, viz. the three ratios $A : B : C : D$, shews that a plane may be made to satisfy three conditions, provided each condition is one which gives only one relation between A, B, C, D. Thus, passing through a given point at a finite or infinite distance is such a condition, but being parallel to a given plane is equivalent to two such conditions.

Equation of a Plane.

68. *To find the equation of a plane in the form $lx + my + nz = p$, in which p is the perpendicular from the origin upon the plane, and l, m, n its direction-cosines.*

A plane may be considered as the locus of a straight line which passes through a given point, and is perpendicular to a given straight line.

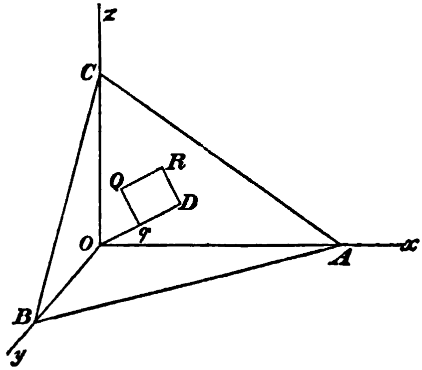
Let $OD = p$ be the perpendicular from the origin upon a plane,



l, m, n its direction-cosines, (x, y, z) any point P in the plane, then, by the definition, PD is perpendicular to OD, and OD is the sum of projections of the coordinates of P on OD; $\therefore lx + my + nz = p$, which is the equation of the plane in the form required, in which, if the axes be rectangular, $l^2 + m^2 + n^2 = 1$.

69. *Interpretation of the expression $p - lx - my - nz$.*

The equation $p - lx - my - nz = 0$ represents a plane, in which



which p is the perpendicular from the origin, and l, m, n are its direction-cosines.

Let ABC be this plane, and suppose OD, QR to be drawn perpendicular to it, in the direction defined by (l, m, n), from the origin, and from the point $Q(x, y, z)$, and join RD, which will be perpendicular to OD. Let $QR = q$, and project x, y, z and q on OD, then $p = lx + my + nz + q$; $\therefore q = p - lx - my - nz$.

Hence, the expression $p - lx - my - nz$ represents the perpendicular drawn from (x, y, z) upon the plane

$$p - lx - my - nz = 0,$$

estimated positive in the direction defined by (l, m, n).

70. *To find the angle between two planes whose equations are given.*

Let $Lx + My + Nz = D$, and $L'x + M'y + N'z = D'$,

be the given equations; then L, M, N and L', M', N' are proportional respectively to the direction-cosines of the normals; but the angle between two planes is equal to the angle between their normals, hence the cosine of the angle between the planes is

$$\frac{LL' + MM' + NN'}{\sqrt{(L^2 + M^2 + N^2)}\sqrt{(L'^2 + M'^2 + N'^2)}}.$$

The conditions of parallelism and perpendicularity are therefore respectively $L/L' = M/M' = N/N'$, and $LL' + MM' + NN' = 0$.

The student may also deduce the conditions of parallelism from the consideration that parallel planes intersect in a straight line at infinity, or directly from the parallelism of the normals.

71. *To determine the perpendicular from a point (f, g, h) upon a plane whose equation is $Ax + By + Cz + D = 0$.*

If we compare the equation $Ax + By + Cz + D = 0$ with the equation of the plane in the form $lx + my + nz - p = 0$; then

$$l/A = m/B = n/C = p/-D = \pm(A^2 + B^2 + C^2)^{-\frac{1}{2}},$$

where, if the ambiguous sign be so taken that p shall be an absolute length, l, m, n will be completely determined.

The perpendicular from (f, g, h) upon the plane, estimated positive when drawn in the direction defined by these cosines, is

$$p - lf - mg - nh = \frac{Af + Bg + Ch + D}{\mp\sqrt{(A^2 + B^2 + C^2)}},$$

the sign being chosen which is the same as that of D.

72. *To find the distance from a given point to a given plane, measured in any given direction.*

Let the equation of the plane be $Ax + By + Cz + D = 0$, and let (f, g, h) be the given point, (l, m, n) the given direction, l, m, n

being direction-cosines for rectangular axes, and direction-ratios for oblique.

The equations of a line drawn through (f, g, h) in the given direction are $(x-f)/l=(y-g)/m=(z-h)/n=r$, and where this straight line meets the plane,

$$A(f+lr)+B(g+mr)+C(h+mr)+D=0;$$

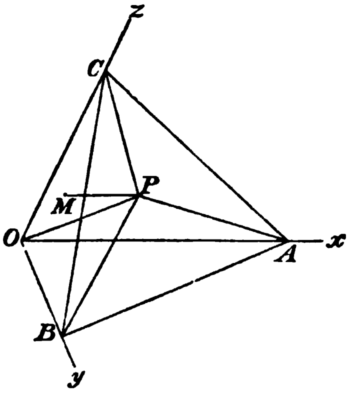
$$\therefore \text{ the required distance is } -\frac{Af+Bg+Ch+D}{Al+Bm+Cn}.$$

73. *To find the equation of a plane in the form* $x/a+y/b+z/c=1$.

The general equation of a plane is $Ax+By+Cz+D=0$, and, if a, b, c be parts of the axes of x, y, z intercepted between the origin and the plane, the plane cuts the axis of x in the point $(a, 0, 0)$; $\therefore Aa+D=0$, similarly, $Bb+D=0$ and $Cc+D=0$, hence, the equation of the plane is $x/a+y/b+z/c=1$.

74. The following direct investigation of this form of the equation is worth noticing.

Let $OA=a$, $OB=b$, $OC=c$ be the intercepts on the axes of x, y, z by the plane ABC, and let PA, PB, PC, PO be drawn from the point $P(x, y, z)$ in the plane.



Draw PM parallel to xO, meeting yOz in M. Since the pyramids $POBC$, $AOBC$ are on the same base,

$$\text{vol}\,POBC : \text{vol}\,OABC :: PM : AO :: x : a;$$

$$\therefore\; x/a=\text{vol}\,POBC/\text{vol}\,OABC; \text{ similarly, } y/b=\text{vol}\,POCA/\text{vol}\,OABC,$$

$$\text{and } z/c=\text{vol}\,POAB/\text{vol}\,OABC,$$

$$\text{and } \text{vol}\,POBC+\text{vol}\,POCA+\text{vol}\,POAB=\text{vol}\,OABC;$$

$$\therefore\; x/a+y/b+z/c=1.$$

The student is recommended to investigate this equation by the employment of a figure in which P lies in another compartment, as $x'y'z$, of the coordinate planes, taking care to interpret the geometrical into algebraical distances.

75. If q be the perpendicular from a point $Q(x, y, z)$ on the plane ABC estimated in the direction of p, the perpendicular from O on the plane,

$$q/p = \text{vol } QABC/\text{vol } OABC = 1 - x/a - y/b - z/c.$$

76. *To find the equation of the plane in the form* $z = px + qy + c$.

Consider the plane as a surface generated by a straight line which moves subject to the conditions that it always intersects one given straight line and is parallel to another.

Let the equations of the line which it intersects be

$$z = px + c, \quad y = 0; \tag{1}$$

and those of the line to which it is parallel $z = qy$, $x = 0$, the equations of the moving line will therefore be of the form

$$z = qy + \beta, \quad x = \alpha; \tag{2}$$

and, since the two lines, whose equations are (1) and (2), intersect, $\beta = p\alpha + c$; therefore, for every point in the plane, $z - qy = px + c$; that is, the equation of the plane is $z = px + qy + c$.

In this form of the equation, c is the intercept on the axis of z cut off by the plane, p, q are the tangents of the angles made respectively with the axes of x and y by the traces on the planes of zx, yz, if the coordinates be rectangular; and the ratios of the sines of the angles made with the axes in those planes, if the coordinates be oblique.

77. *To find the polar equation of a plane.*

Let c, α, β be the polar coordinates of the foot of the perpendicular from the origin on the plane; r, θ, ϕ those of any point in the plane, then if ψ be the angle between the lines joining these points to the origin, $c = r\cos\psi$,

and $\cos\psi = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\phi - \beta)$, Art. 40,

whence $c/r = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\phi - \beta)$,

the most convenient form of the equation of a plane when referred to polar coordinates.

Planes under Particular Conditions.

78. *Equation of a plane passing through a given point.*

Let a, b, c be the coordinates of the given point, and the equation of the plane $Ax + By + Cz + D = 0$, then since (a, b, c) is a point in this plane, $Aa + Bb + Cc + D = 0$, or, eliminating D,

$$A(x - a) + B(y - b) + C(z - c) = 0$$

is the general equation of a plane passing through the point (a, b, c).

79. *Equation of a plane passing through a point determined by the intersection of three given planes.*

If the point be given by the equations of three planes,

$$u = 0, \quad v = 0, \quad w = 0, \tag{1}$$

passing through it and not intersecting in one straight line, then $\lambda u + \mu v + \nu w = 0$ will be the general equation of a plane passing through that point, for it is satisfied by the values of x, y, z, which are given by the equations (1) taken simultaneously, and therefore passes through the intersection of these planes, which is the given point; and since this equation is of the first degree, and involves two arbitrary constants, namely, the ratios $\lambda : \mu : \nu$, it is the general equation of a plane passing through the given point.

If the three planes, $u = 0$, $v = 0$, $w = 0$, intersect in a straight line, then these equations, and therefore the equation $\lambda u + \mu v + \nu w = 0$, will be simultaneously satisfied for all points lying in that straight line. Hence, $\lambda u + \mu v + \nu w = 0$ cannot be the *general* equation of a plane passing through a given point. The position of a point is not, in this case, completely determined by the given equations, but only the fact that it lies on a certain straight line.

80. *Equation of a plane passing through two given points.*

Let (a, b, c), (a', b', c') be the given points; the equation of a plane passing through (a, b, c) is

$$A(x-a) + B(y-b) + C(z-c) = 0.$$

If this plane also pass through (a', b', c'), we shall have

$$A(a'-a) + B(b'-b) + C(c'-c) = 0,$$

which is the condition to which $A : B : C$ are subject; or, the equation of the plane may be written

$$\lambda \frac{x-a}{a'-a} + \mu \frac{y-b}{b'-b} + \nu \frac{z-c}{c'-c} = 0,$$

λ, μ, ν being subject to the condition $\lambda + \mu + \nu = 0$.

It is easily seen that if the points be given by the two systems of planes,

$$u = 0, \quad v = 0, \quad w = 0, \quad \text{and} \quad u = a, \quad v = b, \quad w = c,$$

that the equation of the plane will be $\lambda u + \mu v + \nu w = 0$, subject to the condition $\lambda a + \mu b + \nu c = 0$.

81. *Equation of a plane passing through the line of intersection of two planes.*

If $u = 0$, $v = 0$ be the equations of the two planes, the equation $\lambda u + \mu v = 0$ will represent a plane passing through their line of intersection; and since this equation involves one arbitrary constant $(\lambda : \mu)$, it will be the *general* equation of a plane passing through the straight line which is given by the two planes.

82. *To find the equations of two planes which form an harmonic system with two given planes.*

These two planes must pass through the line of intersection of the given planes, and divide the angles between them, so that the sines of the angles made by each with the given planes shall be in the same ratio.

Let $u=0$, $v=0$ be the equations of the given planes, and let ρ, σ be multipliers, such that ρu and σv are reduced to the form $p-lx-my-nz$; in this form they are the perpendiculars from (x, y, z) on the given planes. Hence, it is evident that $\rho u : \pm\sigma v$ are each numerically equal to the given ratio.

The forms of the equations are therefore $u-kv=0$ and $u+kv=0$.

83. *Equation of a plane passing through three given points.*

Let the three given points be (a, b, c), (a', b', c'), and (a'', b'', c''); the equation of a plane passing through the first of these points is of the form

$$\lambda(x-a)+\mu(y-b)+\nu(z-c)=0;$$

and since it passes through the other points,

$$\lambda(a'-a)+\mu(b'-b)+\nu(c'-c)=0,$$
$$\lambda(a''-a)+\mu(b''-b)+\nu(c''-c)=0;$$

$$\therefore \text{ the equation is } \begin{vmatrix} x-a, & y-b, & z-c \\ a'-a, & b'-b, & c'-c \\ a''-a, & b''-b, & c''-c \end{vmatrix}=0.$$

If one of the points as (a'', b'', c'') pass to infinity in the direction (l, m, n), $a''-a$, &c. may be replaced by l, &c.

The following method has the advantage of interpreting the coefficients geometrically.

84. Let the equation of the plane passing through the three points be $lx+my+nz=p$; and let A be the area of the triangle formed by joining the three points; A_x, A_y, A_z, the projections on the coordinate planes, and since $A_x=lA$, &c., the equation may be written

$$xA_x+yA_y+zA_z=pA=3V, \qquad (1)$$

where V is the volume of the pyramid whose base is the triangle A and vertex the origin.

Substituting the coordinates of the points in the general equation of a plane, and eliminating the constants,

$$\begin{vmatrix} x, & y, & z, & 1 \\ a, & b, & c, & 1 \\ a', & b', & c', & 1 \\ a'', & b'', & c'', & 1 \end{vmatrix}=0, \qquad (2)$$

which by Art. 33 may be written

$$2A_x x+2A_y y+2A_z z=\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}=6V \text{ by } (1).$$

The equation becomes nugatory if $A = 0$ or the three points lie in a straight line. This also appears from the vanishing of the minors in (2), for, if $A_x = 0$,

$$b(c' - c'') + b'(c'' - c) + b''(c - c') = 0,$$

$$\text{or} \quad (b - b')(c' - c'') - (b' - b'')(c - c') = 0,$$

$$\therefore \frac{b - b'}{b' - b''} = \frac{c - c'}{c' - c''} = \frac{a - a'}{a' - a''} \text{ similarly,}$$

which are the conditions that the three points lie in a straight line.

85. *To find the volume of the tetrahedron, the coordinates of the angular points of which are given.*

Referring to Art. 71, if (a''', b''', c''') be any fourth point, p' the perpendicular from it upon the plane of the triangle,

$$p' = p - la''' - mb''' - nc''';$$

$$\therefore p'A = pA - a'''A_x - b'''A_y - c'''A_z;$$

$$\therefore 6V' = \begin{vmatrix} a, & b, & c, & 1 \\ a', & b', & c', & 1 \\ a'', & b'', & c'', & 1 \\ a''', & b''', & c''', & 1 \end{vmatrix},$$

where V' is the volume required.

86. *To find the equation of a plane passing through a given point and parallel to a given plane.*

If (a, b, c) be the given point, and l, m, n the direction-cosines of a normal to the given plane, the equation of the proposed plane will be $l(x - a) + m(y - b) + n(z - c) = 0$.

87. *To find the equation of a plane which passes through two given points and is parallel to a given straight line.*

The last condition may be looked upon as giving a third point at an infinite distance, as in Art. 83, or as stating that the normal to the plane is perpendicular to the given line; in either way the third equation of Art. 83 is replaced by $\lambda l + \mu m + \nu n = 0$, and the equation of the plane is

$$\begin{vmatrix} x - a, & a' - a, & l \\ y - b, & b' - b, & m \\ z - c, & c' - c, & n \end{vmatrix} = 0.$$

This equation will be identical if $(a' - a)/l = (b' - b)/m = (c' - c)/n$, which are the conditions that the given straight line may be parallel to the line joining the two given points. Every plane passing through the two points will necessarily be parallel to the given straight line. The required equation will then be the equation of any plane passing through the two given points.

88. *To find the equation of a plane passing through a given point and parallel to two given straight lines.*

The last conditions are equivalent to statements, either that the plane passes through two points at an infinite distance in the

directions (l, m, n) and (l', m', n'), or that the normal to the plane is perpendicular to those two directions; the second and third equations of Art. 83 are replaced by $\lambda l + \mu m + \nu n = 0$, $\lambda l' + \mu m' + \nu n' = 0$, and the equation of the plane is $\begin{vmatrix} x-a, & l, & l' \\ y-b, & m, & m' \\ z-c, & n, & n' \end{vmatrix} = 0.$

If $l/l' = m/m' = n/n'$, this equation will be satisfied for all values of x, y, z; that is, if the given straight lines be parallel, there will be an infinite number of planes satisfying the given conditions, the direction of the normal to the required plane being indeterminate.

89. *To find the equation of a plane which contains one given straight line, and is parallel to another, not in the same plane.*

This proposition is that of the last article in another form, for let the equations of the given straight lines be

$(x-a)/l = (y-b)/m = (z-c)/n$, and $(x-a')/l' = (y-b')/m' = (z-c')/n'$;

the plane which contains the first line passes through the point (a, b, c), also its normal is perpendicular to each of the lines, and the equation is

$$(x-a)(mn'-m'n) + (y-b)(nl'-n'l) + (z-c)(lm'-l'm) = 0.$$

The equation of the plane containing the second and parallel to the first is

$$(x-a')(mn'-m'n) + (y-b')(nl'-n'l) + (z-c')(lm'-l'm) = 0.$$

The shortest distance of the lines is the difference of the perpendiculars from the origin, estimated in one direction, giving the same result as in Art. 62.

90. *To find the equation of a plane equidistant from two given straight lines, not in the same plane.*

Let the equations of the two given straight lines be

$(x-a)/l = (y-b)/m = (z-c)/n = r$, (1) and $(x-a')/l' = (y-b')/m' = (z-c')/n' = r'$, (2)

(x_1, y_1, z_1) a point in (1), (x_2, y_2, z_2) a point in (2), (ξ, η, ζ) the middle point of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2).

$$\text{Then, } 2\xi = x_1 + x_2 = a + a' + lr + l'r',$$
$$2\eta = y_1 + y_2 = b + b' + mr + m'r'.$$
$$2\zeta = z_1 + z_2 = c + c' + nr + n'r';$$

and eliminating r and r', we obtain, for the locus of (ξ, η, ζ), the equation

$$(2\xi - a - a')(mn' - m'n) + (2\eta - b - b')(nl' - n'l) + (2\zeta - c - c')(lm' - l'm) = 0. \quad (3)$$

The plane represented by this equation bisects all lines joining any point of (1) to any point of (2), and therefore bisects the shortest distance between them; and since the direction-cosines of the normal to (3) are proportional to $mn'-m'n$, $nl'-n'l$, $lm'-l'm$, the normal is parallel to the shortest distance between the lines, Art. 63. Hence this plane bisects at right angles the shortest distance between the lines, which is clear from the geometry.

91. *To determine the conditions necessary and sufficient in order that the general homogeneous equation of the second degree may represent two real or imaginary planes.*

Let the general equation be written

$$u_2 \equiv ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0.$$

If a be finite, the equation is equivalent to

$$(ax + c'y + b'z)^2 = (c'^2 - ab)\, y^2 + 2\,(b'c' - aa')\, yz + (b'^2 - ac)\, z^2. \quad (1)$$

But, if the equation represent two planes, x must be capable of being expressed as a linear function of y and z, and this can only happen when the second side of the above equation is a square, and therefore of the form $(py + qz)^2$, and the two planes will have equations $ax + c'y + b'z = \pm\,(py + qz)$; every point of the line of intersection of the two planes will therefore satisfy $ax + c'y + b'z = 0$ and $py + qz = 0$.

By solving with respect to y and z, if b and c be finite, we obtain similar results, hence, for every point in the line of intersection,

$$\begin{aligned} ax + c'y + b'z &= 0, \\ c'x + by + a'z &= 0, \\ b'x + a'y + cz &= 0; \end{aligned} \quad (2)$$

therefore, by eliminating x, y, and z,

$$\begin{vmatrix} a, & c', & b' \\ c', & b, & a' \\ b', & a', & c \end{vmatrix} = 0,$$

$$\text{or } H(u_2) = abc + 2a'b'c' - aa'^2 - bb'^2 - cc'^2 = 0, \quad (3)$$

this is the condition for real or imaginary planes; it might also be obtained from the consideration that the right side of equation (1) must be a complete square, viz.

$$(b'c' - aa')^2 = (c'^2 - ab)\,(b'^2 - ca),$$

$$\text{or } a\,(abc + 2a'b'c' - aa'^2 - bb'^2 - cc'^2) = 0,$$

which, since a is finite, gives the same result.

The symmetrical form of $H(u_2)$ shews that the result would have been obtained in this way whether a, b, or c were finite.

If none of them be finite, it is easily seen that a', b', or c' must be zero, and the equation will still hold.

It would be correct to say that (3) must hold however small a, b, or c are, and therefore when any or all vanish.

In order that the planes may be real, it is necessary that $c'^2 - ab$, $b'^2 - ac$, and, similarly, $a'^2 - bc$ shall not be negative.

92. *When the general equation of the second degree represents two planes, to find the equations of their line of intersection in a symmetrical form.*

Any two of the equations (2) given in the last Article are equations of the line of intersection. If we eliminate z from the first two of these equations, and x from the last two, we obtain the symmetrical equations of the line

$$x\,(b'c' - aa') = y\,(c'a' - bb') = z\,(a'b' - cc').$$

93. *Four planes have a common line of intersection, to prove that the pencil formed by their intersection by any plane has a constant anharmonic ratio.*

Take any two planes, and let their line of intersection meet the four planes in A, B, C, D, and let them meet the common line in O, O'. The anharmonic ratios of the pencils formed are therefore equal, each being $[ABCD]$, hence the ratio is constant for all planes.

94. *Four planes have a common line of intersection, to prove that the range of the four points of intersection with all cross lines will have the same anharmonic ratio.*

For a plane through any cross line forms, by its intersection with the four planes, a pencil of constant anharmonic ratio.

VII.

(1) The equation of a plane passing through the origin, and containing the straight line $(x-a)/l=(y-b)/m=(z-c)/n$ is $x(bn-cm)+y(cl-an)+z(am-bl)=0$. Hence, find the equations of the straight line passing through the origin, and intersecting two given straight lines; and examine the case in which the straight lines are parallel.

(2) Find the equation of the plane passing through the points (a, b, c), (b, c, a), (c, a, b), and shew that the planes, each of which passes through two of the points and is perpendicular to the former plane, intersect it in the sides of an equilateral triangle.

(3) The equation of a plane passing through the origin, and containing the straight line whose equations are $x+2y+3z+4=2x+3y+4z+1=3x+4y+z+2$, is $x+y-2z=0$.

(4) The equations of three planes are $x+2y-3z=1$, $2x-3y+5z=3$, and $7x-y-z=2$. Shew that the equation of a plane, equally inclined to the three axes, and passing through their common point, is $x+y+z=6$.

(5) Shew that the locus of a point dividing the distance between any two points on the two straight lines $(x-a)/l=(y-b)/m=(z-c)/n$ and $(x-a')/l'=(y-b')/m'=(z-c')/n'$, in the ratio $\lambda':\lambda$, is the plane whose equation is

$$(mn'-m'n)\{x-(\lambda a+\lambda a')/(\lambda+\lambda')\}+\&c.=0.$$

(6) Employ Art. 37 to shew that the equation $Ax+By+Cz=D$ represents a plane, according to Euclid's definition.

(7) The edges of a parallelepiped meeting in a point A are a, b, c, and a plane is drawn cutting off parts a', b', c' from these edges; prove that the plane will cut the diagonal AB in a point B', such that $AB=(a/a'+b/b'+c/c')\,AB'$.

(8) The equation of a plane passing through two straight lines

$$(x-a)/a'=(y-b)/b'=(z-c)/c', \quad (x-a')/a=(y-b')/b=(z-c')/c,$$
is $(bc'-b'c)\,x+(ca'-c'a)\,y+(ab'-a'b)\,z=0$.

Give a geometrical interpretation of the equations.

(9) Shew that the three planes

$$lx+my+nz=0, \quad (m+n)\,x+(n+l)\,y+(l+m)\,z=0, \quad x+y+z=0,$$

intersect in one straight line $x/(m-n)=y/(n-l)=z/(l-m)$.

(10) Shew that if the straight lines

$$x/a = y/\beta = z/\gamma, \quad x/aa = y/b\beta = z/c\gamma, \quad x/l = y/m = z/n,$$

lie in one plane, then $l(b-c)/a + m(c-a)/\beta + n(a-b)/\gamma = 0$.

(11) The equation of any plane containing the straight line

$$(x-a)/l = (y-b)/m = (z-c)/n \text{ is } (x-a)\lambda/l + (y-b)\mu/m + (z-c)\nu/n = 0,$$

λ, μ, ν being connected by the equation $\lambda + \mu + \nu = 0$. Hence find the equation of a plane containing one given straight line and parallel to another.

(12) The equations of two lines are

$$x = y + 2a = 6(z-a) \text{ and } x + a = 2y = -12z.$$

Find the two planes, each containing one line and parallel to the other, and thence shew that the shortest distance of the lines is $2a$.

(13) The angular points of a tetrahedron are (1, 2, 3), (2, 3, 4), (3, 4, 1), and (4, 1, 2); find the equations of its faces, and shew that two of the dihedral angles are right angles, two supplementary and one 60°. Also, that the perpendiculars from the angular points on the opposite faces are $4/\sqrt{6}$, $4/\sqrt{6}$, $2\sqrt{2}$, $2\sqrt{2}$.

(14) Find the coordinates of the centre of perpendiculars of the triangle which the coordinate planes cut from the plane $x/a + y/b + z/c = 1$.

VIII.

(1) The equation of a plane passing through the origin and containing the straight line $(a + mz - ny)/l = (\beta + nx - lz)/m = (\gamma + ly - mx)/n$

$$\text{is } (l^2 + m^2 + n^2)(ax + \beta y + \gamma z) = (la + m\beta + n\gamma)(lx + my + nz).$$

(2) Find the equation of the plane passing through the two parallel lines $(x-a)/l = (y-b)/m = (z-c)/n$; $(x-a')/l = (y-b')/m = (z-c')/n$; and explain the result when $(a-a')/l = (b-b')/m = (c-c')/n$.

(3) Shew that the line represented by the equations

$$(a + mz - ny)/(m-n) = (b + nx - lz)/(n-l) = (c + ly - mx)/(l-m),$$

is at an infinite distance in the plane $x(m-n) + y(n-l) + z(l-m) = 0$, unless $la + mb + nc = 0$, when it is indeterminate.

(4) Shew that the plane containing the line $y/b + z/c = 1$, $x = 0$, and parallel to the line $x/a - z/c = 1$, $y = 0$, is $x/a - y/b - z/c + 1 = 0$: and, if $2d$ be the shortest distance between the lines, shew that $d^{-2} = a^{-2} + b^{-2} + c^{-2}$.

(5) The equation of the planes which pass through the straight line $x/l = y/m = z/n$, and make an angle a with the plane $l'x + m'y + n'z = 0$, is

$$\{l'(ny - mz) + m'(lz - nx) + n'(mx - ly)\}^2$$
$$= \cos^2 a\,(l'^2 + m'^2 + n'^2)\{(ny - mz)^2 + (lz - nx)^2 + (mx - ly)^2\}.$$

What limitation is there to the value of a? Shew that for the limiting values the two planes coincide.

(6) Shew that the equation of the two planes inclined to the plane of xy at an angle a, and containing the line $y = 0$, $z\cos\beta = x\sin\beta$, is

$$z^2 + (x^2 + y^2)\tan^2\beta - 2zx\tan\beta = y^2\tan^2 a.$$

(7) If A, A'; B, B'; C, C' are fixed points in any three fixed straight lines passing through a point, the intersections of the planes $ABC, A'B'C'$; $A'BC, AB'C'$; $AB'C, A'BC'$; and $ABC', A'B'C$ are four straight lines lying in a plane dividing each of the fixed lines harmonically.

(8) Prove that the planes, bisecting the angles between those whose equations are $(b-c)\,x+(c-a)\,y+(a-b)\,z=0$ and $(b+c-2a)\,x+(c+a-2b)\,y+(a+b-2c)\,z=0$, will cut the plane of zx in lines making angles of 15° with the axes of x and z respectively, if a, b, c be in arithmetic progression.

(9) l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3, are the direction-cosines of three planes at right angles to one another, and p_1, p_2, p_3 are the perpendiculars from the origin upon these planes; prove that the locus of a point equally distant from these three planes is the line

$$\frac{x-(l_1p_1+l_2p_2+l_3p_3)}{l_1+l_2+l_3}=\frac{y-(m_1p_1+m_2p_2+m_3p_3)}{m_1+m_2+m_3}=\frac{z-(n_1p_1+n_2p_2+n_3p_3)}{n_1+n_2+n_3}.$$

(10) A straight line moves parallel to a fixed plane, and intersects two fixed straight lines not in the same plane; prove that the locus of a point, which divides the part intercepted in a constant ratio, is a straight line.

(11) The equations $(ax+c'y+b'z)/x=(c'x+by+a'z)/y=(b'x+a'y+cz)/z$ represent, in general, three straight lines mutually at right angles; but, if $a-b'c'/a'=b-c'a'/b'=c-a'b'/c'$, they represent a plane and a straight line perpendicular to that plane.

(12) Prove that all straight lines which intersect the three $y-z=1$, $x=0$; $z-x=1$, $y=0$; $x-y=1$, $z=0$, lie on the surface whose equation is

$$x^2+y^2+z^2-2yz-2zx-2xy=1;$$

also that the two which intersect $y=z=-x$ as well are inclined at an angle whose cosine is $\frac{2}{3}$.

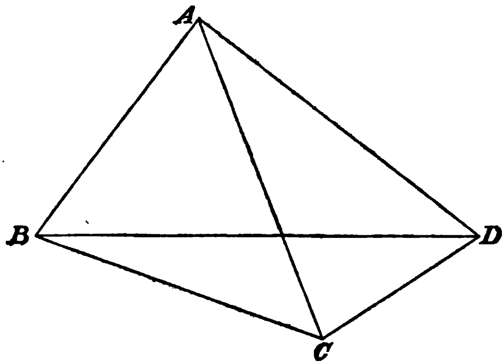
CHAPTER VI.

QUADRIPLANAR AND TETRAHEDRAL COORDINATES.

95. We now proceed to describe other systems of coordinates, which are employed in cases in which it is an object to express the relations between lines, planes, surfaces and curves by means of equations which are homogeneous in form, on account of the facilities which such forms present in the application of theorems of higher algebra.

Four-Plane or Quadriplanar System.

96. In the quadriplanar coordinate system, four planes are fixed upon as planes of reference, which form, by their intersections, a pyramid or tetrahedron $ABCD$. The position of a point is determined in this system by the algebraical distances x, y, z, w from the four planes respectively opposite to the vertices A, B, C, D, these distances being all absolute distances when the point is within the tetrahedron.



Hence, for a point in the compartment between the plane ACD and the other three produced, y will be negative and x, z, w positive; between BAC, CAD, and DAB, produced through A, x will be positive and y, z, w all negative.

If a be positive, $x=a$ will be the equation of a plane parallel to BCD, at a distance a from it, on the side towards A; $x=-a$ that of a plane on the opposite side at the same distance.

97. In this system of coordinates the following peculiarity must be observed, viz., that any three of the coordinates x, y, z, w are sufficient to determine the position of the point, since, when x, y, z are given, three planes are determined parallel to the faces opposite to A, B, C which intersect in the point, and so determine its position completely.

Hence, when x, y, z are given, w ought to be known from the geometry of the figure, and we proceed to determine the relation between the coordinates in this system.

Relation of Coordinates in the Four-Plane System.

98. Let V be the volume of the tetrahedron contained by the four fixed planes, A, B, C, D the areas of the triangular faces.

If the point P, whose coordinates are x, y, z, w, be joined by straight lines to the angular points of the tetrahedron, four pyramids will be formed, whose vertices will be at P, and whose bases will be the faces of the tetrahedron.

The algebraical sum of these four pyramids will make up the volume of the tetrahedron; therefore, remembering that the volume of a pyramid is one-third of the base × the altitude,

$$Ax + By + Cz + Dw = 3V = Ap_0 = Bq_0 = Cr_0 = Ds_0,$$

p_0, q_0, r_0, s_0 being the perpendiculars from the angular points on the opposite faces, whence, when any three of the coordinates of a point are given, the fourth may be found.

The object of the introduction of a fourth coordinate, in this system, is the same as that for which trilinear coordinates are employed in Plane Geometry, viz. to obtain equations homogeneous with reference to the coordinates, and thus to arrive at symmetrical results.

By means of the equation given above, any equation which does not appear in a homogeneous form can be reduced to such a form immediately.

Thus the equation $x = a$ of a plane may be reduced to the homogeneous form $x = a\,(x/p_0 + y/q_0 + z/r_0 + w/s_0)$.

Tetrahedral Coordinates.

99. It is evident that the relation between the coordinates given in the last Article would be much simplified if we were to select as coordinates x/p_0, y/q_0, z/r_0, w/s_0.

Such a system of coordinates is called a system of *tetrahedral coordinates*, each coordinate being the ratio of the pyramid, whose base is a face of the tetrahedron and vertex the point considered, to the volume of the fundamental tetrahedron, sign being of course always regarded.

If (x, y, z, w) represent a point in this system,

$$x + y + z + w = 1,$$

and any given equation involving four-plane coordinates may be transformed into the corresponding equation in tetrahedral coordinates by writing p_0x, q_0y, r_0z, s_0w for x, y, z, w.

Since both these systems are never employed in the same discussions, it is unecessary to adopt a different notation for the coordinates.

100. It may be shewn, as in Art. 37, that the four-plane coordinates of a point which divides the line joining two points (x, y, z, w) and (x', y', z', w') in the ratio $\mu : \lambda$ are $(\lambda x + \mu x')/(\lambda + \mu)$, &c.; and the same result will be true, if the coordinates be tetrahedral.

101. *To find the distance of two given points in tetrahedral coordinates.*

Let (x, y, z, w) and (x', y', z', w') be two given points P, Q. The square of the distance between them is easily seen to be of the second degree in terms of $x - x'$, $y - y'$, $z - z'$, $w - w'$.

But $$x + y + z + w = 1 = x' + y' + z' + w';$$
$$\therefore\ (x - x') + (y - y') + (z - z') + (w - w') = 0;$$
$$\therefore\ (x - x')^2 = -(x - x')(y - y') - \ldots,$$
and similarly for $(y - y')^2$, &c.

Hence, the square of the distance can be expressed in terms of the six products $(x - x')(y - y')$, &c.

Let γ be the coefficient of $(x - x')(y - y')$, and let us apply the expression to find the distance AB; now the coordinates of A and B are 1, 0, 0, 0, and 0, 1, 0, 0, hence every product but one vanishes, $\therefore AB^2 = -\gamma$, and
$$-PQ^2 = AB^2(x - x')(y - y') + AC^2(x - x')(z - z') + \ldots.$$

102. Hence we may obtain the equation of a sphere, the coordinates of whose centre are f, g, h, k, and whose radius is r,
$$-r^2 = a^2(y-g)(z-h) + b^2(z-h)(x-f) + c^2(x-f)(y-g)$$
$$+ a'^2(x-f)(w-k) + b'^2(y-g)(w-k) + c'^2(z-h)(w-k),$$
a, b, c being the sides of ABC opposite to A, B, C; a', b', c' the edges DA, DB, DC respectively opposite to a, b, c.

The Straight Line.

103. *To find the equations of a straight line in four-plane coordinates.*

If (f, g, h, k) be a fixed point in a straight line, (x, y, z, w) any other point, R the distance between them, λ, μ, ν, ρ the cosines of the angles between the straight line and the normals to the corresponding faces of the tetrahedron, $x - f = \lambda R$, &c.

Therefore the equations of the straight line are
$$(x - f)/\lambda = (y - g)/\mu = (z - h)/\nu = (w - k)/\rho = R,$$

where, since two equations are sufficient to determine the line, there must be a relation between λ, μ, ν, ρ.

$$\text{Now } (x-f)/p_0 + (y-g)/q_0 + (z-h)/r_0 + (w-k)/s_0 = 0,$$

$$\therefore \text{ the relation is } \lambda/p_0 + \mu/q_0 + \nu/r_0 + \rho/s_0 = 0.$$

Another relation, not homogeneous, may be obtained from the value of R, in Art. 101, changed to four-plane coordinates.

$$\frac{a^2}{q_0 r_0}\mu\nu + \frac{b^2}{r_0 p_0}\nu\lambda + \frac{c^2}{p_0 q_0}\lambda\mu + \frac{a'^2}{p_0 s_0}\lambda\rho + \frac{b'^2}{q_0 s_0}\mu\rho + \frac{c'^2}{r_0 s_0}\nu\rho = -1.$$

In this form of the equations, λ, μ, ν, ρ may be called the direction-cosines of the line.

In tetrahedral coordinates the corresponding equations are, for the straight line $(x-f)/\lambda = (y-g)/\mu = (z-h)/\nu = (w-k)/\rho = R/\sigma$, with the conditions $\lambda + \mu + \nu + \rho = 0$,

$$\text{and } a^2\mu\nu + b^2\nu\lambda + c^2\lambda\mu + a'^2\lambda\rho + b'^2\mu\rho + c'^2\nu\rho = -\sigma^2,$$

$\lambda p_0/\sigma$, $\mu q_0/\sigma$, $\nu r_0/\sigma$, $\rho s_0/\sigma$ being the direction-cosines.

104. The general equations of the straight line may be written in tetrahedral coordinates

$$(x-f)/L = (y-g)/M = (z-h)/N = (w-k)/R,$$

where L, M, N, R satisfy the equation $L+M+N+R=0$; and it may be observed that, if two equations $(x-f)/L = (y-g)/M = (z-h)/N$ be given, the fourth member may be derived from it.

If the straight line pass through one of the angular points, as A, $(1, 0, 0, 0)$ $(x-1)/L = y/M = z/N = w/R$.

If it join the middle points of AB, CD, viz. $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ and $(0, 0, \frac{1}{2}, \frac{1}{2})$, $(x-\frac{1}{2})/-\frac{1}{2} = (y-\frac{1}{2})/-\frac{1}{2} = z/\frac{1}{2} = w/\frac{1}{2}$; or $x = y$ and $z = w$.

105. *To find the angle between two straight lines whose equations are given in tetrahedral coordinates.*

Let the equations be $(x-f)/\lambda = \ldots = R/\sigma$, and $(x-f')/\lambda' = \ldots = R'/\sigma'$.

Take two straight lines parallel to these, passing through D and meeting ABC in P, P'.

The equation of DP is $x/\lambda = y/\mu = z/\nu = (w-1)/\rho = R/\sigma$, and the coordinates of P are $-\lambda/\rho$, $-\mu/\rho$, $-\nu/\rho$, 0, and $DP = -\sigma/\rho$, and similarly for P' and DP'; $\therefore -PP'^2 = a^2(\mu/\rho - \mu'/\rho')(\nu/\rho - \nu'/\rho') + b^2(\nu/\rho - \nu'/\rho')(\lambda/\rho - \lambda'/\rho') + \ldots = -\sigma^2/\rho^2 - \sigma'^2/\rho'^2 + 2\cos PDP'.\sigma\sigma'/\rho\rho'$, whence, substituting the values of σ^2 and σ'^2 (Art. 103), we obtain $\pm 2\sigma\sigma' \cos PDP' = a^2(\mu\nu' + \mu'\nu) + \ldots + a'^2(\lambda\rho' + \lambda'\rho) + \ldots$.

106. As an example of the use of this formula, we will find the angle between AD and BC, whose lengths are a' and a. For AD, $y=0$ and $z=0$, and for BC, $x=0$, $w=0$, and the values of $\lambda, \mu, \ldots, \lambda', \mu', \ldots$ may be taken respectively, $1, 0, 0, -1$ and $0, 1, -1, 0$, $\therefore \sigma^2 = a'^2$ and $\sigma'^2 = a^2$, and, if θ be the acute angle between those lines, $2aa'\cos\theta = (b^2 + b'^2) \sim (c^2 + c'^2)$.

107. The condition of perpendicularity of two straight lines given in tetrahedral coordinates $(x-f)/L = \&c.$ and $(x-f')/L' = \&c.$,

$$\text{is } a^2(MN' + M'N) + \ldots + a'^2(LR' + L'R) + \ldots = 0.$$

It may be seen, as an example, that, if the lines joining the middle points of a, a' and b, b' be perpendicular, c and c' will be equal.

The Plane.

108. *The general equation of the first degree represents a plane.*

Let $Ax + By + Cz + Dw = 0$ be any equation of the first degree.

If arbitrary points (f, g, h, k), $(f', g', h', k',)$ be taken, which satisfy the equation, the coordinates of *any* point P in the line joining them are proportional to

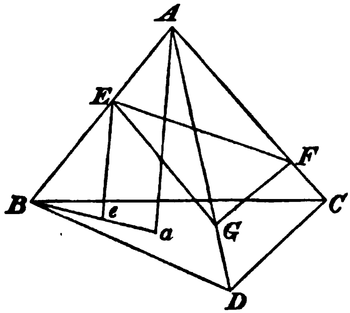
$$\lambda f + \mu f', \ \lambda g + \mu g', \ \ldots,$$

and since
$$Af + Bg + Ch + Dk = 0,$$
$$Af' + Bg' + Ch' + Dk' = 0;$$
$$\therefore A(\lambda f + \mu f') + B(\lambda g + \mu g') + \ldots = 0;$$

therefore the coordinates of P will satisfy the equation, and the whole straight line joining any two arbitrary points will lie in the locus of the equation, which is therefore a plane.

109. *Geometrical interpretation of the constants in the equation of a plane* $\lambda x + \mu y + \nu z + \rho w = 0$.

Let E be the point in which the plane EFG cuts AB, its four-plane coordinates being $x', y', 0, 0$; $\therefore \lambda x' + \mu y' = 0$.



Draw Ee, Aa perpendicular to BCD, $\therefore Ee/Aa = EB/AB$, or $x'/p_0 = EB/AB$; and similarly $y'/q_0 = EA/BA = AE/AB$; also, if p, q be the perpendiculars from A, B upon the given plane, estimated in the same direction, $p/AE = -q/EB$; $\therefore px'/p_0 + qy'/q_0 = 0$, $\lambda p_0/p = \mu q_0/q$, and similarly each $= \nu r_0/r = \rho s_0/s$, and the equation of the plane is $xp/p_0 + yq/q_0 + zr/r_0 + ws/s_0 = 0$. In tetrahedral coordinates the equation is $px + qy + rz + sw = 0$.

110. *To find the equation of a plane at an infinite distance.*

If the plane be at an infinite distance, $p = q = r = s$, and the equation in tetrahedral coordinates becomes $x + y + z + w = 0$.

111. *To find the conditions of parallelism of two planes, whose equations are given.*

Let the equations of the two planes be

$$\lambda x + \mu y + \nu z + \rho w = 0 \text{ and } \lambda' x + \mu' y + \nu' z + \rho' w = 0,$$

these planes intersect in the plane at an infinite distance, whose equation is $x + y + z + w = 0$, using tetrahedral coordinates.

From the three equations we deduce

$$(\lambda - \rho)\, x + (\mu - \rho)\, y + (\nu - \rho)\, z = 0,$$
$$(\lambda' - \rho')\, x + (\mu' - \rho')\, y + (\nu' - \rho')\, z = 0,$$

which are satisfied by an infinite number of values of the ratios $x : y : z$, they must therefore be identical equations; it follows that the equations of condition may be written

$$\left\| \begin{matrix} \lambda, & \mu, & \nu, & \rho \\ \lambda', & \mu', & \nu', & \rho' \\ 1, & 1, & 1, & 1 \end{matrix} \right\| = 0.$$

They also follow immediately from $p - p' = q - q' = \ldots$.

112. *To find the length of the perpendicular from a given point upon a plane given in four-plane or tetrahedral coordinates.*

Let the equation of the plane with four-plane coordinates be

$$\lambda x + \mu y + \nu z + \rho w = 0,$$

and let (x', y', z', w') be the given point.

Suppose the whole system to be referred to rectangular Cartesian coordinate axes, and let the equations of the four planes of reference be $l_1\xi + m_1\eta + n_1\zeta = p_1$, $l_2\xi + m_2\eta + n_2\zeta = p_2$ &c., then the equation of the given plane will become

$$(\lambda l_1 + \mu l_2 + \nu l_3 + \rho l_4)\, \xi + \ldots = \lambda p_1 + \mu p_2 + \nu p_3 + \rho p_4,$$

hence, by Art. 71, the perpendicular required will be

$$(\lambda x' + \mu y' + \nu z' + \rho w')/\sigma \qquad (1),$$

where $\sigma^2 = (\lambda l_1 + \mu l_2 + \nu l_3 + \rho l_4)^2 + \ldots$

$$= \lambda^2 (l_1^2 + m_1^2 + n_1^2) + \ldots + 2\lambda\mu\, (l_1 l_2 + m_1 m_2 + n_1 n_2) + \ldots$$
$$= \lambda^2 + \mu^2 + \nu^2 + \rho^2 - 2\lambda\mu \cos(AB) - \ldots \qquad (2),$$

(AB) being written for the dihedral angle between the faces of the tetrahedron opposite to A and B.

Cor. If p, q, r, s be the perpendiculars from A, B, C, D on the given plane, p being the perpendicular from $(p_0, 0, 0, 0)$, the expression (1) will give $p = \lambda p_0/\sigma$, $q = \mu q_0/\sigma$, &c., (3), whence the constants are interpreted as in Art. 109, and the left side of the equation of the plane in either of the forms given at the end of that article represents the length of the perpendicular from any point (x, y, z, w).

113. The method which we have adopted in the last Article shews that the quadric function $\lambda^2 + \mu^2 + \ldots - 2\lambda\mu \cos(AB) - \ldots$ is reducible by transformation to three squares, the condition of which is that the discriminant vanishes, or that

$$\begin{vmatrix} -1, & \cos(AB), & \cos(AC), & \cos(AD) \\ \cos(AB), & -1, & \cos(BC), & \cos(BD) \\ \cos(AC), & \cos(BC), & -1, & \cos(CD) \\ \cos(AD), & \cos(BD), & \cos(CD), & -1 \end{vmatrix} = 0.$$

Also, since each of the three squares is positive, there is only one system of values which reduces the function to zero, viz. that which belongs to a plane at an infinite distance, for which $p = q = r = s$, whence, by (3) Art. 112, Cor.

$$\lambda p_0 = \mu q_0 = \nu r_0 = \rho s_0.$$

That the discriminant vanishes, may be shewn independently by projecting any three of the faces of the tetrahedron on the fourth, and obtaining the determinant from the four equations similar to $A - B\cos(AB) - C\cos(AC) - D\cos(AD) = 0$.

114. *To find the angle between two planes whose equations are given.*

Let the equations of the planes be

$$\lambda x + \mu y + \nu z + \rho w = 0, \text{ and } \lambda' x + \mu' y + \nu' z + \rho' w = 0,$$

using the same method as in Art. 112, if θ be the angle between the planes

$$\sigma\sigma' \cos\theta = (\lambda l_1 + \mu l_2 + \nu l_3 + \rho l_4)(\lambda' l_1 + \mu' l_2 + \nu' l_3 + \rho' l_4) + \dots$$
$$= \lambda\lambda' + \mu\mu' + \nu\nu' + \rho\rho' - (\lambda\mu' + \lambda'\mu)\cos(AB) - \dots.$$

IX.

(1) Shew that for every point in a plane through the edge AB bisecting the angle between the planes CAB, DAB,

$z - w = 0$, if the angle be the internal angle,

$z + w = 0$, ... external ....

(2) Shew that for every point in a plane drawn through the vertex A parallel to the opposite face, $By + Cz + Dw = 0$; or, with tetrahedral coordinates, $y + z + w = 0$.

(3) If AO be drawn perpendicular to the opposite face BCD, then for any point in AO, $By/\Delta COD = Cz/\Delta DOB = Dw/\Delta BOC = AO - x$.

(4) Every point in a plane through CD parallel to AB satisfies the equation, in tetrahedral coordinates, $x + y = 0$.

(5) A point is determined in tetrahedral coordinates by the equations $lx = my = nz = rw$; what plane is represented by the equation $my = nz$, and what straight line by the equations $my = nz = rw$?

(6) At any point in the straight line joining the first points of trisection of AB and CD, the tetrahedral coordinates satisfy $x = 2y$, $z = 2w$.

(7) Shew that the coordinates (tetrahedral) of the centre of gravity of the fundamental tetrahedron are given by $x = y = z = w$.

(8) The three straight lines joining the middle points of opposite edges of a tetrahedron meet in its centre of gravity.

X.

(1) A plane cuts each of the six edges of a tetrahedron; another point is taken in each edge, so as to cut it harmonically; prove that the six planes through these latter points and the opposite edges of the tetrahedron intersect in one point.

(2) If the equations of a point O be $x/l = y/m = z/n = w/r$, and AO, BO, CO, DO be joined and produced to A', B', C', D', such that O bisects the lines AA', &c., the tetrahedral coordinates of the point A' will satisfy the equations $2x/(l - m - n - r) = y/m = z/n = w/r = 2/(l + m + n + r)$, and similarly for B', C', D'.

(3) The line joining the centres of the two spheres which touch the faces of the tetrahedron $ABCD$ opposite to A, B respectively, and the other faces produced, will intersect the edge CD in a point P, such that

$$CP : PD :: \Delta ACB : \Delta ADB,$$

and the edge AB (produced) in a point Q, such that $AQ : BQ :: \Delta CAD : \Delta CBD$.

(4) If two opposite edges of a tetrahedron be trisected, and the points of trisection be joined by two lines in either order, shew that the line which bisects these lines will also bisect two other opposite edges.

(5) l, l' are the lengths of two of the lines joining the middle points of opposite edges of a tetrahedron, ω the angle between these lines, a, a' those edges of the tetrahedron which are not met by either of the lines,

$$4ll' \cos\omega = a^2 - a'^2.$$

(6) A point O is taken within a tetrahedron $ABCD$, so as to be the centre of gravity of equal masses placed at the feet of the perpendiculars let fall from O on the faces; prove that the distances of O from the several faces are proportional respectively to those faces.

(7) Shew that the reciprocals of the radii of the spheres which can be drawn to touch the four faces of a tetrahedron are the positive values of the expression $\pm p_0^{-1} \pm q_0^{-1} \pm r_0^{-1} \pm s_0^{-1}$, p_0, q_0, r_0, s_0 being the perpendiculars from the angular points upon the opposite faces.

(8) Lines are drawn from the angular points of a tetrahedron, through the centre of the sphere circumscribing the tetrahedron, to meet the opposite faces; prove that the sum of their reciprocals is three times the reciprocal of the radius of the sphere.

(9) The inscribed sphere of a tetrahedron $ABCD$ touches the faces in A', B', C', D'; prove that AA', BB', CC', DD' will meet in a point, if

$$\cos\tfrac{1}{2}a \cos\tfrac{1}{2}\alpha = \cos\tfrac{1}{2}b \cos\tfrac{1}{2}\beta = \cos\tfrac{1}{2}c \cos\tfrac{1}{2}\gamma;$$

where a, α; b, β; c, γ are pairs of dihedral angles at opposite edges.

(10) Prove that the direction-cosines of the normal to a plane, whose equation is $px + qy + rz + sw = 0$ in tetrahedral coordinates, are

$$p/p_0 - \cos(AB)\, q/q_0 - \cos(AC)\, r/r_0 - \cos(AD)\, s/s_0, \text{ \&c.}$$

CHAPTER VII.

FOUR-POINT COORDINATE SYSTEM.

115. In the *Four-Plane Coordinate System*, the position of a point is given by its algebraical distances from four fundamental planes, given in position, which do not pass through one point, so that they form the plane faces of a tetrahedron of finite volume.

The position of a plane is given by a relation between the four-plane coordinates, which exists for every point which lies in that plane.

In the *Four-Point Coordinate System*, the position of a plane is given by its distances from four fundamental points, given in position, which do not all lie in one plane, so that they form the angular points of a tetrahedron of finite volume.

These distances are called *Point Coordinates* of the plane.

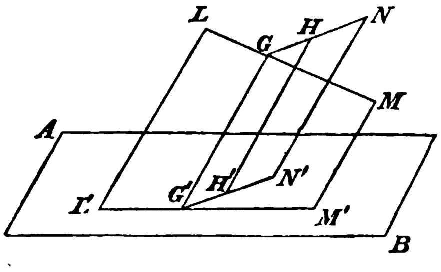
An infinite number of planes can be drawn through any given point, and it can be shewn that the point coordinates of each of these planes satisfy a linear equation; this equation determines the position of the point, and is called the equation of the point.

116. *When the position of a point relative to fixed points is known, to find its distance from a plane whose distances from the fixed points are given.*

Let L, M be two points, and let a point G be taken in the straight line joining them, such that $\lambda.LG = \mu.MG$.

Let LL', MM', GG' be parallel lines, drawn in a given direction, meeting a given plane AB in L', G', M', then it is evident that

$$(LL' - GG')/LG = (GG' - MM')/MG,$$



$$\text{and } \lambda.(LL' - GG') = \mu.(GG' - MM');$$

$$\therefore \lambda.LL' + \mu.MM' = (\lambda + \mu)\, GG'. \qquad (1)$$

If N be any other point, and H be taken in GN so that $(\lambda + \mu)\, GH = \nu.NH$, and if HH', NN' be drawn parallel to LL',

$$\text{then } \nu.NN' + (\lambda + \mu)\, GG' = (\lambda + \mu + \nu)\, HH';$$

$$\therefore \lambda.LL' + \mu.MM' + \nu NN' = (\lambda + \mu + \nu)\, HH'. \qquad (2)$$

If there be four fundamental points L, M, N, R, and K be taken in HR, such that $(\lambda + \mu + \nu)\, HK = \rho.RK$, and RR', KK' be drawn parallel to LL' meeting the plane AB in R', K', then

$$\rho.RR' + (\lambda + \mu + \nu)\, HH' = (\lambda + \mu + \nu + \rho)\, KK';$$

$$\therefore \lambda.LL' + \mu.MM' + \nu.NN' + \rho.RR' = (\lambda + \mu + \nu + \rho)\, KK'. \quad (3)$$

Thus, the position of the point K relative to four fundamental points is given by the quantities λ, μ, ν, ρ, and it may be denoted by $(\lambda, \mu, \nu, \rho)$; and the distance of the point thus determined from a plane, estimated in any direction, is known by (3) in terms of the distances of the four fundamental points estimated in the same direction. The equations (1) and (2) determine the same thing for two and three fundamental points respectively.

117. If the formula for the position of the centre of gravity of any number of particles be assumed, the results of the preceding article will be obtained at once, by considering masses proportional to λ, μ, ν, ρ placed at the fundamental points, the point K, Art. 114, being the centre of gravity of these particles, where the masses may be supposed negative if necessary.

118. *To find the equation of a point in four-point coordinates.*

If the four points lie in a plane, then, by the construction of Art. 116, it is obvious that the point K, being in the line HR, will lie in the same plane with the four points. This accounts for the restriction with respect to the fundamental points, that they shall not lie in one plane, because the equation obtained would then always denote a point in the same plane, and could not be the general equation of a point in space.

If A, B, C, D be any four points which do not lie in a plane, p, q, r, s the perpendicular distances of a plane from these points, estimated in the same direction, $(\lambda, \mu, \nu, \rho)$ a point P, with reference to these fundamental points, and t the perpendicular from P upon the plane, we shall have, by (3), Art. 116,

$$\lambda p + \mu q + \nu r + \rho s = (\lambda + \mu + \nu + \rho)\, t,$$

and $t = 0$ for every plane passing through P,

$$\therefore \lambda p + \mu q + \nu r + \rho s = 0.$$

Hence, upon the same principle, upon which, in the four-plane coordinates, x, y, z, w being the coordinates of any point,

$$\lambda x + \mu y + \nu z + \rho w = 0$$

is the equation of that plane, in which a series of points lie, whose coordinates satisfy the equation; so, p, q, r, s being the coordinates of a plane in the four-point system of coordinates, $\lambda p+\mu q+\nu r+\rho s=0$ is the equation of that point, through which all planes pass whose coordinates satisfy the equation.

119. *To interpret the constants in the equation of a point.*

Let the equation of the point P be $\lambda p+\mu q+\nu r+\rho s=0$; the four-point coordinates of BCD are p_0, 0, 0, 0, hence, the perpendicular on BCD from P is, by Art. 118, $\lambda p_0/(\lambda+\mu+\nu+\rho)$, therefore $\lambda/(\lambda+\mu+\nu+\rho)$ is the corresponding tetrahedral coordinate of P for the same fundamental tetrahedron; so that λ, μ, ν, ρ are proportional to the tetrahedral coordinates of the point whose equation is given, which may therefore be written as in Art. 109,

$$xp+yq+zr+ws=0.$$

In this form the first member of the equation is the perpendicular from the point given by the equation, also denoted by (x, y, z, w), upon the plane whose four-point coordinates are p, q, r, s.

120. *To find the equation of a point which divides the straight line joining two points, whose equations are given, in a given ratio.*

Let the equations of the two points P, P' be

$$xp+yq+zr+ws=0, \quad \text{and} \quad x'p+y'q+z'r+w's=0,$$

and let Q be a point in PP', such that $PQ : QP' :: m : l$; for every plane passing through Q the perpendiculars from P, P' are in the same ratio, and observing these perpendiculars are drawn in opposite directions if Q be between P and P', we have the equation required

$$l(xp+yq+zr+ws)+m(x'p+y'q+z'r+w's)=0.$$

121. *To find the equation of a point at an infinite distance.*

Let the equation of a point be $\lambda p+\mu q+\nu r+\rho s=0$, the perpendicular distance from this point on any plane (p, q, r, s) is $(\lambda p+\mu q+\nu r+\rho s)/(\lambda+\mu+\nu+\rho)$. If this point be at an infinite distance, we must have the condition $\lambda+\mu+\nu+\rho=0$, which expresses that, if (x, y, z, w) be the point in tetrahedral coordinates, $x+y+z+w=0$, or that the point lies in the plane at an infinite distance, Art. 110.

122. The signs of the constants in the equation of a point in the general form $\lambda p+\mu q+\nu r+\rho s=0$, in order that the point may lie in the different portions of space cut off by the indefinite planes which form the faces of the fundamental tetrahedron, can be obtained by considering that λ/σ, μ/σ, ν/σ, ρ/σ are the tetrahedral coordinates of the point, if $\sigma=\lambda+\mu+\nu+\rho$.

123. *To find the distance between two points, whose four-point coordinates are given.*

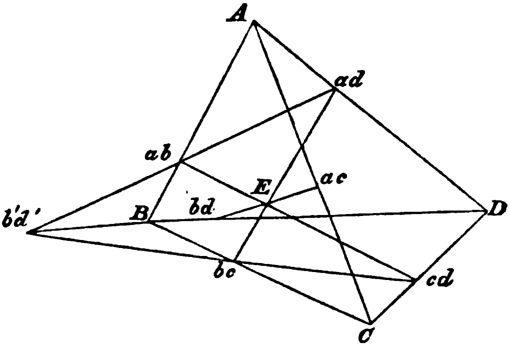
The distance between two points P, P', whose equations are $\lambda p+\mu q+\nu r+\rho s=0$, and $\lambda'p+\mu'q+\nu'r+\rho's=0$, can be found from Art. 101, by considering that λ/σ, μ/σ, ν/σ, ρ/σ, and λ'/σ', μ'/σ', ν'/σ', ρ'/σ', are tetrahedral coordinates of two points; $\therefore PP'^2=\Sigma\{(\lambda'/\sigma'-\lambda/\sigma)(\mu/\sigma-\mu'/\sigma')AB^2\}$.

124. *To shew that the straight lines joining the middle points of opposite edges of a tetrahedron intersect and bisect each other.*

The equation of the middle point of AB is $p+q=0$ (Art. 120), and of the middle point of CD is $r+s=0$; therefore the equation of the middle point of the line joining these is $p+q+r+s=0$, which for the same reason bisects the lines joining the middle points of the other opposite edges.

125. The student is recommended to examine carefully the processes employed in the following applications of point-coordinates.

Let $\lambda p+\mu q+\nu r+\rho s=0$ be the equation of any point E, and let planes be



drawn through this point and each of the edges; and let (ab) denote the point in which the plane ECD meets AB, and similarly for the other edges.

The point E lies in the straight line joining (ab), for which $\lambda p+\mu q=0$, and (cd), for which $\nu r+\rho s=0$, since its equation is of the form

$$L(\lambda p+\mu q)+M(\nu r+\rho s)=0.$$

$$\text{Since for}\quad \begin{array}{ll} (ab), & \left.\begin{array}{l}\lambda p+\mu q=0,\\ \lambda p+\rho s=0,\end{array}\right\} \\ (ad), & \\ (bc), & \left.\begin{array}{l}\mu q+\nu r=0,\\ \nu r+\rho s=0,\end{array}\right\} \\ (cd), & \end{array}$$

the straight lines joining these pairs of points meet BD in a point $(b'd')$ whose equation is $\mu q=\rho s$, and the equation of (bd) is $\mu q+\rho s=0$; therefore $b'd'$, bd divide BD harmonically.

Similarly, the lines joining (ab), (ac) and (bd), (cd) intersect BC in $(b'c')$, for which $\mu q=\nu r$; and the straight line (ab), (cd) meets the plane passing through A and the points $(b'c')$, $(b'd')$ in the point whose equation is

$$2\lambda p+2\mu q-\nu r-\rho s=0,$$

since this equation may be written $2\lambda p+(\mu q-\nu r)+(\mu q-\rho s)=0$.

Again, the equation $2\lambda p+\mu q+\nu r=0$, being of the form

$$Lp+Ms+N(\lambda p+\mu q+\nu r+\rho s)=0,$$

represents a point in the plane ADE, and, being of the form

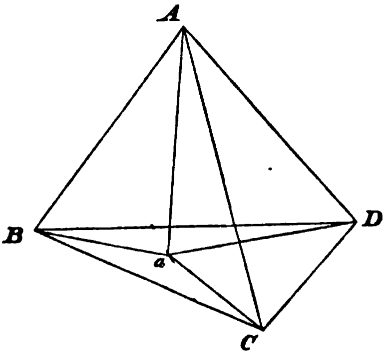
$$L(\lambda p+\mu q)+M(\lambda p+\nu r)=0,$$

the point lies on the line joining (ab), (ac), and is obviously in the plane ABC.

Let the straight lines AE, BE meet the opposite faces in A', B'; the equations of these points are $\mu q+\nu r+\rho s=0$, and $\lambda p+\nu r+\rho s=0$, and therefore $A'B'$ intersects AB in the point $\lambda p-\mu q=0$, the same point in which (ac) (bc), (ad) (bd), meet AB.

126. *To find the inclinations of a plane, whose coordinates are given, to the faces of the fundamental tetrahedron.*

Let p, q, r, s be the coordinates of a plane, λ, μ, ν, ρ the cosines of the angles between the direction in which the coordinates are measured, and the perpendiculars from A, B, C, D on the opposite faces, and let $Aa = p_0$ be the perpendicular from A on BCD.



The tetrahedral coordinates of a are 0, $\cos(AB)\,p_0/q_0$, $\cos(AC)\,p_0/r_0$, $\cos(AD)\,p_0/s_0$; the equation of the point a is therefore

$$q\cos(AB)p_0/q_0 + r\cos(AC)\,p_0/r_0 + s\cos(AD)\,p_0/s_0 = 0,$$

and, the coefficients being tetrahedral coordinates, the first member is the perpendicular from a on the plane (p, q, r, s), Art. 112, and therefore $= p - p_0\lambda$, $\therefore \lambda = p/p_0 - \cos(AB)\,q/q_0 - \cos(AC)\,r/r_0 - \cos(AD)\,s/s_0$; similar values may be obtained for μ, ν, ρ.

127. *To find the relation between the four-point coordinates of a plane.*

The tetrahedral coordinates of P, the foot of the perpendicular from A on the plane (p, q, r, s) are $(p_0 - p\lambda)/p_0$, $-p\mu/q_0$, $-p\nu/r_0$, $-p\rho/s_0$; therefore, from the equation of the point P, we obtain $\lambda p/p_0 + \mu q/q_0 + \nu r/r_0 + \rho s/s_0 = 1$. Hence, substituting the values of λ, μ, ν, ρ found above, we obtain the relation between the coordinates $p^2/p_0^2 + q^2/q_0^2 + r^2/r_0^2 + s^2/s_0^2 - 2\cos(AB)\,pq/p_0q_0 - \ldots = 1$. See (2) and (3), Art. 112.

128. If the plane be at an infinite distance the left side of the above equation will vanish, the only system of values for which this will be the case being $p = q = r = s$, Art. 113.

Since the coordinates are equal and of infinite magnitude, the expression for λ in Art. 126 gives

$$0 = 1/p_0 - \cos(AB)/q_0 - \cos(AC)/r_0 - \cos(AD)/s_0.$$

$$\text{or } 0 = A - B\cos(AB) - C\cos(AC) - D\cos(AD).$$

We have also from Art. 127 the linear relation $\lambda/p_0 + \mu/q_0 + \nu/r_0 + \rho/s_0 = 0$.

129. The equation which connects the four-point coordinates of a plane is of the second degree, whereas the corresponding equation for four-plane or tetrahedral coordinates is of the first degree.

The reason of this equation being of the second degree should be explained.

If three four-point coordinates of a plane be given, suppose p, q, r, this plane must touch three spheres whose radii are p, q, r, centres A, B, C; and, if we suppose the most general case, there will be eight such planes, two for which the three spheres lie on the same side, in which cases p, q, r will be of

the same sign, and six for which one of the spheres lies on the opposite side to the other two, in which cases two of the coordinates p, q, r will be of opposite sign to the third.

Whether p, q, r be of the same sign, or p be of opposite sign to q and r, there are two positions of the touching plane; that is, there are two values of s, viz. the perpendiculars in these two positions; the equation must, therefore, be of the second degree in s, and similarly for p, q, and r.

Hence, although, when three tetrahedral coordinates are given, the fourth is fully determined by the equation of condition, this is not the case in four-point coordinates.

130. *To find the coordinates of a plane which passes through the intersections of two planes whose coordinates are given.*

Let (p', q', r', s') (p'', q'', r'', s'') be the planes U' and U'', and let (p, q, r, s) be the plane V passing through their line of intersection. The perpendiculars from the fundamental point A on the three planes all lie in a plane, and the relation between these three may be found from the trilinear coordinates corresponding to an evanescent fundamental triangle, whose angles are the angles between the planes, or the supplement of those angles; hence

$$p \sin(U', U'') = p' \sin(U'', V) + p'' \sin(U', V);$$

$$\therefore\ p = lp' + mp'', \text{ where } l^2 \pm 2lm \cos(U', U'') + m^2 = 1,$$

and similarly for q, r, s.

131. If $Lp + Mq + Nr + Rs = 0$ be an equation involving one variable t in the first degree, it may be considered as the equation of a line, since it may be put into the form $u + tv = 0$, and by varying t we may obtain every point in the line joining the points $u = 0$, $v = 0$.

132. If $Lp + Mq + Nr + Rs = 0$ be an equation involving two variables t, t' in the first degree, it may be considered as the equation of a plane, since it may be put in the form $u + tv + t'w = 0$, and by varying t, t' we may obtain every point in the plane passing through the three points $u = 0$, $v = 0$, and $w = 0$.

XI.

(1) The equation of the centre of gravity of the face ABC is $p + q + r = 0$. Hence, shew that the lines joining the vertices with the centres of gravity of the opposite faces meet in a point.

(2) The equation of the centre of the circle circumscribing the triangle ABC is $p \sin 2A + q \sin 2B + r \sin 2C = 0$.

(3) The coordinates of the plane passing through the centres of gravity of the faces ACD, ADB, and ABC, are given by the equations

$$-\tfrac{1}{2}p = q = r = s = \tfrac{1}{3}p_0.$$

(4) If P be any point in BD, Q, R points in AC, such that

$$AQ : QC :: DP : PB :: CR : RA,$$

then PQ and PR will intersect the lines joining the middle points of BC, AD, and AB, CD respectively, and divide them in the same ratio as AC.

(5) If through the middle points of the edges BC, CD, DB straight lines be drawn parallel respectively to the opposite edges, these straight lines will meet in a point; and the line joining this point with A will pass through the centre of gravity of the pyramid.

(6) The equation of the centre of gravity of the surface of the tetrahedro is $(A+B+C+D)(p+q+r+s)=Ap+Bq+Cr+Ds$.

(7) Shew that the equations of the centres of the eight spheres which touc the faces or the faces produced of the fundamental tetrahedron are included i

$$Ap \pm Bq \pm Cr \pm Ds = 0.$$

XII.

(1) The points B, C, D are joined to the centres of gravity of the opposit faces, and the joining lines produced to points b, c, d, so that B, b, &c., ar equidistant from the corresponding faces, prove that the coordinates of th plane bcd are given by the equation $-2p=q=r=s$, and that this plane divide the edges AB, AC, AD in the ratio $1:2$.

(2) If points be taken in the lines joining B, C, D to the centres of gravit of the opposite faces, dividing them in the ratio $m:n$, the plane containin these points will divide the edges AB, AC, AD in the ratio $m:2m+3n$.

(3) If through any point P straight lines AP, BP, CP, DP be draw meeting the opposite faces in a, b, c, d, the straight lines AB, ab will intersect and their point of intersection and the point in which Cd meets AB wil divide AB harmonically.

(4) The straight lines joining D to the intersection of AB, ab, and A to th intersection of DB, db, will intersect in a point lying on Bc.

(5) The centre O of the inscribed sphere lies on the line joining G, H th centres of gravity of the volume and of the surface of the tetrahedron; als shew that $OG=3GH$.

(6) Shew that the cosine of the angle between two planes whose coor- dinates are given, is $p'p''/p_0^2+\dots-(p'q''+p''q')\cos(AB)/p_0q_0-\dots$.

CHAPTER VIII.

LOCI OF EQUATIONS. TANGENTIAL EQUATIONS OF SURFACES. TORSES. DUAL INTERPRETATION OF EQUATIONS. BOOTHIAN COORDINATES.

Tangential Equations of Surfaces and Torses.

133. If p, q, r, s be the coordinates of a plane referred to a four-point system, to find what is represented by the general equation $F(p, q, r, s) = 0$, we observe that there are generally an infinite number of planes, the coordinates of each of which satisfy the equation, and that these planes envelope a surface of which the equation is called the tangential equation, from the circumstance that each plane of the system is a tangent plane to the surface, and surface may be called the envelope locus of the equation.

As in a quadriplanar or tetrahedral system, if we take three points on the locus of $\phi(x, y, z, w) = 0$, the plane containing these three points will ultimately be a tangent plane to the locus, if the three points become ultimately coincident, so, if we take three planes touching the envelope locus of $F(p, q, r, s) = 0$, the point in which these three planes intersect will be ultimately the point of contact of these planes when they become coincident.

Thus, $p = f$ is the tangential equation of a sphere whose centre is the fundamental point A.

In the case of the linear equation $\lambda p + \mu q + \nu r + \rho s = 0$, the envelope degenerates into a point, through which all the planes pass which correspond to the various solutions of the equation.

134. Again, if we take two surfaces represented in the tetrahedral system by $\phi(x, y, z, w) = 0$, and $\phi_1(x, y, z, w) = 0$, the coordinates of every point in their curve of intersection will satisfy both equations, and the two equations will determine the position of this curve; so if $F(p, q, r, s) = 0$, and $F_1(p, q, r, s) = 0$ represent two surfaces in the four-point system, the coordinates of every plane which touches both surfaces satisfy the two equations, and the series of planes so determined have for their envelope a particular kind of surface, called a *Developable Surface*, or, according to Cayley, a *Torse*, which touches the two envelope loci of the equations, the reason of the term developable being as follows:

If we take three consecutive planes P, Q, R, each of which touches the two loci S, S_1, of the equations $F(p, q, r, s) = 0$, $F_1(p, q, r, s) = 0$, Q will be intersected by P and R in two straight

lines (P, Q) and (Q, R), and in the limit the portion of Q intercepted between these lines will ultimately be a portion of the envelope of the planes; similar portions of P and R will, with the former, constitute three elements of the envelope, each of which will touch both S and S_1; and this envelope is called a developable surface, because the three elements can be developed into one plane by turning them about the lines (P, Q), (Q, R); and the same is true for all the elements.

According to this interpretation, the developable surface touching two spheres, $p=f$, $q=g$, is a system of two cones, of which one will be imaginary if the spheres intersect.

135. It may appear, from what has been said, that, since two equations, as well as one, in the four-point system determine a surface, the case of this system is not analogous to that of the three or four-plane system. But the two kinds of surfaces in the point system are really as distinct from one another as the surface and curve in the plane system.

For, as in the four-plane system one equation represents the limitation that the current point must remain on a surface, and the second equation confines the motion on that surface to positions such that it also remains on a second surface; so in the four-point system one equation limits the motion of a plane to positions in which it touches a surface, and the second equation allows the plane to touch that surface only in such a manner that it also touches a second surface.

Stated in this way the analogy is complete, the point moving in the direction of a line, the plane turning round a line, to gain the consecutive position.

136. As a simple example of the use of the tangential equation of a surface, we will consider the properties of the *Poles of Similitude* of four spheres; the poles of similitude of two spheres being the vertices of the two cones which envelope both spheres, points from which the lengths of the tangents to the spheres are proportional to their radii.

These poles of similitude are called *internal* or *external* poles, according as they lie on the line joining the centres, or on this line produced.

137. *To find the relative positions of the internal or external poles of similitude of four spheres.*

Let the centres of the spheres be taken for the fundamental points A, B, C, and D, and let their radii be r_1, r_2, r_3, r_4.

The tangential equations of the spheres are $p=r_1$, $q=r_2$, $r=r_3$, $s=r_4$.

The external and internal poles of similitude of the spheres (A) and (B) have equations $p/r_1 \mp q/r_2 = 0$.

i. The external poles of (AB), (AC), and (AD) lie in a plane whose coordinates are connected by the equations $p/r_1 = q/r_2 = r/r_3 = s/r_4$, which evidently contains also the external poles of (BC), (CD) and (DB).

ii. The coordinates of the plane containing the external poles of (AB) and (AC) and the internal pole of (AD) satisfy the equations $p/r_1 = q/r_2 = r/r_3 = -s/r_4$, and the same plane evidently contains the external pole of (BC) and the internal poles of (BD) and (CD).

iii. The coordinates of the plane containing the external pole of (AB) and the internal poles of (AC) and (AD) satisfy the equations

$$p/r_1 = q/r_2 = -r/r_3 = -s/r_4,$$

and this plane evidently contains the external pole of (CD) and the internal poles of (BC) and (BD).

Hence one plane contains the six external poles, four planes contain each three external and three internal poles, and three contain each two external and four internal poles.

The poles of similitude lie in eight planes, which are called planes of similitude, each of which passes through six poles of similitude situated three and three in four straight lines.

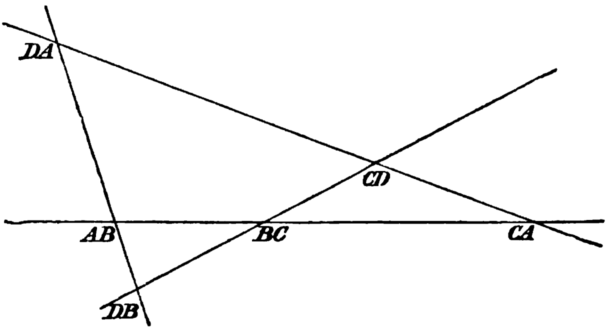
Thus for the six external poles

$$p/r_1 = q/r_2 = r/r_3 = s/r_4, \quad \text{and} \quad q/r_2 - r/r_3 = p/r_1 - r/r_3 - (p/r_1 - q/r_2) = 0;$$

therefore the external pole of (BC) lies in the line joining those of (AB) and (AC).

Similarly it lies in the lines joining those of (DB) and (DC).

Hence, the six external poles lie in the sides of a plane quadrilateral, as in the figure.



Dual Interpretations of Equations.

138. By what has preceded, we see that all homogeneous equations and systems of two equations in four variables $\alpha, \beta, \gamma, \delta$, admit of a *dual* interpretation, according as we conceive the four variables to be tetrahedral or four-point coordinates.

Thus $\lambda\alpha + \mu\beta + \nu\gamma + \rho\delta = 0$ is the equation, in these two methods of viewing it, of a plane or of a point.

So if $\alpha, \beta, \gamma, \delta$ be tetrahedral coordinates, the equation $F(\alpha, \beta, \gamma, \delta) = 0$ gives a surface on which every point lies, whose tetrahedral coordinates satisfy the equation, while if they be point coordinates, the equation gives a surface touched by every plane whose coordinates satisfy the equation.

Two such equations may in like manner be interpreted to define (1) a curve, which is the intersection of the two surfaces represented by the separate equations; or (2) a torse enveloped by all planes which touch both surfaces represented by the separate equations.

Thus, the dual results given by the method of Reciprocal

Polars, which will be seen to apply to three as well as to two dimensions, may be obtained by giving this dual interpretation to all our equations.

We cannot, however, in a volume of moderate compass pretend to include all the dual results to which our equations might give rise, but must confine ourselves to a development of the methods most generally useful.

Boothian Coordinates.

139. There is another system of coordinates which bears the same relation to four-point coordinates as the Cartesian to the tetrahedral system; these coordinates have been called Boothian from Dr. Booth, who first published the method in Dublin.*

We have seen that the equation of a plane in one form is $\alpha x + \beta y + \gamma z = 1$, where α, β, γ are the reciprocals of the intercepts on the coordinate axes; if this plane pass through a point P, (f, g, h), then $\alpha f + \beta g + \gamma h = 1$ is an equation between α, β, and γ, which will be true for all planes passing through P. α, β, γ are called Boothian coordinates of a plane, and any equation of the first degree in α, β, γ expresses that the plane passes through a certain fixed point, and may be considered the equation of that point.

Any equation whatever between α, β, γ will express that the plane touches a certain fixed surface, and may be considered the equation of that surface.

Thus, we know that the equation of a sphere is $x^2 + y^2 + z^2 = r^2$, and that the distance of the plane $lx + my + nz = r$ from the origin is r; the plane therefore touches the sphere, and if the equation of the plane be written $\alpha x + \beta y + \gamma z = 1$, then we have $\alpha^2 + \beta^2 + \gamma^2 = (l^2 + m^2 + n^2) r^{-2} = r^{-2}$, which is the Boothian equation of the sphere and is of the same form as the Cartesian.

140. *Cartesian coordinates are a particular case of tetrahedral, and Boothian of four-point coordinates.*

If we imagine the plane ABC of the tetrahedron of reference to move off to infinity, and make the corresponding changes in our equations, any equation between x, y, z, w will become one between ξ, η, ζ ordinary Cartesian coordinates, and any equation between p, q, r, s will become one between α, β, γ Boothian coordinates of a plane.

Thus take the equation $px + qy + rz + sw = 0$, where (x, y, z, w) are tetrahedral coordinates of any point, and (p, q, r, s) four-point coordinates of any plane through (x, y, z, w). If the plane meet DA, DB, DC in a, b, c, and ξ, η, ζ be the Cartesian coordinates of (x, y, z, w) referred to the planes meeting in D, then $x = \xi/DA$, $y = \eta/DB$, $z = \zeta/DC$, $w = 1 - \xi/DA - \eta/DB - \zeta/DC$, whence the equation becomes $(s-p)\,\xi/DA + (s-q)\eta/DB + (s-r)\zeta/DC = s$; but $p/s = Aa/Da$, hence $(s-p)/s \,.\, DA = 1/Da = \alpha$, and the equation of the plane becomes $\alpha\xi + \beta\eta + \gamma\zeta = 1$.

141. Any equation in which (x, y, z, w) are involved will generally have the coefficients of the different terms functions of DA, BC, ... edges of the tetrahedron of reference, so that although for any finite point (ξ, η, ζ) we have, when ABC moves off to infinity, $x = 0$, $y = 0$, $z = 0$, $w = 1$, yet we shall get a limiting equation between ξ, η, ζ when we have made the substitutions above and take DA, DB, DC, ... all infinite. So, for any equation in p, q, r, s, the coordinates of a plane at a finite distance from D, although in the limit s/p, s/q, s/r are all equal and zero, yet by making

$$p = s(1 - \alpha \,.\, DA), \quad q = s(1 - \beta \,.\, DB), \quad r = s(1 - \gamma \,.\, DC),$$

* Both Chasles and Plücker seem to have conceived the idea previously. Briot and Bouquet, *Géom. Anal.* p. 388.

we shall obtain a finite resulting equation in α, β, γ. But such transformation is very seldom of much advantage. It is, however, frequently convenient to render Cartesian or Boothian equations homogeneous by multiplying such terms as require it by w, w^2, ... or by δ, δ^2, ..., these being at some subsequent stage put equal to unity.

XIII.

(1) State some properties of the loci of the following equations, whether α, β, γ, δ be regarded as tetrahedral or four-point coordinates.

i. $l\alpha\beta = m\gamma\delta$. ii. $(\alpha+\beta)^2 = n\gamma\delta$. iii. $l/\alpha + m/\beta + n/\gamma + r/\delta = 0$.

(2) If $u=0$ be the equation of a surface of the second degree, $v=0$ that of a plane, referred to tetrahedral coordinates, $v^2 = Au$ is the equation of a surface touching the surface $u=0$ along the section made by $v=0$.

Give the interpretation when the tetrahedral are replaced by four-point coordinates.

(3) Prove that if the fundamental points be in the angles of a regular tetrahedron, the tangential equations of the spheres circumscribing and inscribed in the tetrahedron will be respectively

$$p^2 + q^2 + r^2 + s^2 - qr - rp - pq - sp - sq - sr = 0,$$
$$\text{and } qr + rp + pq + sp + sq + sr = 0.$$

(4) Shew that in the same case the envelope locus of the equation

$$p^2 + q^2 + r^2 + s^2 - 2qr - 2rp - 2pq - 2sp - 2sq - 2sr = 0$$

will touch each of the edges of the fundamental tetrahedron.

(5) In the last problem, find the two tangent planes parallel to one of the faces of the tetrahedron, and shew that their distances from that face will be $\frac{1}{2}a(\sqrt{3}\pm 1)/\sqrt{6}$, a being the length of an edge.

(6) If two surfaces given by tetrahedral coordinates intersect in two plane curves, what is the corresponding property of the torse in the dual interpretation?

(7) Prove that the Boothian equation of a sphere passing through the origin of rectangular axes is of the form

$$\alpha^2 + \beta^2 + \gamma^2 = (r^{-1} - l\alpha - m\beta - n\gamma)^2,$$

l, m, n being the direction-cosines of the radius r drawn through the origin.

(8) Two spheres of radii r, s, pass through the origin, and have their centres on the axes of x and y respectively; shew that the torse or developable surface enveloping both is a cone whose vertex has the Boothian equation $\alpha - \beta = r^{-1} - s^{-1}$.

CHAPTER IX.

TRANSFORMATION OF COORDINATES.

142. THE investigation of the properties of a surface represented by a given equation is often rendered more convenient by referring it to a different system of coordinate axes, in the choice of which we must be guided by the nature of the investigation proposed.

We proceed to obtain the working formulæ by which such a transformation may be effected.

143. *To change the origin of coordinates from one point to another, without altering the direction of the axes.*

Let (f, g, h) be the new origin referred to the primary system. If (x, y, z), (x', y', z') represent the position of the same point P referred to the first and second systems respectively, since the algebraical distance of the plane of $y'z'$ from that of yz is f, and x, x' are the algebraical distances of P from the planes of yz and $y'z'$, we have $x = x' + f$, and similarly $y = y' + g$, $z = z' + h$; or, suppressing the accents, the transformation is effected by writing $x + f$, $y + g$, $z + h$, for x, y, z.

144. Since the formulæ thus obtained involve three arbitrary constants, we can generally by this transformation make the coefficients of three terms in the resulting equation vanish, but, since the coefficients of the terms of highest dimensions are unaltered, none of the three terms, so eliminated, can be of the same dimensions as the degree of the equation. Thus, in an equation of the second degree, we can generally destroy the terms of one dimension in x, y, z; in an equation of the third degree three of the terms of two dimensions, and so on with equations of higher degrees. If, however, the terms, whose coefficients we desire to destroy, differ by more than one dimension from the degree of the equation, the equations for determining f, g, h in order to effect this result will rise to second, third, or higher degrees. For, if $F(x, y, z) = 0$ be an equation of the n^{th} degree, the transformed equation will be $F(x+f, y+g, z+h) = 0$, and the terms of the $(n-r)^{\text{th}}$ degree in the expansion can be represented in the form

$$\frac{1}{(n-r)!}\left(x\frac{d}{df} + y\frac{d}{dg} + z\frac{d}{dh}\right)^{n-r} F(f, g, h),$$

he coefficients of any term in which will involve $F(f, g, h)$ ifferentiated $n-r$ times with respect to the quantities f, g, h; ence the resulting equation for destroying any such term will be f the r^{th} degree. If three terms are to be destroyed, it is necessary hat the three corresponding equations should be consistent; it nay happen that these equations are not independent, in which ase if two terms are made to disappear the third term will disappear ıt the same time, and we shall be able to get rid of a fourth term.

145. *To transform from one system of coordinates to another ystem having the same origin, both systems being rectangular.*

Let Ox, Oy, Oz be the first system, Ox', Oy', Oz' the second; et a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3 be the direction-cosines of Ox', Oy', Oz', referred to Ox, Oy, Oz; and x, y, z; x', y', z' coordinates of he same point in the two systems.

Then the algebraic distance of the point from the plane of yz is x; but, projecting the broken line $x'+y'+z'$, this same listance is $a_1x' + a_2y' + a_3z'$. Hence

$$\left.\begin{aligned} x &= a_1x' + a_2y' + a_3z', \\ \text{and similarly, } y &= b_1x' + b_2y' + b_3z', \\ z &= c_1x' + c_2y' + c_3z', \end{aligned}\right\} \qquad (1)$$

the formulæ required.

The nine constants introduced in these results are connected by six equations of condition, expressing that the two systems of coordinates are rectangular, for since Ox, Oy, Oz are each two at right angles, we have the system of equations

$$\left.\begin{aligned} a_1^2 + b_1^2 + c_1^2 &= 1, \\ a_2^2 + b_2^2 + c_2^2 &= 1, \\ a_3^2 + b_3^2 + c_3^2 &= 1, \end{aligned}\right\} \qquad (A)$$

and by reason of Ox', Oy', Oz' being also at right angles, the system

$$\left.\begin{aligned} a_2a_3 + b_2b_3 + c_2c_3 &= 0, \\ a_3a_1 + b_3b_1 + c_3c_1 &= 0, \\ a_1a_2 + b_1b_2 + c_1c_2 &= 0. \end{aligned}\right\} \qquad (B)$$

The number of disposable constants in this transformation is therefore only three.

The relations (A), (B) subsisting among the nine constants involved in these formulæ may be replaced by

$$\left.\begin{aligned} a_1^2 + a_2^2 + a_3^2 &= 1, \\ b_1^2 + b_2^2 + b_3^2 &= 1, \\ c_1^2 + c_2^2 + c_3^2 &= 1, \end{aligned}\right\} \qquad (A')$$

$$\left.\begin{aligned} b_1c_1 + b_2c_2 + b_3c_3 &= 0, \\ c_1a_1 + c_2a_2 + c_3a_3 &= 0, \\ a_1b_1 + a_2b_2 + a_3b_3 &= 0, \end{aligned}\right\} \qquad (B')$$

if we consider Ox', Oy', Oz' the primary system of axes, in which case the direction-cosines of Ox, Oy, Oz will be a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3. The equations (A') and (B'), obtained from the same facts as the equations (A) and (B), are of course deducible from them. Either system may be obtained from the identity $x^2+y^2+z^2 \equiv x'^2+y'^2+z'^2$, by substituting for x, y, z their equivalents given in equations (1), or similarly for x', y', z'.

146. The relations between these constants may also be expressed in the following convenient form.

From the equations $a_1a_2+b_1b_2+c_1c_2=0$, $a_3a_1+b_3b_1+c_3c_1=0$, we obtain $a_1/(b_2c_3-b_3c_2)=b_1/(c_2a_3-c_3a_2)=c_1/(a_2b_3-a_3b_2)$,

each member of these equations is therefore equal to

$$\frac{(a_1^2+b_1^2+c_1^2)^{\frac{1}{2}}}{\{(b_2c_3-b_3c_2)^2+(c_2a_3-c_3a_2)^2+(a_2b_3-a_3b_2)^2\}^{\frac{1}{2}}}$$

$$=\frac{(a_1^2+b_1^2+c_1^2)^{\frac{1}{2}}}{\{(a_2^2+b_2^2+c_2^2)(a_3^2+b_3^2+c_3^2)-(a_2a_3+b_2b_3+c_2c_3)^2\}^{\frac{1}{2}}}=\pm 1,$$

by equations (A), (B).

In a similar manner, we obtain

$$a_2/(b_3c_1-b_1c_3)=b_2/(c_3a_1-c_1a_3)=c_2/(a_3b_1-a_1b_3)=\pm 1,$$

$$a_3/(b_1c_2-b_2c_1)=b_3/(c_1a_2-c_2a_1)=c_3/(a_1b_2-a_2b_1)=\pm 1.$$

By using the equations (A'), (B') in a similar manner, we obtain

$$a_1/(b_2c_3-b_3c_2)=a_2/(b_3c_1-b_1c_3)=a_3/(b_1c_2-b_2c_1),$$

which shews that the ambiguities in the three systems of equations, here obtained, must be taken all of the same sign.

Any two of these three systems of equations may be taken as completely expressing the relations between the nine constants; the third system being deducible from the other two, as well as the equations (A), (B), (A') and (B').

147. The equation $b_2c_3-b_3c_2=\pm a_1$ may be shewn by projections as follows.

Let P, Q be points in Oy', Oz' at unit distance from O, the coordinates of P and Q are a_2, b_2, c_2 and a_3, b_3, c_3. Then, by Art. 33, twice the area of the projection $OP'Q'$ of the triangle OPQ on the plane of yz is

$$\pm\begin{vmatrix} 0, & 0, & 1 \\ b_2, & c_2, & 1 \\ b_3, & c_3, & 1 \end{vmatrix}=\pm(b_2c_3-b_3c_2),$$

$\pm$ as $OP'Q'$ is in the direction Oyz or Ozy, that is of the motion of the hands of a watch, or in the contrary direction, also $2\Delta OPQ=1$, $\therefore\ 2\Delta OP'Q'=\pm a_1$, $\pm$ as $\angle xOx'$ is acute or obtuse.

148. *To find the equations of the straight lines which bisect the angles between two straight lines given by the equations*

$$lx + my + nz = 0 \quad and \quad ax^2 + by^2 + cz^2 = 0.$$

Choosing the axes, so that the plane of $x'y'$ shall contain the given lines, the axes of x' and y' being two bisectors, the equations of the given lines will be $z' = 0$, and $\lambda^2 x'^2 - \mu^2 y'^2 = 0$. (1)

The formulæ of transformation give

$$x = \alpha x' + \alpha' y' + lz', \quad y = \beta x' + \beta' y' + mz', \quad z = \gamma x' + \gamma' y' + nz';$$

hence, by (1), the second given equation becomes

$$a(\alpha x' + \alpha' y')^2 + b(\beta x' + \beta' y')^2 + c(\gamma x' + \gamma' y')^2 = 0,$$

identical with $\lambda^2 x'^2 - \mu^2 y'^2 = 0$; $\therefore a\alpha\alpha' + b\beta\beta' + c\gamma\gamma' = 0$; and since the bisectors are at right angles, $\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$, $\therefore \alpha\alpha'/(b-c) = \beta\beta'/(c-a) = \gamma\gamma'/(a-b)$. (2)

From these equations a variety of forms may be obtained for the equations of the bisecting lines; thus, if we require the forms $Ayz + Bzx + Cxy = 0$, and $lx + my + nz = 0$, since the first equation must reduce to the form $x'y' = 0$, when $z' = 0$, we have the relations $A\beta\gamma + B\gamma\alpha + C\alpha\beta = 0$, and $A\beta'\gamma' + B\gamma'\alpha' + C\alpha'\beta' = 0$;

$$\therefore \quad A : B : C = \alpha\alpha'(\beta\gamma' - \beta'\gamma) : \beta\beta'(\gamma\alpha' - \gamma'\alpha) : \gamma\gamma'(\alpha\beta' - \alpha'\beta)$$
$$= (b-c)\,l : (c-a)\,m : (a-b)\,n \quad \text{by (2) and Art, 146.}$$

hence, the equations of the bisecting lines may be written,

$$(b-c)\,l/x + (c-a)\,m/y + (a-b)\,n/z = 0, \quad \text{and} \quad lx + my + nz = 0.$$

149. The method given above solves the problem directly by transformation of coordinates, but the result may be obtained indirectly by considering the directions of the bisecting lines as those of the axes of a line-hyperbola. Thus $r^2 = x^2 + y^2 + z^2$ is a maximum subject to the conditions $lx + \ldots = 0$ and $ax^2 + \ldots = \alpha$, whence we obtain the equations $(\alpha - ar^2)\,x/l = (\alpha - br)\,y/m = (\alpha - cr^2)\,z/n$, and $(b-c)\,l/x + (c-a)\,m/y + (a-b)\,n/z = 0$, whether α be finite or zero.

150. *Euler's formulæ for transforming from one system of rectangular coordinates to another having the same origin.*

There being in the formulæ already obtained for this purpose nine constants connected by six invariable relations, it must be possible to obtain formulæ to effect this transformation which shall involve only three constants. The three chosen by Euler for this purpose are (1) the angle which the intersection of the planes of xy and $x'y'$ makes with the axis of x, (2) the angle made by the same straight line with the axis of x', (3) the angle between the planes of xy and $x'y'$.

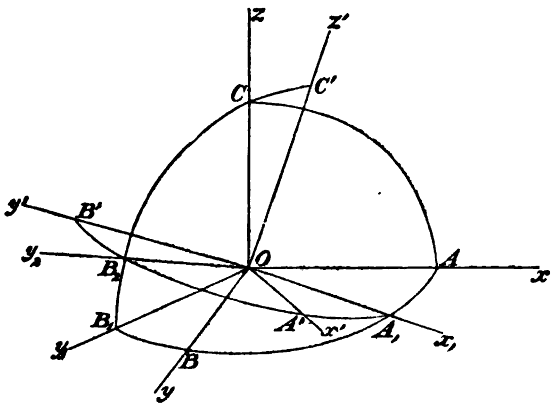
Let Ox, Oy, Oz be the original; Ox', Oy', Oz' the transformed axes of coordinates; Ox_1 the intersection of the planes of xy, $x'y'$; $xOx_1 = \phi$, $x'Ox_1 = \psi$, $zOz' = \theta$, which is the same as the angle between the planes of xy, $x'y'$.

The transformations may be effected by successive transformations, each in one plane—

i. through an angle ϕ, in the plane of xy, from Ox, Oy to Ox_1, Oy_1;

ii. through an angle θ, in the plane of y_1z, from Oy_1, Oz to Oy_2, Oz';

iii. through an angle ψ, in the plane of y_2x_1, from Ox_1, Oy_2 to Ox', Oy'.



The formulæ for these transformations are, using the same suffix for any one of the coordinates as for the corresponding axis,

$$\left.\begin{aligned} x &= x_1 \cos\phi - y_1 \sin\phi, \\ y &= x_1 \sin\phi + y_1 \cos\phi, \end{aligned}\right\}$$

$$\left.\begin{aligned} y_1 &= y_2 \cos\theta - z' \sin\theta, \\ z &= y_2 \sin\theta + z' \cos\theta, \end{aligned}\right\}$$

$$\left.\begin{aligned} x_1 &= x' \cos\psi - y' \sin\psi, \\ y_2 &= x' \sin\psi + y' \cos\psi, \end{aligned}\right\}$$

from which we obtain, by successive substitutions,

$$x = x'(\cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta)$$
$$- y'(\cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta) + z'\sin\phi\sin\theta,$$
$$y = x'(\sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta)$$
$$- y'(\sin\phi\sin\psi - \cos\phi\cos\psi\cos\theta) - z'\cos\phi\sin\theta,$$
$$z = x'\sin\psi\sin\theta + y'\cos\psi\sin\theta + z'\cos\theta.$$

These formulæ might be established without successive transformation by Spherical Trigonometry, but this is left for the exercise of the student.

151. These formulæ are too complicated and unsymmetrical to be generally employed; a modification of them, however, is useful in determining the nature of any proposed plane section of a surface. We may in that case, by using the first two transformations, make the plane of x_1y_2 coincide with the proposed plane section, and then, making $z' = 0$, obtain the equation of the section in that plane.

The formulæ in this case are easily obtained independently, viz. $x = x'\cos\phi - y'\cos\theta\sin\phi$, $y = x'\sin\phi + y'\cos\theta\cos\phi$, $z = y'\sin\theta$, and by effecting these substitutions we may obtain the equation of the plane section of the surface.

If the equation of the plane be $lx+my+nz=0$, and the curve of intersection with the surface $f(x, y, z)=0$ be required, we shall have $\cos\theta=n/\sqrt{(l^2+m^2+n^2)}$, and $\sin\phi/(-l)=\cos\phi/m$, and the equation of the curve of intersection will be

$$f(x\cos\phi-y\cos\theta\sin\phi,\ x\sin\phi+y\cos\theta\cos\phi,\ y\sin\theta)=0.$$

152. As an example of this method, we will examine the position of a plane passing through the origin, when its intersection with the surface $ax^2+by^2+cz^2=1$ is a circle.

Let the intersection of the plane with the plane of xy make an angle ϕ with Ox, and let θ be its inclination to the plane of xy.

The equation of the section will be

$$a(x\cos\phi-y\cos\theta\sin\phi)^2+b(x\sin\phi+y\cos\theta\cos\phi)^2+cy^2\sin^2\theta=1,$$

if this be a circle, the coefficient of $xy=0$, and those of x^2 and y^2 will be equal and positive; $\therefore\ (a-b)\cos\theta\sin 2\phi=0$, and

$$a\cos^2\phi+b\sin^2\phi=(a\sin^2\phi+b\cos^2\phi)\cos^2\theta+c\sin^2\theta>0,$$

we shall therefore obtain the following systems of solutions if a, b, c be unequal:

i. $\cos\theta=0$, $a\cos^2\phi+b\sin^2\phi=c>0$, or $\cos^2\phi/(b-c)=\sin^2\phi/(c-a)$.

ii. $\phi=0$, $a=b\cos^2\theta+c\sin^2\theta>0$, or $\cos^2\theta/(c-a)=\sin^2\theta/(a-b)$.

iii. $\phi=\frac{1}{2}\pi$, $b=a\cos^2\theta+c\sin^2\theta>0$, or $\cos^2\theta/(c-b)=\sin^2\theta\,(b-a)$.

If a, b, c be of the same sign and in order of magnitude, iii will be the only admissible solution, and the cutting plane must pass through the axis corresponding to the mean coefficient.

If a and b be positive, c negative, $b>a$, ii will be the only solution.

If a be positive, b and c negative, there will be no plane circular section through the origin.

153. *Transformation from one system of coordinates to another having the same origin, both systems being oblique.*

Let Ox, Oy, Oz and Ox', Oy', Oz' be the two systems; On, On', On'' the normals respectively to yz, zx, and xy, and let nx denote the angle nOx, and so for the others. Then the algebraical distance of a point whose coordinates in the two systems are respectively x, y, z and x', y', z' from the plane of yz, is $x\cos nx$, and is also $x'\cos nx'+y'\cos ny'+z'\cos nz'$,

$$\therefore\ x\cos nx=x'\cos nx'+y'\cos ny'+z'\cos nz',$$

and similarly for y and z.

154. *Transformation from any one system of axes to any other.*

If we wish in any of the above transformations of the directions of the axes also to remove the origin, we may first remove the origin to the point (f, g, h), retaining the directions of the axes. This will give $x=x_1+f$, $y=y_1+g$, $z=z_1+h$, x_1, y_1, z_1 being the coordinates of a point (x, y, z) referred to the system of axes through the new origin parallel to the primary system. Now changing the direction by transformations of the form

$$x_1=a_1x'+a_2y'+a_3z',\ \&c.,$$

we see that the most general transformation possible is obtained by

formulæ of the form

$$x = f + a_1x' + a_2y' + a_3z',$$
$$y = g + b_1x' + b_2y' + b_3z',$$
$$z = h + c_1x' + c_2y' + c_3z'.$$

155. *To shew that the degree of an equation cannot be chang[ed] by transformation of coordinates.*

We can now prove the important proposition, that the degree [of] an equation cannot be altered by any transformation of coordinate[s]; the degree of an equation meaning the greatest number which ca[n] be obtained by adding the indices of the coordinates involved in an[y] term. For let $Ax^py^qz^r$ be a term in an equation of the n^{th} degre[e] such that $p+q+r=n$: this will be a type of all the terms of th[e] n^{th} degree involved in the equation, any one of which may b[e] obtained by assigning to A, p, q, r suitable values. Now on an[y] transformation this term becomes

$$A\,(f + a_1x' + a_2y' + a_3z')^p\,(g + b_1x' + b_2y' + b_3z')^q\,(h + c_1x' + c_2y' + c_3z')$$

and no term in this product rises beyond the degree $p+q+r$ or n. Hence the degree of an equation cannot be raised by transformatio[n] of coordinates; nor can it be depressed, for if by any transformatio[n] the degree be depressed, then on re-transformation the degree [of] the equation so depressed would be raised to its original valu[e] which we have seen to be impossible.

156. *Relations between coefficients of a ternary quadric befo[re] and after transformation of coordinates.*

We notice here that in the case of quadric functions, relation[s] between the coefficients in the original and transformed function[s] may be obtained without the use of the formulæ of transformatio[n].

The method of obtaining these relations depends upon th[e] consideration that, if a quadric be the product of two linear factor[s] it will still be so after any transformation of coordinates has bee[n] effected.

The square of the distance of any point (x, y, z) from the origi[n] being $x^2+y^2+z^2$, if the axes be rectangular, this expression will b[e] unaltered in form when a change is made from one set of rectangula[r] axes to another having the same origin.

Let $u \equiv ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy$ be any ternar[y] quadric, and let h be supposed so chosen that $h(x^2+y^2+z^2) - u$ shall be the product of two linear functions; the condition tha[t] this shall be the case is, Art. 91,

$$(h-a)(h-b)(h-c) - a'^2(h-a) - b'^2(h-b) - c'^2(h-c) - 2a'b'c' = 0$$

shewing that there are generally three such values of h.

Suppose now that, on transformation to another system o[f] rectangular axes, u becomes

$$v \equiv \alpha x^2 + \beta y^2 + \gamma z^2 + 2\alpha' yz + 2\beta' zx + 2\gamma' xy,$$

then $h(x^2+y^2+z^2)-v$ will for the same values of h be the product of two linear factors;

$$\therefore\ (h-\alpha)(h-\beta)(h-\gamma)$$
$$-\alpha'^2(h-\alpha)-\beta'^2(h-\beta)-\gamma'^2(h-\gamma)-2\alpha'\beta'\gamma'=0.$$

The two cubics being satisfied by the same values of h, we may equate the coefficients and obtain three relations

$$a+b+c=\alpha+\beta+\gamma,$$
$$bc+ca+ab-a'^2-b'^2-c'^2=\beta\gamma+\gamma\alpha+\alpha\beta-\alpha'^2-\beta'^2-\gamma'^2,$$
$$abc-aa'^2-bb'^2-cc'^2+2a'b'c'=\alpha\beta\gamma-\alpha\alpha'^2-\beta\beta'^2-\gamma\gamma'^2+2\alpha'\beta'\gamma',$$

these three functions of the coefficients, $a+b+c$, &c. are called invariants of the quadric, being equal to the same functions of the corresponding coefficients of the transformed quadric.

157. In the more general case of transformation from any system of axes to any other, the square of the distance of (x, y, z) from the origin being

$$x^2+y^2+z^2+2yz\cos\lambda+2zx\cos\mu+2xy\cos\nu,$$

it is easily seen that the two cubics which determine h will be of the form

$$(h-a)(h-b)(h-c)-(h\cos\lambda-a')^2(h-a)-\ldots$$
$$+2(h\cos\lambda-a')(h\cos\mu-b')(h\cos\nu-c')=0,$$

and the corresponding invariants are the ratios of the coefficients to the coefficient of h^3.

158. *To transform from rectangular to polar coordinates.*

In the cases in which polar coordinates are required to be used, we may first transform the axes so that the axis of z is parallel to the line from which θ is measured, and the plane of zx parallel to the plane from which ϕ is measured. If when referred to these axes the coordinates of the pole be f, g, h, the formulæ expressing the rectangular in terms of the polar coordinates will be

$$x=f+r\sin\theta\cos\phi,\quad y=g+r\sin\theta\sin\phi,\quad z=h+r\cos\theta.$$

159. *To transform from a four-plane to a three-plane coordinate system.*

This is effected by the substitution of $p-lx-my-nz$ for x, and by similar substitutions for y, z, w.

If the three planes terminating in D be taken for the three-plane system, and l, m, n be the sines of the angles which the edges DA, DB, DC make with the planes DBC, DCA, DAB respectively, we shall have to write lx, my, nz for x, y, z, and $s_0(1-lx/p_0-my/q_0-nz/r_0)$ for w to effect the transformation.

160. *To transform from one four-point coordinate system to another.*

If the equations of the fundamental points of the second system, referred to the first, be $\lambda p + \mu q + \nu r + \rho s = 0$, &c., and p, q, r, s; p', q', r', s', be the coordinates of any plane in the two systems,

$$p' = (\lambda p + \mu q + \nu r + \rho s)/(\lambda + \mu + \nu + \rho), \text{ \&c. Art. 116,}$$

from which equations the formulæ required for transformation can be deduced.

XIV.

(1) If $l_1m_1n_1$, $l_2m_2n_2$, $l_3m_3n_3$ be the direction-cosines of a system of rectangular axes, and if $a/l_1+b/m_1+c/n_1=0$, and $a/l_2+b/m_2+c/n_2=0$, then will $a/l_3+b/m_3+c/n_3=0$, and $a:b:c :: l_1l_2l_3 : m_1m_2m_3 : n_1n_2n_3$.

(2) If $al_1^2 + bm_1^2 + cn_1^2 = al_2^2 + bm_2^2 + cn_2^2 = al_3^2 + bm_3^2 + cn_3^2 = 0$,

shew that $l_1^2 - m_1^2 : l_2^2 - m_2^2 : l_3^2 - m_3^2 :: m_1^2 - n_1^2 : m_2^2 - n_2^2 : m_3^2 - n_3^2$,

and that $l_1(m_2n_3 + m_3n_2) + l_2(m_3n_1 + m_1n_3) + l_3(m_1n_2 + m_2n_1) = 0$.

(3) Transform the equation $yz + zx + xy = a^2$, referred to rectangular axes, to an equation referred to another system, one of which makes equal angles with the original axes.

(4) Shew that, by the same transformation as in the last problem, the equation $x^2 + y^2 + z^2 + yz + zx + xy = a^2$ is reduced to the form $4x^2 + y^2 + z^2 = 2a^2$.

(5) Prove both analytically and geometrically that, if the three straight lines, each of which is perpendicular to two of three other straight lines, be perpendicular to each other, the other three straight lines will be at right angles to each other.

(6) The straight lines bisecting the angles between the straight lines given by the equations $lx + my + nz = 0$, $ax^2 + 2bxy + cy^2 = 0$, lie in the two planes

$$x^2\{alm - b(n^2 + l^2)\} + xy\{a(m^2 + n^2) - c(n^2 + l^2)\} - y^2(clm - b(m^2 + n^2)\} = 0.$$

(7) The equations of the straight lines bisecting the angles between the straight lines given by the equations $lx + my + nz = 0$, $ax^2 + by^2 + cz^2 = 0$, may be put into the form $lx + my + nz = 0$, and

$$\begin{aligned} &l^2x^2\{-l^2(b-c) + m^2(c-a) + n^2(a-b)\} \\ &+ m^2y^2\{\;\;l^2(b-c) - m^2(c-a) + n^2(a-b)\} \\ &+ n^2z^2\{\;\;l^2(b-c) + m^2(c-a) - n^2(a-b)\} = 0. \end{aligned}$$

(8) The straight lines bisecting the angles between the two lines given by the equations $lx + my + nz = 0$, $ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0$, lie on the cone $x^2(c'n - b'm) + \ldots + \ldots + yz\{c'm - b'n + (c-b)l\} + \ldots + \ldots = 0$.

(9) If $ax^2 + by^2 + cz^2$ become $\alpha\xi^2 + \beta\eta^2 + \gamma\zeta^2$ by any transformation of coordinates, the positive and negative coefficients will be in like number in the two expressions.

(10) Employ the method of Art. 156 to reduce the equation

$$x^2 + y^2 + \tfrac{1}{2}yz + zx = a^2, \quad \text{to the form} \quad x^2 + \tfrac{5}{4}y^2 - \tfrac{1}{4}z^2 = a^2.$$

(11) Shew, by transformation of four-point coordinates, that the centre of gravity of a tetrahedron is also the centre of gravity of the tetrahedron formed by joining the centres of gravity of the faces.

(12) Shew, by the same method, that the centre of gravity of the surface of a tetrahedron is the centre of the sphere inscribed in the tetrahedron formed by joining the centres of gravity of the faces.

CHAPTER X.

ON CERTAIN SURFACES OF THE SECOND DEGREE.

161. **Before** proceeding to discuss the general equation of the second degree, we think it advisable for the student to render himself familiar with some of the properties of the surfaces which are represented by the general equation. We shall therefore introduce him to the equations of these surfaces in their simplest forms, in which the axes of coordinates being in the direction of lines symmetrically situated with regard to these surfaces, the nature and properties of the surfaces will be more easily deduced. We hope that by following this plan we shall assist the student to understand more clearly the methods adopted in the general equations.

For this purpose we shall give geometrical definitions of the surfaces, and deduce equations from those definitions; and we shall shew *vice versâ* how from these equations the geometrical construction of those surfaces can be deduced.

The Sphere.

162. *To find the equation of a sphere.*

Def. A sphere is the locus of a point, whose distance from a fixed point is constant. The fixed point is the centre and the constant distance the radius of the sphere.

Let (a, b, c) be the centre of the sphere, d the radius, (x, y, z) any point on the sphere; $\therefore (x-a)^2+(y-b)^2+(z-c)^2=d^2$.

This equation may be written in the general form

$$x^2+y^2+z^2+Ax+By+Cz+D=0,$$

the equation required.

163. Since the general equation of the sphere contains four arbitrary constants, the sphere may be made to satisfy four specific conditions.

It may be seen from geometrical considerations that, when four conditions are given, there may be only one sphere, or a limited number, or an infinite number of spheres, which satisfy the equations; at the same time the four conditions must be consistent with the nature of a sphere, and if this be the case, and the

conditions be independent, there must be a limited number of spheres satisfying those conditions. For example, if four points be given through which a sphere is to pass, no three points can lie in one straight line; and if four points lie in one plane, they must also lie in a circle, otherwise no sphere could pass through them, and if such a condition be satisfied, an infinite number of spheres can be constructed, each of which will contain the circle in which the four points lie; if the four points do not lie in a plane, so that the four conditions to be satisfied are independent, the sphere will be completely determined.

Again, if four planes be given, each of which is to be touched by the sphere, no three of these must have one line of intersection, and the four cannot pass through one point, except under a condition, and in that case an infinite number of spheres can be drawn, touching the four planes. In other cases, eight spheres can be drawn satisfying the conditions.

Equation of a sphere under specific conditions.

164. *To find the equation of a sphere passing through a given point.*

Let (a, b, c) be the given point, and the equation of the sphere

$$x^2+y^2+z^2+Ax+By+Cz+D=0;$$

$$\therefore a^2+b^2+c^2+Aa+Bb+Cc+D=0;$$

$$\therefore x^2+y^2+z^2+A(x-a)+B(y-b)+C(z-c)=a^2+b^2+c^2$$

is the equation required.

If the given point be the origin, the equation will become

$$x^2+y^2+z^2+Ax+By+Cz=0,$$

and the sphere may be made to satisfy three more conditions.

165. ***To find the equation of a sphere which passes through two given points in the axis of z.***

Let c_1, c_2 be the distances of the given points from O; when $x=0$ and $y=0$, the equation must become $(z-c_1)(z-c_2)=0$; therefore the equation of the sphere is $x^2+y^2+(z-c_1)(z-c_2)+Ax+By=0$.

If the sphere touch the axis of z, $c_1=c_2=\gamma$,

$$\therefore x^2+y^2+z^2+Ax+By-2\gamma z+\gamma^2=0.$$

166. ***To find the equations of spheres which touch the three coordinate axes.***

Let the equation of any sphere touching the three axes be

$$x^2+y^2+z^2+2Ax+2By+2Cz+D=0.$$

Since it touches the axis of x, let a be the distance from the origin; therefore when $y=0$ and $z=0$, $x^2+2Ax+D=0$, the roots of which are each equal to $\pm a$;

$$\therefore A=\mp a \text{ and } D=a^2.$$

Similarly, $y^2+2By+a^2$ is a complete square;

$$\therefore B=\mp a; \text{ so also } C=\mp a,$$

and the equations of the spheres which satisfy the given conditions are $x^2+y^2+z^2\mp 2ax\mp 2ay\mp 2az+a^2=0$, which are eight in number for any given value of a, corresponding to the different compartments of the coordinate planes.

167. *To find the equation of a sphere touching the plane of xy in a given point.*

Since the sphere meets the plane of xy only in the given point $(a, b, 0)$, when $z=0$, the equation must reduce to $(x-a)^2+(y-b)^2=0$; therefore the required equation of the sphere is $(x-a)^2+(y-b)^2+z^2+2Cz=0$.

168. *Interpretation of the expression*

$$(x-a)^2+(y-b)^2+(z-c)^2-d^2$$

in the equation of a sphere.

Let the equation of the sphere be

$$(x-a)^2+(y-b)^2+(z-c)^2-d^2=0,$$

and (x', y', z') be any point Q, C the centre of the sphere, and let a straight line through Q intersect the sphere in the points P, P', and have for its equations $(x-x')/l=(y-y')/m=(z-z')/n=r$; therefore at the points P and P'

$$(lr+x'-a)^2+(mr+y'-b)^2+(nr+z'-c)^2-d^2=0,$$

if r_1, r_2 be the roots of this equation,

$$r_1r_2=(x'-a)^2+(y'-b)^2+(z'-c)^2-d^2;$$

therefore the left side of the equation for any point (x', y', z') is

$$QP.QP', \text{ or } -QP.QP',$$

according as Q is without or within the sphere.

If Q be without the sphere it will be the square of a tangent drawn from Q to the sphere.

If Q be within, it will be the square of the radius of the small circle on the sphere whose centre is Q.

Def. The product of the segments QP, QP' is called the *power* of the sphere with respect to Q.

Cor. All tangents drawn from an external point to the sphere are equal.

169. *To find the equation of the radical plane of two spheres.*

Def. The locus of points, with respect to which the powers of two spheres are equal, is a plane called the *radical plane*.

Let the equations of the two spheres be

$$(x-a)^2+(y-b)^2+(z-c)^2-d^2\equiv u=0,$$

$$\text{and } (x-a')^2+(y-b')^2+(z-c')^2-d'^2\equiv u'=0.$$

The equation of the radical plane is therefore $u-u'=0$.

170. *To shew that the six radical planes of four spheres intersect in one point.*

Let $u=0$, $u'=0$, $u''=0$, $u'''=0$ be the equations of the four spheres in the above form.

The equations of the six radical planes are given by $u=u'=u''=u'''$, which intersect in one point determined by these equations.

Def. The point of intersection of the six radical planes is called the *radical centre* of the four spheres.

Cylindrical Surfaces.

171. It has been seen that the locus of an equation $F(x, y) = 0$ which involves only two of the cordinates, is a cylindrical surface of which the generating lines are parallel to the axis of the omitted coordinate. We shall now shew how to obtain the equation of certain cylindrical surfaces in which the generating lines are in a general direction.

172. *To find the equation of the cylindrical surface, whose generating lines are in a given direction and guiding curve a central conic traced on the plane of xy.*

Let the equations of the guiding conic be

$$ax^2 + by^2 = 1, \text{ and } z = 0; \tag{1}$$

and (l, m, n) the direction of the generating lines.

Let the equations of any generating line be

$$nx = lz + \alpha, \text{ and } ny = mz + \beta. \tag{2}$$

At the point of intersection of the generating line with the guiding curve, the values of x, y, z in (1) and (2) being the same, we obtain as a general equation, after eliminating x, y and z, $a\alpha^2 + b\beta^2 = n^2$ (3), and since this is true for all positions of the generating line, eliminating α, β between (2) and (3),

$$a(nx - lz)^2 + b(ny - mz)^2 = n^2,$$

is true for every point in the cylindrical surface, and is therefore its equation.

Conical Surfaces.

173. DEF. A *conical surface* is a surface generated by a straight line which constantly passes through a given point, called the vertex, and is subject to some other condition.

174. *To find the equations of a conical surface, whose vertex is the origin, generated by a straight line, of which a guiding curve is a central conic, whose centre is in the axis of z, and plane parallel to the plane of xy.*

Let the equations of the guiding conic be $ax^2 + by^2 = 1$, and $z=c$ (1), those of a generating line in any position, $x=\alpha z$, $y=\beta z$. (2)

Eliminating x, y, z, the coordinates of the point in which the generating line meets the guiding curve, which therefore satisfy (1) and (2) simultaneously, we have $a\alpha^2c^2 + b\beta^2c^2 = 1$ (3); and since this equation is true for every position of the generating line, eliminating α, β from (2) and (3), we obtain $ax^2 + by^2 = z^2/c^2$, which is the required equation of the surface.

175. *To find the equation of the conical surface, whose vertex is any given point, and of which the section by the plane of xy is a central conic whose axes are on the axes of x and y.*

Let the coordinates of the vertex be f, g, h, and the equations of the guiding conic be $z=0$, and $ax^2+by^2=1$; and let the equations of any generating line be

$$x-f=\alpha(z-h), \text{ and } y-g=\beta(z-h); \qquad (1)$$

where this line meets the ellipse $x=f-\alpha h$, $y=g-\beta h$;

$$\therefore\ a(f-\alpha h)^2+b(gz-hy)^2=(z-h)^2; \qquad (2)$$

eliminating α and β from (1) and (2), we obtain for every point in the surface $a(fz-hx)^2+b(gz-hy)^2=(z-h)^2$, which is the equation required.

COR. 1. If l, m, n be the direction-cosines of any generating line, $a(fn-hl)^2+b(gn-hm)^2=n^2$.

COR. 2. The equation of an oblique circular cone is

$$(fz-hx)^2+(gz-hy)^2=a^2(z-h)^2,$$

a being the radius of the circle in the plane of xy.

176. *To shew that there are two systems of circular sections of any oblique circular cone.*

When the circle which guides the motion of the generating line has equations $z=0$, $x^2+y^2=a^2$, the cone will be perfectly general, if we take the vertex in the plane of zx, and therefore $f, 0, h$ for its coordinates.

The equation of the cone will then be, as in the last Article,

$$(fz-hx)^2+h^2y^2=a^2(z-h)^2;$$

this may be written in the form

$$h^2(x^2+y^2+z^2-a^2)=z\{2fhx-(f^2-h^2-a^2)z-2ha^2\}.$$

Hence, if the conical surface be cut by either of the planes

$$z=\alpha, \text{ or } 2fhx-(f^2-h^2-a^2)z-2ha^2=\beta,$$

the points of intersection will satisfy an equation of the form

$$x^2+y^2+z^2+2Ax+2Bz+D=0,$$

for all values of α and β, and the sections will therefore be plane sections of a sphere.

Therefore, there are two series of circular sections made by two systems of parallel planes.

177. The trace of the cone on the plane of zx, putting $y=0$, has for its equation $(fz-hx)^2-a^2(z-h)^2=0$, being the two generating lines which lie in that plane; and the equation of two

planes in opposite systems, giving circular sections, is

$$(z-\alpha)\{2fhx-(f^2-h^2-a^2)z-2ha^2-\beta\}=0;$$

adding these equations we have $h^2(x^2+z^2+Ax+Bz+C)=0$, which shews that the four points, in which these generating lines meet the two circular sections, lie in a circle; hence the first system of planes makes the same angle with one generating line which the second system does with the other.

The Spheroids.

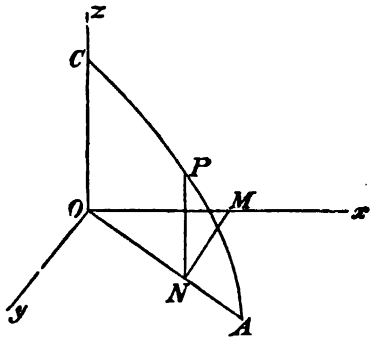
178. DEF. A *spheroid* may be generated by the revolution of an ellipse about either axis.

If the axis of revolution be the minor axis, the surface is called an *Oblate Spheroid*, and if the major axis, a *Prolate Spheroid*.

179. *To find the equation of a spheroid.*

Let the centre be taken as origin, the axis of revolution that of z, and let P be a point (x, y, z) in the ellipse CPA, which is the position of the revolving ellipse, when inclined at any angle to the plane zx, $OM=x$, $MN=y$, $NP=z$, $OA=a$, $OC=c$;

$$\therefore\ ON^2/a^2+NP^2/c^2=1,\ \text{ or } (x^2+y^2)/a^2+z^2/c^2=1.$$

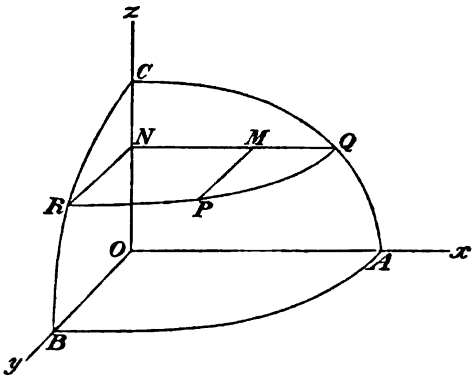


This is the equation of an oblate or prolate spheroid according as c is less or greater than a.

The Ellipsoid.

180. DEF. An *ellipsoid* may be generated by the motion of a variable ellipse, which moves so that its plane is always parallel to a fixed plane, and which changes its form so that its vertices lie in two ellipses having a common axis traced on planes perpendicular to each other, and to the fixed plane.

181. *To find the equation of an ellipsoid.*



Let QRN be a variable ellipse in any position, Q, R being its vertices lying in two ellipses AC, BC, traced on perpendicular planes, taken for those of zx and yz; the plane of xy, to which the variable ellipse is parallel, being the plane containing the semi-axes OA, OB.

Let a, c, and b, c, be the semi-axes of AC and BC, and (x, y, z) any point P in QR, and let PM be perpendicular to QN.

Then, $y^2/RN^2 + x^2/QN^2 = 1$, and, since Q is a point in the ellipse AC, $QN^2/a^2 = 1 - z^2/c^2$, similarly $RN^2/b^2 = 1 - z^2/c^2$,

$$\therefore x^2/a^2 + y^2/b^2 + z^2/c^2 = 1,$$

which is the equation required.

182. *To construct the surface whose equation is*

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

Let the surface be cut by a plane whose equation is $z = \gamma$; the projection of the curve of intersection on the plane of xy has the equation $x^2/a^2 + y^2/b^2 = 1 - \gamma^2/c^2$, therefore the curve is an ellipse whose semi-axes α, β are given by the equations

$$\alpha^2/a^2 = 1 - \gamma^2/c^2 = \beta^2/b^2;$$

hence the vertices lie in the two ellipses which are the traces of the surface on the planes of zx, yz.

Also, since $\alpha/a = \beta/b$, the variable ellipse remains always similar to a given ellipse, which is the trace on the plane of xy.

The surface may therefore be generated by the motion of a variable ellipse, whose plane, &c., see Def. Art. 180.

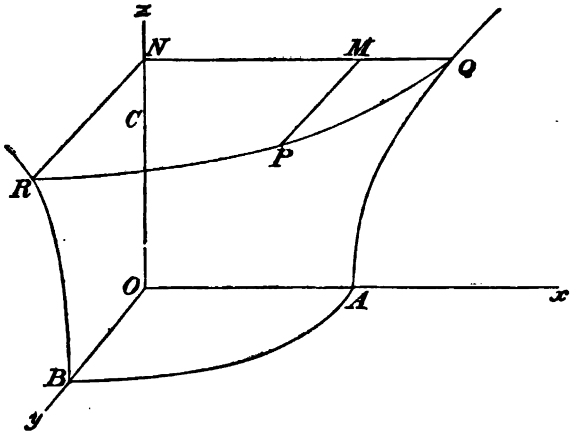
The Hyperboloid of one Sheet.

183. Def. The *hyperboloid of one sheet* may be generated by the motion of a variable ellipse, which moves so that its plane is

always parallel to a fixed plane, and which changes its form so that its vertices always lie in two hyperbolas traced on planes perpendicular to each other and to the fixed plane, these hyperbolas having a common conjugate axis.

184. *To find the equation of a hyperboloid of one sheet.*

Let AQ, BR be the hyperbolas traced on the two perpendicular planes taken for the planes of zx, yz, OC their common conjugate semi-axis being the direction of the axis of z.



Let QPR be the variable ellipse in any position, P any point (x, y, z) in it, QN, RN its semi-axes. Draw PM perpendicular to QN, then $MN = x$, $PM = y$, $ON = z$, and $x^2/QN^2 + y^2/RN^2 = 1$, also, since Q, R are points in the hyperbolas, if $OA = a$, $OB = b$, and $OC = c$, $QN^2/a^2 = z^2/c^2 + 1 = RN^2/b^2$, $\therefore$ $x^2/a^2 + y^2/b^2 = z^2/c^2 + 1$, or $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$, which is the equation of the hyperboloid of one sheet.

185. *To construct the surface which is the locus of the equation*

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1.$$

Let the surface be cut by a plane whose equation is $z = \gamma$, then the projection of the curve of intersection upon the plane of xy has for its equation $x^2/a^2 + y^2/b^2 = 1 + \gamma^2/c^2$, which is the equation of an ellipse, whose semi-axes α, β are given by the equations $\alpha^2/a^2 = 1 + \gamma^2/c^2 = \beta^2/b^2$, therefore the vertices of the ellipse lie respectively on the hyperbolas which are the traces of the surface on the planes of zx, yz.

Also, since $\alpha/a = \beta/b$, this ellipse is always similar to the ellipse which is the trace of the surface on the plane of xy.

Hence the locus may be generated by the motion of a variable ellipse which moves, &c. See Def. Art. 183.

186. The locus may also be generated by the motion of a hyperbola, for, if the surface be cut by a plane parallel to the plane of yz, whose equation is $x = \alpha$, the curve of intersection will be a hyperbola, the equation of whose projection on the plane of yz will be $y^2/b^2 - z^2/c^2 = 1 - \alpha^2/a^2$; let β, γ be the semi-axes of this projection.

If $\alpha < a$, $\beta^2/b^2 = \gamma^2/c^2 = 1 - \alpha^2/a^2$, hence the extremities of the transverse axis 2β will lie on the ellipse, which is the trace on the plane of xy.

If $\alpha > a$, $\beta^2/b^2 = \gamma^2/c^2 = \alpha^2/a^2 - 1$, hence the extremities of the transverse axis 2γ will lie on the hyperbola, which is the trace on the plane of zx.

187. *To find the form of the surface at an infinite distance.*

If z be increased indefinitely, we have ultimately

$$x^2/a^2 + y^2/b^2 = z^2/c^2\,(1 + c^2/z^2) = z^2/c^2.$$

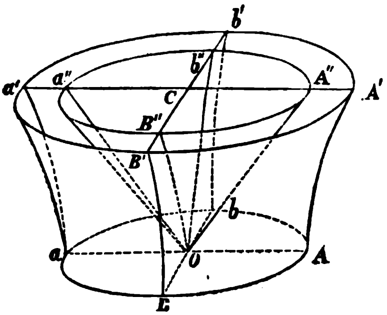
Let this surface and the hyperboloid be cut by a straight line drawn parallel to Oz through a point $(x', y', 0)$, and let z_1, z_2, be the corresponding values of z, then

$$x'^2/a^2 + y'^2/b^2 = z_1^{\,2}/c^2, \text{ and } x'^2/a^2 + y'^2/b^2 = z_2^{\,2}/c^2 + 1,$$

$$\therefore\ (z_1^{\,2} - z_2^{\,2})/c^2 = 1, \text{ and } z_1 - z_2 = c^2/(z_1 + z_2);$$

if x', or y', or both, and therefore z_1 and z_2, be indefinitely increased, $z_1 - z_2$ will diminish indefinitely, and ultimately vanish; $\therefore\ x^2/a^2 + y^2/b^2 = z^2/c^2$ is the equation of an asymptotic surface, which lies further from the plane of xy than the hyperboloid.

This asymptotic surface is a cone, for, if it be cut by any plane whose equation is $x/a = \cos\theta\, z/c$, all the points of intersection will lie in the planes $y/b = \pm \sin\theta\, z/c$. The surface is therefore capable of being generated by a straight line which passes through the origin, and is guided by the ellipse whose equations are $x^2/a^2 + y^2/b^2 = 1$, and $z = c$.



The figure shews the position of the conical asymptote relative to the hyperboloid.

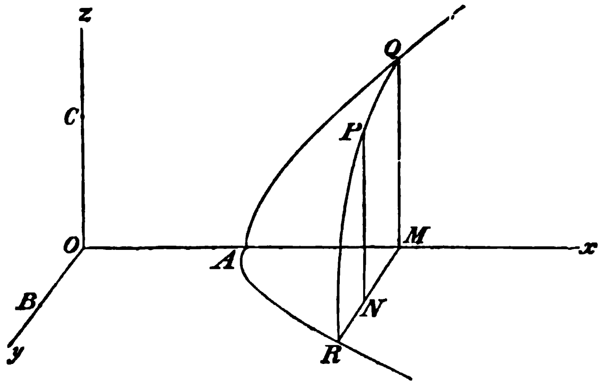
$ABab$ is the principal elliptic section, $A'B'a'b'$, $A''B''a''b''$ the sections of the hyperboloid and cone made by a plane parallel to that principal section, at a distance $OC = c$.

The Hyperboloid of two sheets.

188. DEF. The *hyperboloid of two sheets* may be generated by the motion of a variable ellipse, which moves so that its plane is always parallel to a fixed plane, and which changes its form, so that its vertices lie always on two hyperbolas having a common transverse axis, traced upon two planes perpendicular to each other and to the fixed plane.

189. *To find the equation of the hyperboloid of two sheets.*

Let AQ, AR be the hyperbolas traced on two perpendicular planes, taken for the planes of zx, xy, and having the common transverse semi-axis OA, and let QPR be the variable ellipse in any position, whose axes are QM, RM, parallel to the plane of yz. Take P any point (x, y, z) in the ellipse, and draw PN perpendicular to RM, then $OM = x$, $MN = y$, and $NP = z$; therefore, since P is in the ellipse, $y^2/RM^2 + z^2/QM^2 = 1$;



and if a, c and a, b be the semi-axes of the two hyperbolas AQ, AR, then $RM^2/b^2 = x^2/a^2 - 1 = QM^2/c^2$;

$$\therefore\ y^2/b^2 + z^2/c^2 = x^2/a^2 - 1, \text{ or } x^2/a^2 - y^2/b^2 - z^2/c^2 = 1,$$

which is the equation of the hyperboloid of two sheets.

190. *To construct the locus of the surface whose equation is*

$$x^2/a^2 - y^2/b^2 - z^2/c^2 = 1.$$

Let the surface be cut by a plane whose equation is $x = \alpha$; the equation of the projection of the curve of intersection on the plane of yz is $y^2/b^2 + z^2/c^2 = \alpha^2/a^2 - 1$, which, if $\alpha > a$, is the equation of an ellipse whose semi-axes β, γ are given by the equations

$$\beta^2/b^2 = \alpha^2/a^2 - 1 = \gamma^2/c^2,$$

therefore the vertices of the ellipse lie in two hyperbolas, whose equations are $x^2/a^2 - y^2/b^2 = 1$, and $x^2/a^2 - z^2/c^2 = 1$, which are the traces of the surface on the planes of xy, zx, having a common transverse axis in the line Ox; and since $\beta/b = \gamma/c$, this ellipse is always similar to a given ellipse, axes $2b$, $2c$.

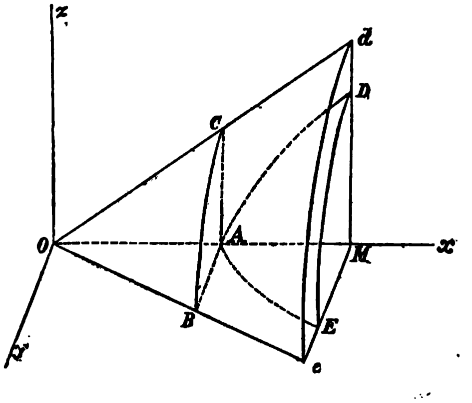
Hence the locus may be constructed by the motion of a variable ellipse which, &c. See Def. Art. 188.

191. The locus may also be generated by the motion of a hyperbola; for, if the surface be cut by a plane parallel to the plane of xy, whose equation is $z = \gamma$, the curve of intersection will be a hyperbola, the equations of whose projection on the plane of xy will be $x^2/a^2 - y^2/b^2 = 1 + \gamma^2/c^2$, which may be written $x^2/\alpha^2 - y^2/\beta^2 = 1$, whose transverse and conjugate semi-axes will satisfy the equations $\alpha^2/a^2 = 1 + \gamma^2/c^2 = \beta^2/b^2$. Hence the tranverse axis will have its extremities in the hyperbola, which is the trace on the plane of zx, and the hyperbolic section will be similar to the trace on the plane of xy.

192. *To find the form of the hyperboloid of two sheets at an infinite distance.*

If x increase indefinitely, the equation $x^2/a^2 = y^2/b^2 + z^2/c^2 + 1$ shews that y, or z, or both, will also be increased indefinitely, and the equation becomes $x^2/a^2 = y^2b^2 + z^2/c^2$ ultimately.

Let the hyperboloid, and the surface represented by this equation, be cut by a straight line parallel to the axis of x, drawn through the point $(0, y', z')$, and let x_1, x_2 be the corresponding



values of x; as in Art. 187, $x_2 - x_1$ diminishes indefinitely, and ultimately vanishes as y', or z', or both, increase indefinitely; hence the hyperboloid of two sheets continually approximates to the form of the surface whose equation is $x^2/a^2 = y^2/b^2 + z^2/c^2$, which is therefore called an asymptotic surface.

Also, as before, this surface can be generated by straight lines drawn through the origin, guided by the ellipse whose equations are $y^2/b^2 + z^2/c^2 = 1$, $x = a$.

This asymptotic surface is therefore a cone on an elliptic base, and lies nearer to the plane of yz than the hyperboloid, since $x_1^2 < x_2^2$.

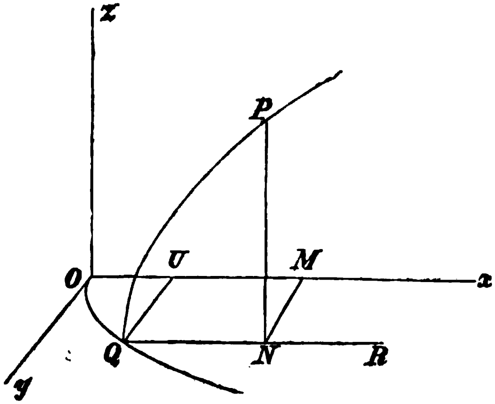
Its position relative to the hyperboloid is shewn in the figure, in which BC is the section made by a plane parallel to yz through the extremity of the transverse axis, and DE, de are sections of the hyperboloid and conical asymptote, made by a plane parallel to yz.

The Elliptic Paraboloid.

193. Def. The *elliptic paraboloid* may be generated by the motion of a given parabola, whose vertex lies in a parabola traced upon a fixed plane, to which its plane is always perpendicular, the axes of the two parabolas being parallel, and the concavities turned in the same direction.

194. *To find the equation of the elliptic paraboloid.*

Let xOy be the plane on which the fixed parabola OQ is traced, Ox the axis of OQ; QR the axis of the moveable parabola QP, P any point (x, y, z) in the parabola.



Draw PN perpendicular to QR, and QU, NM to Ox, then since P is a point in QP, if l, l' be the latera recta of OQ and QP,

$PN^2 = l'.QN$, and $QU^2 = l.OU$;

$$\therefore\ y^2/l + z^2/l' = OU + QN = OM = x,$$

which is the equation of the elliptic paraboloid.

195. *To construct the locus of the equation* $y^2/l + z^2/l' = x$.

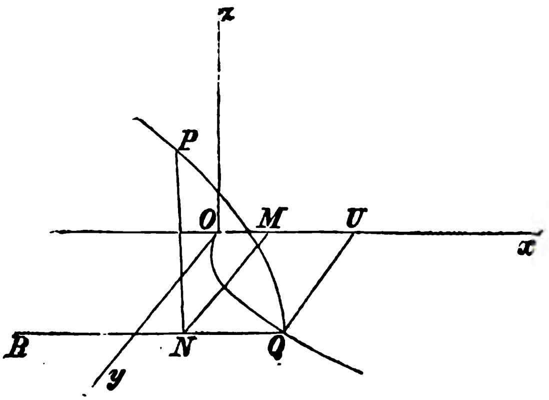
Let the locus be cut by a plane, whose equation is $y = \beta$, the projection of the curve of intersection upon the plane of zx has for its equation $z^2 - l'(x - \beta^2/l)$, which represents a parabola whose axis is parallel to the axis of x, the coordinates of whose vertex are β^2/l, β, 0; therefore the vertex of the parabolic section lies in the parabola whose equation is $y^2 = lx$, which is the trace on the plane of xy; therefore the locus may be constructed by the motion of a parabola, whose vertex, &c. See Def. Art. 193.

The Hyperbolic Paraboloid.

196. Def. The *hyperbolic paraboloid* may be generated by the motion of a parabola, whose vertex lies in a parabola traced upon a fixed plane, to which its plane is perpendicular, the axes of the two parabolas being parallel, and the concavities turned in opposite directions.

197. *To find the equation of the hyperbolic paraboloid.*

Let xOy be the plane upon which the fixed parabola is drawn, Ox the direction of the axis of the parabola; let QR be the axis of the moveable parabola QP, parallel to Ox, measured in the direction contrary to Ox.



Draw PN perpendicular to QR, and QU, NM to Ox; then, if P be any point (x, y, z) in QP, $OM = x$, $MN = y$, and $NP = z$. Let l, l' be the latera recta of OQ, QP;

$$\text{therefore, } PN^2 = l'.QN, \text{ and } QU^2 = l.OU,$$

and $QU^2/l - PN^2/l' = OU - QN = OM$;

$\therefore y^2/l - z^2/l' = x$ is the equation of the hyperbolic paraboloid.

198. *To construct the locus of the equation $y^2/l - z^2/l' = x$.*

Let the locus of the equation be cut by the plane, whose equation is $y = \beta$; the projection of the curve of intersection upon the plane of zx has for its equation $z^2 = l'(\beta^2/l - x)$, which represents a parabola, whose axis is measured in the direction contrary to Ox, and the algebraical distance of whose vertex from the plane yOz is β^2/l; therefore the section by the plane $y = \beta$ is a parabola, whose latus rectum is l' and the coordinates of whose vertex are β^2/l, β, 0; or, the vertex lies in a parabola traced upon the plane of xy, whose equation is $y^2 = lx$.

Hence the locus may be generated by the motion of a parabola, whose vertex, &c. See Def. Art. 196.

199. The locus may also be generated by the motion of an hyperbola; for if it be cut by a plane parallel to that of yz on the positive side, whose equation is $x = \alpha$, the equation of the projection of the curve of intersection on the plane of yz will be $y^2/l - z^2/l' = \alpha$; whose transverse and conjugate semi-axes, β, γ, will satisfy the equations $\beta^2 = l\alpha$ and $\gamma^2 = -l'\alpha$, the extremities of the transverse axis will lie in the trace on the plane of xy, and the conjugate axis will be equal to the double ordinate of the trace on the plane of zx corresponding to $x = -\alpha$.

If it be cut by a plane parallel to yz on the negative side, the section will be an hyperbola whose transverse axis will be in the direction of Oz.

If $\alpha = 0$, the hyperbolas will degenerate into two straight lines, which is the intermediate form in the transition.

200. *To find the form of the hyperbolic paraboloid at an infinite distance.*

If y and z be indefinitely increased while $x : z$ remains finite,

$$y^2/l = z^2(1 + l'x/z^2)/l' = z^2/l' \text{ ultimately}; \ \therefore y/\sqrt{l} = \pm z/\sqrt{l'},$$

and if these planes and the hyperbolic paraboloid be cut by a straight line parallel to Oy, drawn through a point $(x', 0, z')$, y_1, y_2 the corresponding values of y will be given by the equations

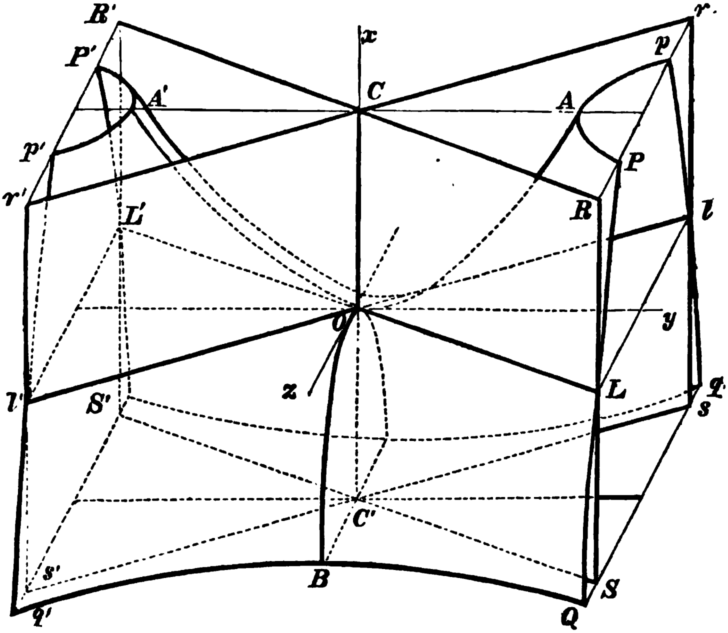
$$y_1^2/l = z'^2/l_1', \text{ and } y_2^2/l - z'^2/l = x';$$

$$\therefore y_2^2 - y_1^2 = lx', \text{ or } y_2 - y_1 = lx'/(y_1 + y_2).$$

Therefore, if x' remain finite or small compared with y_1 or y_2, $y_2 - y_1$ will diminish as z' increases, and will ultimately vanish; and the two planes, whose equations are $y/\sqrt{l} = \pm z/\sqrt{l'}$, will give the form of the surface at an infinite distance for finite values of x, or for values of x which are small compared with y or z.

These planes will not form an asymptotic surface, except for

points at which x vanishes compared with y or z, since $y_2 - y_1$ will not ultimately vanish in any other case, and similarly for $z_2 - z_1$.



The figure is intended to shew the position of the asymptotic planes with reference to the hyperbolic paraboloid.

Ox is parallel to the axis of the generating parabola, of which OB is one position in the plane of zx.

PAp, $P'A'p'$ are opposite branches of a hyperbolic section perpendicular to Ox, the asymptotes of which RCR', rCr' are sections of the asymptotic surface, AA' the transverse axis being parallel to Oy.

LL', ll' are the traces on the plane of yz of both the paraboloid and its asymptotic surface.

QBq' is a branch of a hyperbolic section on the negative side of Ox, the two asymptotes of which $SC'S'$, $sC's'$ are sections of the asymptotic surface, and the transverse axis BC' is parallel to Oz.

201. *To shew that the elliptic and hyperbolic paraboloid are particular cases of the ellipsoid, or one of the hyperboloids.*

Let $x^2/a^2 \pm y^2/b^2 \pm z^2/c^2 = 1$ be the equation of an ellipsoid or hyperboloid, and let the origin be removed to the point $(-a, 0, 0)$.

The transformed equation is $x^2/a^2 \pm y^2/b^2 \pm z^2/c^2 = 2x/a$.

Let a, b, c become infinite, while b^2/a, c^2/a remain finite quantities, and denote these by l and l'.

The equation may then be written $x^2/a \pm y^2/l \pm z^2/l' = 2x$, which has for its limit, when a is infinite, $\pm y^2/l \pm z^2/l' = 2x$, which is the equation of an elliptic or hyperbolic paraboloid. The assumption that b^2/a and c^2/a remain finite is the same thing as assuming that the latera recta of the traces on the planes xy, zx, respectively, remain finite when the axes become infinite, and the corresponding ellipses or hyperbolas become parabolas.

It is obvious from the above, that the elliptic paraboloid is a limiting case of either the ellipsoid or the hyperboloid of two sheets, and the hyperbolic paraboloid of the hyperboloid of one sheet.

202. The surfaces of the second degree, which we have been discussing, have equations of the two forms,

$$Ax^2 + By^2 + Cz^2 = D, \tag{1}$$

$$\text{and } By^2 + Cz^2 = Ax; \tag{2}$$

and it will be shewn in a succeeding chapter that the equations of all surfaces of the second degree may by transformation of coordinates be reduced to one of these two forms.

The first form of equation includes all surfaces which have a centre at a *finite* distance, and the second those which have a centre at an *infinite* distance.

In the equation (1), if $-x$, $-y$, $-z$ be written respectively for x, y, z, the equation will not be altered; therefore if (x, y, z) be a point in the surface, $(-x, -y, -z)$ also will be a point in it, so that if POP' be any chord through the origin O, the chord will be bisected in O, and O will be a centre of the surface.

Also, for any values of y and z, the values of x are equal and of opposite signs, therefore the plane of yz bisects the chords which are drawn perpendicular to it; and a plane which bisects the chords drawn perpendicular to it is called a *principal plane* of the surface.

Hence the planes xy, yz and zx are principal planes of the surface.

It is evident that the planes of zx, xy are principal planes of the surfaces whose equations are of the form (2).

The sections made by the principal planes are called *principal sections*.

That the surface represented by (2) may be considered to have a centre at an infinite distance may be shewn by considering this equation as the limiting form of (1) when the origin is transferred to a point $(-\alpha, 0, 0)$, α being determined by the equation $A\alpha^2 = D$. The equation (1) will then assume the form $Ax^2 + By^2 + Cz^2 = 2A\alpha x$, and this surface has a centre on the axis of x, at distance α from the origin.

Now, if we suppose A to vanish, while $A\alpha$ remains finite, an

equation of the form (2) is the result. But to satisfy these conditions a must be infinitely great; hence a surface represented by (2) has a centre at an infinite distance on the axis of x, and also a third principal section, parallel to the plane of yz, at an infinite distance.

203. Considering the peculiar importance of the properties of surfaces of the second degree, and their frequent occurrence in the solution of problems, and the establishment of theorems, in all departments of physical science, we have adopted a special term derived from the term *Conic*, invented by Salmon for the locus of the equation of the second degree in Plane Geometry.

DEF. The locus of the general equation of the second degree is called a *Conicoid*.*

XV.

(1) A straight line is drawn through a fixed point O, meeting a fixed plane in Q, and in this straight line is taken a point P such that $OP.OQ$ is equal to a given quantity; shew that P lies on a sphere passing through O, whose centre lies on the perpendicular from O upon the plane.

(2) Investigate the equation of a sphere conceived to be generated by the motion of a variable circle, whose diameter is one of a system of parallel chords of a given circle, to which the plane of the variable circle is perpendicular.

(3) Construct the sphere whose polar equation is $r = a \sin\theta \cos\phi$.

(4) A straight line moves with three fixed points A, B, C in the three coordinate planes; shew that any other fixed point P of the straight line will lie on an ellipsoid whose semi-axes are equal to PA, PB, and BC.

(5) Find the locus of a point whose distance from a given point bears a constant ratio to its distance, (i) from a fixed plane, (ii) from a fixed straight line.

(6) Find the locus of a point which is equidistant from two fixed lines which do not intersect.

(7) The locus of a point, whose distance from a fixed plane is always equal to its distance from a fixed line, is a cone.

(8) Shew that the elliptic paraboloid may be generated by a variable ellipse, the extremities of whose axes lie on two parabolas having a common axis, and whose planes are at right angles to each other.

(9) Shew that an hyperboloid of one or two sheets degenerates into a cone, when its axes become indefinitely small, preserving a finite ratio to each other.

(10) Three straight lines, mutually at right angles, are drawn from the origin to meet the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, shew that, if their lengths be r_1, r_2, r_3, then $r_1^{-2} + r_2^{-2} + r_3^{-2} = a^{-2} + b^{-2} + c^{-2}$.

(11) The curve traced out on the surface $y^2/b + z^2/c = x$, by the extremities of the latera recta of sections made by planes through the axis of x, lies on the cone $y^2 + z^2 = 4x^2$.

(12) The locus of the line of intersection of two planes at right angles to each other, each of which passes through one of two straight lines, inclined at an angle 2α, and whose shortest distance is $2c$, is a hyperboloid of one sheet, one of whose axes is $2c$, and the others are as $\cos\alpha : \sin\alpha$.

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* The reasons for not adopting the term *Quadric*, which is employed by Salmon and approved of by many writers, are given in the Preface.

XVI.

(1) Find the equation of the sphere which stands on the circle $z=0$, $(x-a)^2+(y-b)^2=r^2$, and which touches the plane $y=0$. Shew that the area which it cuts off from the plane of yz is $\pi(b^2-a^2)$; why is this result independent of r?

(2) The surface generated by a straight line, revolving about a fixed straight line, with which it is supposed rigidly connected, will be a cone, or a hyperboloid, according as the straight lines do or do not intersect.

(3) The locus of the middle points of all straight lines passing through a fixed point and terminated by two fixed planes is a hyperbolic cylinder.

(4) Shew that a line which always intersects two given lines, and is perpendicular to one of them, generates a hyperbolic paraboloid. Interpret the result when the two given lines are at right angles to each other.

(5) The ellipse, whose equations are $ax^2+by^2=1$, and $x=mz$, rotates about the axis of z, prove that it always lies on the surface

$$b(x^2+y^2)-(b-a)m^2z^2=1.$$

(6) Prove that the cones on the elliptic base $x^2/a^2+y^2/b^2=1$, $z=0$, whose vertices are on the hyperbola $x^2/(a^2-b^2)-z^2/b^2=1$, $y=0$, are right circular.

(7) Of two equal circles, one is fixed and the other moves parallel to a given plane and intersects the former in two points; prove that the locus of the moving circle is two elliptic cylinders.

(8) If A, B, C be the extremities of the axes of an ellipsoid, and AC, BC the sections containing the least axis, find the equations of the two cones whose vertices are A, B. and bases BC, AC respectively; shew that the cones have a common parabolic section, and if l be the latus-rectum of this parabola, and l_1, l_2 those of the sections AC, BC, then $l^{-2}=l_1^{-2}+l_2^{-2}$.

(9) Through a focus of the spheroid $x^2/a^2+(y^2+z^2)/c^2=1$, $a>c$, a plane is drawn parallel to the axis of z, and making an angle θ with the plane of zx; prove that the section of the surface $x^2/(a^2-\lambda^2)+(y^2+z^2)/(c^2-\lambda^2)=1$ will be parabolic, if $\cos^2\theta=(a^2-\lambda^2)/(a^2-c^2)$, and shew that its latus rectum

$$=2(\lambda^2-c^2)/\sqrt{(a^2-\lambda^2)}.$$

(10) A straight line is projected on a plane which always passes through a given straight line, shew that the locus of the projection is a hyperboloid of one sheet or a plane.

(11) The equation of the surface generated by the revolution of the line $(x-f)/l=(y-g)/m=(z-h)/n$ about the line $x/A=y/B=z/C$ is

$$(f+l\rho)^2+(g+m\rho)^2+(h+n\rho)^2=x^2+y^2+z^2,$$

$$\text{where } \rho(lA+mB+nC)=A(x-f)+B(y-g)+C(z-h).$$

(12) Find the locus of a point through which three straight lines can be drawn mutually at right angles, and passing through the perimeter of a curve whose equations are $z=0$, and $ax^2+by^2=1$.

(13) The trace of an ellipsoid on the plane of xy is AB; shew that a cone which has AB for a guiding curve intersects the ellipsoid in another plane curve, and that the plane of this curve intersects the plane of AB in the polar with respect to AB of the projection of the vertex on that plane.

(14) The axis of the right circular cone, vertex at the origin, which passes through the three lines whose directions are (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) is normal to the plane

$$\begin{vmatrix} x, & y, & z, & 0 \\ l_1, & m_1, & n_1, & 1 \\ l_2, & m_2, & n_2, & 1 \\ l_3, & m_3, & n_3, & 1 \end{vmatrix}=0.$$

CHAPTER XI.

ON GENERATION BY STRAIGHT LINES.

204. In the preceding chapter we have shewn how certain surfaces of the second degree may be generated by the motion of ellipses, hyperbolas and parabolas. In the case of the cylinder and cone we have investigated the equations by supposing them to be generated by the motion of a straight line subject to certain conditions.

We shall in this chapter show that the hyperboloid of one sheet, and the hyperbolic paraboloid, as well as the cone and cylinder, are capable of being generated by the motion of a straight line.

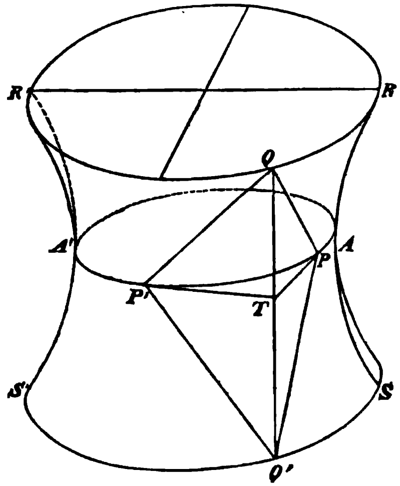
But, before giving the analytical representations of the mode of generation by straight lines, a general geometrical discussion may be found useful.

205. Since a surface of the second degree can be intersected by a straight line in two points only, unless it should turn out that the line lies entirely in the surface, as in the case of a cylinder, it follows that no straight line can intersect a plane section of the surface in more than two points, and that every plane section must therefore be a conic.

Now, if a plane be drawn containing a tangent to the principal elliptic section of the hyperboloid of one sheet and perpendicular to its plane, the curve of intersection with the surface will, in consequence of the flexure of the surface being in opposite directions, be a conic which crosses itself at the point of contact, and the only conic having this property is two intersecting straight lines.

Hence, through every point of the elliptic section two straight lines can be drawn which lie entirely in the surface, and by making the plane travel round the ellipse, such straight lines sweep round the whole surface, which can therefore be generated in two ways by the motion of a straight line.

206. If we take any two positions of the plane through tangents PT, $P'T$ to the principal elliptic section $APP'A'$, their line of intersection, which will be perpendicular to the principal plane, will intersect the hyperboloid in two points only Q, Q', and the two pairs of generating lines will be PQ, PQ' and $P'Q$, $P'Q'$, since no straight line common to the plane PQQ' and the surface can meet QQ' except in Q or Q'.



Thus, *the orthogonal projections of generating lines on this principal plane are tangents to the principal elliptic section; and similarly for the principal hyperbolic sections.*

Also, through every point such as Q, two straight lines can be drawn which lie entirely in the surface; and it is evident that *generating lines of the same system*, such as PQ and $P'Q'$, *do not intersect.*

If a plane be drawn in any direction containing a generating line PQ, the conic of intersection must be two straight lines; hence, the plane will contain another generating line, and these two generating lines will be of opposite systems, since they must intersect at a finite or infinite distance.

Since $P'Q$ is parallel to a generating line of the opposite system, drawn through the other extremity of the diameter through P', the same conical surface will be generated by lines drawn through the centre of the hyperboloid parallel to either system of generating lines, and will moreover be of the second degree.

207. *No straight line, which does not belong to one of the two systems of generating lines, lies on an hyperboloid.*

For, if possible, let a straight line (C) lie entirely on the hyperboloid, then since each system generates the whole hyperboloid, (C) must meet an infinite number of straight lines of each system; let two of these (A) and (B) of opposite systems intersect (C) in two different points, in which case a plane can be drawn through them intersecting the surface in three straight lines; but the section of a surface of the second order by a plane must be a curve of the second degree, therefore no such line as (C) can exist.

208. We leave it to the student to shew that a hyperbolic paraboloid may be generated in a similar way, and that the generating lines are all parallel to one or other of two fixed planes.

It will thus be seen that, since no three lines of a cone of the second degree can be parallel to the same plane, unless the cone split up into two planes, this forms a complete distinction between the two cases in which the generating lines of conicoids are real, viz. the hyperboloid of one sheet and the hyperbolic paraboloid.

209. *To find the surface generated by a straight line which meets three fixed non-intersecting straight lines.*

Let these fixed straight lines be (A), (B), (C); these lines obviously lie on the surface in question.

Now consider any plane through (A); it will meet (B) and (C) in points Q, R, and in these points only, and QR will meet (A) in some point P; so that PQR is the only straight line of the system lying in the plane. Hence this plane meets the surface in two straight lines PQR and (A), which form a group of the second degree. But the section of a surface by a plane is a curve of the same degree as that of the surface. The surface in question is, therefore, of the second degree.

The equation of the locus may be found in a simple form by considering the three given lines as edges of a parallelepiped which do not meet and taking axes through its centre parallel to those edges, the equations of which become $y=b$, $z=-c$; $z=c$, $x=-a$; $x=a$, $y=-b$; now $z+c=\alpha(y-b)$, $z-c=\beta(x+a)$ give planes containing the first two edges, and may be taken for the equations of the generating line, which intersects the third if $c+ba+a\beta=0$,

$$\therefore\ c(x+a)(y-b)+b(z+c)(x+a)+a(y-b)(z-c)=0,$$

$$\text{or } ayz+bzx+cxy+abc=0.$$

210. *To find in what cases a straight line can be drawn through a given point of a conicoid, so as to lie entirely in the surface.*

Let the equation of the conicoid, supposed central, be $ax^2+by^2+cz^2=1$, and let (f, g, h) be the given point, l, m, n the direction cosines of the straight line supposed to satisfy the condition; the coordinates of any other point at a distance r from (f, g, h) are $f+lr$, $g+mr$, $h+nr$; hence the equation

$$a(f+lr)^2+b(g+mr)^2+c(h+nr)^2=1,$$

must be satisfied for all values of r;

$$\therefore\ al^2+bm^2+cn^2=0, \tag{1}$$

$$afl+bgm+chn=0, \tag{2}$$

$$\text{and } af^2+bg^2+ch^2=1. \tag{3}$$

(1) shews that one or two of the quantities a, b, c must be negative; let c be negative; then since, by (1), (2), and (3),

$$(al^2+bm^2)(af^2+bg^2)-(afl+bgm)^2=(1-ch^2)(-cn^2)-c^2h^2n^2,$$

$$ab(gl-fm)^2+cn^2=0; \tag{4}$$

hence, unless a, b, or $c=0$, which are cases of cylindrical surfaces, ab must be positive, and therefore both a and b will be positive, since all three cannot be negative.

Thus the central surface, on which a straight line can lie entirely, must be the hyperboloid of one sheet.

If the surface be non-central and its equation $by^2+cz^2=x$, the equation corresponding to (1) will be $bm^2+cn^2=0$, which shews that the surface must be the hyperbolic paraboloid, since b and c must have opposite signs.

In either case, for every position of the point (f, g, h) there are two straight lines which lie entirely on the surface.

The hyperboloid of one sheet, and the hyperbolic paraboloid, can therefore be generated in two ways by the motion of a straight line.

211. The equations (2) and (4) give the directions of the two straight lines through (f, g, h), namely

$$\frac{l}{\pm\sqrt{(-cb/a)}\,g-cfh}=\frac{m}{\mp\sqrt{(-ca/b)}\,f-cgh}=\frac{n}{af^2+bg^2}.$$

212. *To find the angle between the generating lines which cross at a given point.*

By applying the method of Art. 26 to equations (1) and (2) of Art. 210, or by using the ratios $l:m:n$ given in the last article, it can be shewn that, if ψ be the angle between the generating lines of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ which pass through a point at a distance r from the centre,

$$2abc\cot\psi=p\,(r^2-a^2-b^2+c^2),$$

where p is the perpendicular from the centre upon the plane containing the generating lines.

COR. If the generating lines be at right angles, the points lie in the intersection of the sphere $r^2=a^2+b^2-c^2$ with the hyperboloid.

Analysis of Generating Lines.

213. *To find the generating lines of a hyperboloid of one sheet.*

The equation of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ may be written in the form

$$x^2/a^2+y^2/b^2=(\cos\alpha\pm\sin\alpha\, z/c)^2+(\sin\alpha\mp\cos\alpha\, z/c)^2,$$

$$\therefore\ x/a=\cos\alpha\pm\sin\alpha\, z/c \text{ and } y/b=\sin\alpha\mp\cos\alpha\, z/c\ \ldots\ldots (1)$$

satisfy the equation for all values of α; hence the two straight lines which, for a particular value of α, have these for their equations, lie entirely in the surface.

By the variation of α we obtain two systems of straight lines, which lie entirely in the surface, and either of these systems generates the hyperboloid. These equations may also be written in the form

$$(x-a\cos\alpha)/a\sin\alpha=(y-b\sin\alpha)/(-b\cos\alpha)=\pm z/c,$$

from which equations it is manifest that straight lines drawn

parallel to them through the centre will lie upon the asymptotic cone. Hence also, no three generators of the hyperboloid can be parallel to the same plane.

If $z = 0$, $x = a\cos\alpha$, and $y = b\sin\alpha$; therefore α is the eccentric angle of the point of intersection of the two straight lines (1) with the trace of the hyperboloid on the plane of xy.

214. Any point of the hyperboloid may be represented by the coordinates $a\cos\theta\sec\phi$, $b\sin\theta\sec\phi$, $c\tan\phi$, since these satisfy the equation for all values of θ and ϕ.

If the points represented by these coordinates lie in the same generating line, θ and ϕ must be restricted in value; to shew that either $\theta+\phi$, or $\theta-\phi$ must be constant, it is only necessary to substitute the values of x, y, z in the equations of Art. 213

$$(x/a - \cos\alpha)/\sin\alpha = (y/b - \sin\alpha)/-\cos\alpha = \pm z/c,$$

where α is some constant angle.

215. *The projections of the generating lines upon the principal planes are tangents to the traces on those planes.*

For, the equations of the trace on the plane of zx, and of the projection of a generating line on the same plane being $x^2/a^2 - z^2/c^2 = 1$, and $x/a = \cos\alpha \pm \sin\alpha\, z/c$, the values of z at their points of intersection are given by the equation $1 - (\cos\alpha \pm \sin\alpha\, z/c)^2 + z^2/c^2 = 0$, which has equal roots, the projection is therefore a tangent to the trace. Similarly for the trace on the plane of xy. See Art. 206.

216. *To shew from their equations that two generating lines of the same system do not intersect.*

The equations of two generating lines of the same system

$$\text{are } x/a = \cos\alpha \pm z/c\sin\alpha, \quad y/b = \sin\alpha \mp z/c\cos\alpha,$$
$$\text{and } x/a = \cos\alpha' \pm z/c\sin\alpha', \quad y/b = \sin\alpha' \mp z/c\cos\alpha';$$

if the two lines meet, we shall have at the points of intersection,

$$0 = \cos\alpha - \cos\alpha' \pm z/c\,(\sin\alpha - \sin\alpha'),$$
$$\text{and } 0 = \sin\alpha - \sin\alpha' \mp z/c\,(\cos\alpha - \cos\alpha');$$

$\therefore$ $(\cos\alpha - \cos\alpha')^2 + (\sin\alpha - \sin\alpha')^2 = 0$ will be the condition of intersection, which cannot be satisfied unless $\alpha = \alpha'$.

Similarly it may be shewn that the condition that the two generators of opposite systems defined by α, α' may intersect is $\cos^2\alpha - \cos^2\alpha' + \sin^2\alpha - \sin^2\alpha' = 0$, which is identically true.

217. *To find the locus of the intersection of two generating lines of opposite systems, drawn through points in the principal elliptic section, whose eccentric angles differ by a constant angle.*

Let $\beta + a$, and $\beta - a$, be the eccentric angles of two points in the principal elliptic section, differing by a constant angle $2a$.

The equations of the generating lines of opposite systems are

$$x/a = \cos(\beta + a) \pm z/c \sin(\beta + a), \quad y/b = \sin(\beta + a) \mp z/c \cos(\beta + a),$$

and $$x/a = \cos(\beta - a) \mp z/c \sin(\beta - a), \quad y/b = \sin(\beta - a) \pm z/c \cos(\beta - a).$$

At the points of intersection, by subtracting

$$0 = \sin\beta \sin a \mp z/c \sin\beta \cos a, \quad \text{or } z/c = \pm \tan a,$$

$$\therefore \; x/a = \sec a \cos\beta \text{ and } y/b = \sec a \sin\beta \text{ which give } x^2/a^2 + y^2/b^2 = \sec^2 a.$$

Therefore the locus required is the two elliptic sections, parallel to the plane of xy, which intersect the traces on the planes of zx, yz, at points whose eccentric angles are $\pm a$.

218. The generating lines can also be represented by taking for the parameter the eccentric angle of one of the principal hyperbolic sections instead of that of the elliptic section. Thus the equations of any generating lines may be written

$$x/a = \sec a \pm \tan a \; y/b,$$
$$z/c = \tan a \pm \sec a \; y/b.$$

219. Another simple method is to take the equations in the form $x/a + z/c = \lambda(1 \pm y/b)$ and $x/a - z/c = \lambda^{-1}(1 \mp y/b)$ the double sign, both upper or both lower, corresponding to the two systems of generators.

220. In this manner of considering generating lines there are two pencils of planes passing respectively through the lines $x/a - z/c = 0$, $1 - y/b = 0$, and $x/a + z/c = 0$, $1 + y/b = 0$, and these pencils are homographic, since to each plane of one pencil corresponds one and only one plane of the other, therefore:

Every hyperboloid of one sheet is the locus of the intersection of corresponding planes of two homographic pencils of planes.

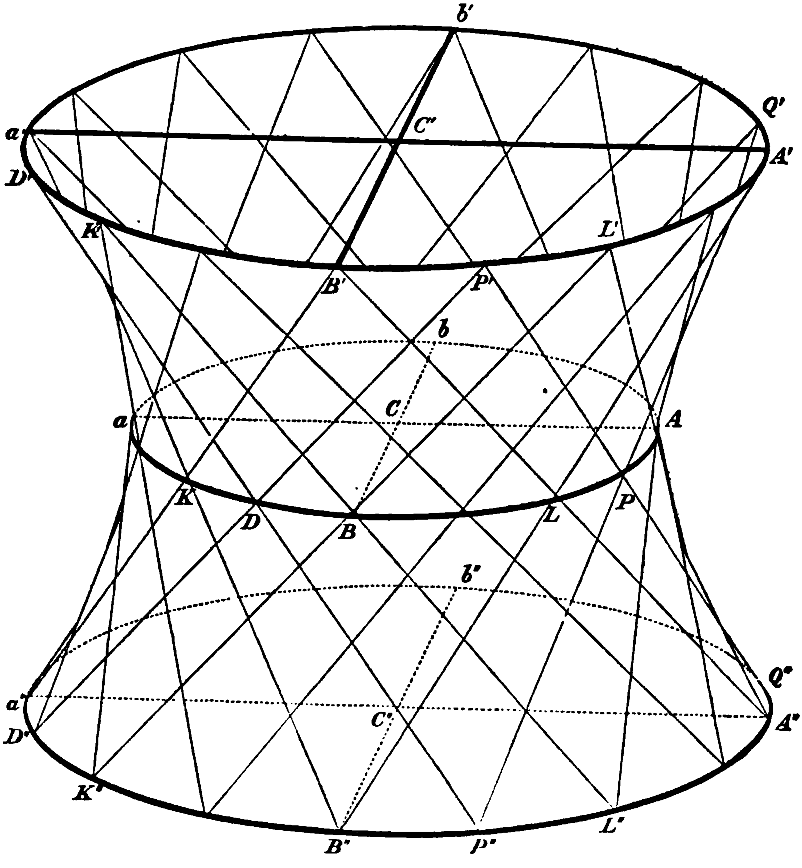
221. If any four planes of one pencil be taken and also the corresponding four planes of the other, the two sets of four planes will have equal anharmonic ratios. For, if λ_1, λ_2, λ_3, λ_4 be the parameters of four planes of the first pencil, their anharmonic ratio will be $(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3) : (\lambda_3 - \lambda_2)(\lambda_1 - \lambda_4)$, which will be unaltered if λ^{-1} be written for λ.

222. The accompanying figure is meant to be a representation of the positions of sixteen generating lines of each system, corresponding to eccentric angles differing by $\frac{1}{8}\pi$. $ABab$ is the principal elliptic section, $A'B'a'b'$ and $A''B''a''b''$ are the parallel elliptic sections which intersect the conjugate axis of the hyperboloid at its extremities C', C'', the axes of which sections are in the ratio $\sqrt{2} : 1$ to the axes of the principal sections.

The generating lines through the extremities of the axes Aa, Bb intersect these two ellipses at points L', K', and L'', K'', whose eccentric angles are $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, *i.e.* at the extremities of equi-conjugate diameters; and those through L, K, the extremities of equi-conjugate diameters of the principal elliptic section pass through the extremities of the axes of the two ellipses.

The two ellipses $A'B'a'$ and $A''B''a''$ are the loci of the intersections of opposite systems of generating lines drawn through the extremities of conjugate diameters of the principal elliptic section.

The figure serves to represent that the intersections of generating lines of opposite systems drawn through points in the principal elliptic section, whose eccentric angles differ by a constant angle, lie in an ellipse, the plane of which is parallel to the principal plane. As, for example, such pairs of generating lines as LB', $P'D$, and BL', PP'.



223. *To find the generating lines of a hyperbolic paraboloid.*

The equation of a hyperbolic paraboloid, $y^2/b^2 - z^2/c^2 = 2x/a$, is satisfied for every point in the lines whose equations are

$$y/b \pm z/c = 2x/\lambda \text{ and } y/b \mp z/c = \lambda/a, \tag{1}$$

whatever be the values of λ; therefore giving λ all values we obtain two series of straight lines, all of which lie entirely in the surface; and these are the two systems of lines, which are rectilinear generators of the paraboloid. The second of the equations (1) shews that the two systems of generators are parallel respectively to the two asymptotic planes whose equations are $y/b \mp z/c = 0$.

224. *To shew that generating lines of a hyperbolic paraboloid of the same system do not intersect, and that those of opposite systems do intersect.*

Let the equations of two lines of the same system be

$$y/b \pm z/c = 2x/\lambda, \quad y/b \mp z/c = \lambda/a,$$
$$\text{and } y/b \pm z/c = 2x/\lambda', \quad y/b \mp z/c = \lambda'/a,$$

if the lines could intersect $\lambda = \lambda'$, which is inadmissible, since the lines are supposed distinct.

Again, changing the order of the signs in the second set of equations, the lines are of opposite systems; if they intersect

$$(\lambda + \lambda')/a = \quad 2y/b = \quad 2x(\lambda^{-1} + \lambda'^{-1}),$$
$$(\lambda - \lambda')/a = \mp 2z/c = -2x(\lambda^{-1} - \lambda'^{-1}),$$

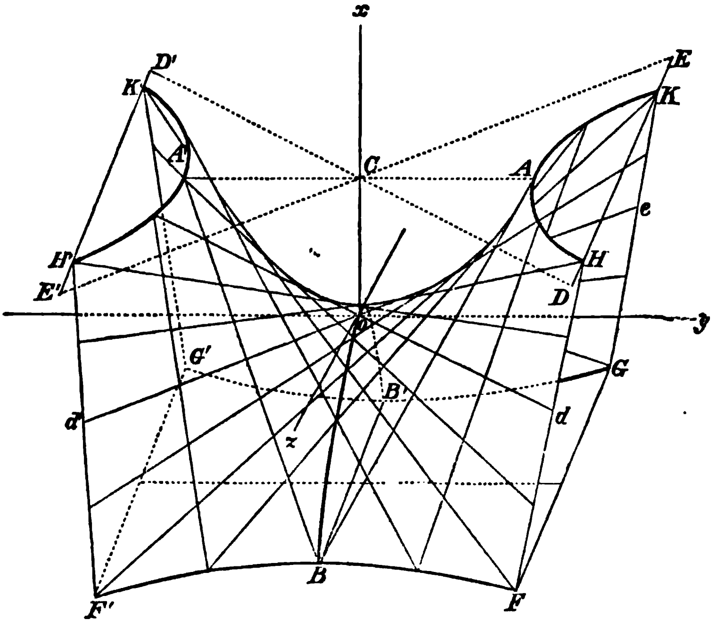
the consistency of these equations proves that the lines of opposite systems intersect.

225. *To shew that the projections of the generating lines on the principal planes are tangents to the principal sections.*

The equations of the generating lines are

$$y/b \pm z/c = 2x/\lambda, \quad y/b \mp z/c = \lambda/a,$$

and that of the projection on the plane of zx is $\pm 2z/c = 2x/\lambda - \lambda/a$, which meets the trace of the paraboloid on the same plane, viz. $z^2/c^2 = -2x/a$, in two coincident points.



226. The figure on the preceding page is intended to represent the manner in which the hyperbolic paraboloid is generated by straight lines.

HAK, $H'A'K'$ are portions of the branches of a hyperbolic section made by a plane parallel to that of yz, cutting Ox on the positive side; ECE', DCD' are the asymptotes of the section.

FBF', $GB'G'$ are portions of the branches of a hyperbolic section parallel to yz on the negative side of Ox.

AOA' and BOB' are the traces on the planes of xy and zx.

The two sections are so chosen that the generating lines through B, an extremity of the transverse axis of one section, pass through A, A', the extremities of the transverse axis of the other.

dO, $d'Oe$ are the traces of the paraboloid on the plane yz, where the hyperbolic section degenerates into two straight lines.

XVII.

(1) Shew, by geometrical considerations, that the locus of intersection of two generating lines drawn through two points in the principal elliptic section of a hyperboloid of one sheet, whose eccentric angles differ by a constant quantity, is two ellipses parallel to the principal plane, at equal distances from it.

(2) Shew that there are two straight lines, and two only, which intersect four straight lines, no three of which are parallel to the same plane, and no two of which intersect.

(3) The eccentric angles of the points in which the principal hyperbolic sections of a hyperboloid of one sheet are met by any generating line are complementary, and that of the point in which it meets the principal elliptic section is equal to one of these.

(4) Find at what points of the principal elliptic section of a hyperboloid the generating lines can be at right angles, and shew that the diameters parallel to the tangents at those points are equal to the length of the conjugate axis.

(5) Prove that the points at a finite distance on a hyperbolic paraboloid, at which the generating lines are at right angles to each other, lie in a plane perpendicular to the axis.

(6) Prove that, if any straight line intersect three straight lines which are all parallel to the same plane without intersecting each other, the intersecting straight line will in all positions be parallel to another fixed plane.

(7) The generating lines of the surface $yz + zx + xy + a^2 = 0$, through the point $(0, am, -a/m)$, are $x(1 \pm m) = am - y = \mp (mz + a)$.

(8) Where the planes $x + y + z = \pm a$ meet the surface $xy + yz + zx + a^2 = 0$, the generating lines of the surface are at right angles to each other.

(9) If a right circular cone have three generators mutually at right angles, the secant of the vertical angle will be -3.

(10) If four generating lines intersect so as to form a quadrilateral, whose angular points taken in order are $(\theta_1\phi_1)$, $(\theta_2\phi_2)$, $(\theta_3\phi_3)$, $(\theta_4\phi_4)$, (see Art. 214), prove that $\theta_1 + \theta_3 = \theta_2 + \theta_4$, and $\phi_1 + \phi_3 = \phi_2 + \phi_4$.

(11) A straight line moves so as to intersect each of the parabolas $y^2 = ax$, $z = 0$; $z^2 = -bx$, $y = 0$; and to be always parallel to one of the planes $y^2/a = z^2/b$; shew that its locus is the paraboloid $y^2/a - z^2/b = x$.

(12) Generating lines of the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ are drawn through points in the plane of xy, whose eccentric angles are $\alpha \pm \beta$, shew that their points of intersection are given by the equations

$$x : y : z : 1 = a \cos\alpha : b \sin\alpha : \pm c \sin\beta : \cos\beta,$$

also that the shortest distance δ between two of the same system is given by

$$4 \sin^2\beta/\delta^2 = \sin^2\alpha/a^2 + \cos^2\alpha/b^2 + \cos^2\beta/c^2.$$

XVIII.

(1) If three generating lines of the same system on a hyperboloid be mutually at right angles, the shortest distance between any two will lie on a generating line.

(2) If two planes be drawn, passing respectively through two generating lines of the same system at the extremities of the major axis of the principal elliptic section, and intersecting in a third generating line, the traces of these planes on either of two fixed planes will be at right angles to each other.

(3) If a ray of light be reflected between two plane mirrors, inclined at any finite angle, shew that all the reflected rays will lie on a hyperboloid of revolution; and find its position.

(4) $ax^2 + by^2 + cz^2 + abc = 0$ is a hyperboloid of one sheet, shew that one system of generators is represented by any two of the equations $la + mz - ny = 0$, $-lz + mb + nx = 0$, and $ly - mx + nc = 0$. What is the other system?

(5) The perpendiculars from the origin on the generating lines of the paraboloid $x^2/a^2 - y^2/b^2 = 2z/c$ lie upon the cones $(x/a \pm y/b)(ax \pm by) + 2z^2 = 0$.

(6) The perpendiculars from the origin upon the generating lines of the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ lie upon the cone

$$(b^2 + c^2)^2\, a^2/x^2 + (c^2 + a^2)\, b^2/y^2 = (a^2 - b^2)^2\, c^2/z^2.$$

(7) The angle between two planes, each passing through the centre, and through one of the generating lines at any point of a hyperboloid, is given by the equation $2r \cot\psi = abc\,(p^{-2} - a^{-2} - b^{-2} + c^{-2})$, r being the distance of the point, and p that of the plane containing the generating lines, from the centre of the hyperboloid.

(8) If 2θ be one of the angles between the perpendiculars from the centre on the generating lines of a hyperboloid which pass through the point $(a \cos\alpha, b \sin\alpha, 0)$, then will

$$c^2 (a^2 - b^2)^2 \cot^2\theta = a^2 (b^2 + c^2)^2 \operatorname{cosec}^2\alpha + b^2 (c^2 + a^2)^2 \sec^2\alpha.$$

(9) The tangent of the angle between the generating lines of the surface $x^2/a - y^2/b = z$, which pass through the point (f, g, h), is

$$\frac{\sqrt{(\frac{1}{4}ab + bf^2/a + ag^2/b)}}{h + \frac{1}{4}(a - b)}.$$

(10) Prove that if r be the distance of any point of the surface $yz + zx + xy + 2a^2 = 0$, from the origin, the cosine of the angle between the two generating lines at that point will be $(r^2 - 6a^2)/(r^2 + 2a^2)$.

(11) The cosine of the angle between the generating lines through the point (x, y, z) of the hyperboloid $x^2/a + y^2/b + z^2/c = 1$ is $(\lambda_2 + \lambda_1)/(\lambda_2 \sim \lambda_1)$, where λ_1, λ_2 are the roots of the equation

$$x^2/a\,(a + \lambda) + y^2/b\,(b + \lambda) + z^2/c\,(c + \lambda) = 0.$$

(12) The straight line which is orthogonal to each of two non-intersecting generators of the hyperboloid $x^2 + y^2 - z^2 = a^2$, becomes a generator of the opposite system when the two non-intersecting generators become consecutive.

(13) The generating lines of the hyperboloid $ax^2 + by^2 + cz^2 = 1$, at any point where it is met by the cone $a^2x^2 + b^2y^2 + c^2z^2 = 0$, are both perpendicular to some other generating line. If the generators be themselves at right angles, the point will lie on the sphere $x^2 + y^2 + z^2 = a^{-1} + b^{-1} + c^{-1}$. Shew that these conditions cannot coexist unless $a + b + c = 0$.

(11) Shew that the shortest distances between generating lines of the same system drawn at the extremities of diameters of the principal elliptic section of a hyperboloid, axes $2a$, $2b$, $2c$, lie on the surfaces whose equations are

$$cxy/(x^2 + y^2) = \pm\, abz/(a^2 - b^2).$$

(15) Shew that, to an eye outside of a hyperboloid of one sheet, every generating line will appear to lie on another.

If the eye be placed upon the surface of the hyperboloid whose equation is $ax^2 + by^2 + cz^2 = 1$, prove that the points, the generating lines through which appear to be perpendicular, will lie on a plane whose equation is

$$(a + b + c)\,(afx + bgy + chz - 1) = 2\,(a^2fx + b^2gy + c^2hz),$$

where (f, g, h) is the position of the eye.

(16) If three generating lines of the same system be the edges of a parallelepiped, shew that the angular points of the parallelepiped which are not on the hyperboloid will lie on the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 + 3 = 0$.

CHAPTER XII.

SIMILAR SURFACES. PLANE SECTIONS OF CONICOIDS. CYCLIC SECTIONS.

227. We shall now consider the nature of the curves in which a plane intersects central and non-central conicoids, and we shall at present consider these surfaces as given by equations in the simplest form, such as have been discussed in the tenth chapter.

We shall examine the special cases in which the section made by a plane is circular, called a *cyclic* section, and the generation of the central conicoids and of the elliptic paraboloid by the motion of a variable circle, the plane of which is parallel to a given plane.

Similar Surfaces.

228. Def. Two surfaces are similar, say U and U', when for *any* point O determined with regard to U, and *any* two radii OP, OQ, another point O', and two radii $O'P'$, $O'Q'$ can be found for U', such that $\angle POQ = \angle P'O'Q'$, and

$$O'P' : O'Q' :: OP : OQ.$$

229. From the definition it follows that if OA, OB, OC be three arbitrary radii at right angles to one another in U, three radii $O'A'$, $O'B'$, $O'C'$ can be found also at right angles satisfying the above proportion, and if the direction cosines of radii OP and $O'P'$ referred to these as axes, in U and U' respectively, be equal, $OP : O'P' :: OA : O'A'$.

The surfaces will be similarly situated when the lines OA, OB, OC are parallel to $O'A'$, $O'B'$, $O'C'$, and in this case O may always be chosen so that O and O' coincide, in which case the surfaces are said to be similarly situated with respect to O; this point will be the intersection of two lines PP', QQ' where P', Q' on one surface correspond to any two points P, Q on the other.

230. The analytical expression is that, if $f(x, y, z) = 0$ be the equation of any surface, that of any similar and similarly situated surface will be

$$f\{\lambda(x-\alpha), \lambda(y-\beta), \lambda(z-\gamma)\} = 0, \text{ where } OP = \lambda O'P'.$$

The number of conditions, which the coefficients of the equations of two surfaces of the n^{th} degree must satisfy is

$$(n+1)(n+2)(n+3)/6 - 5,$$

in order that they may be similar and similarly situated.

The terms of the highest degree in the two equations must be the same, except for a constant factor.

Thus, in the case of the hyperboloids, they are similar if they have similar conical asymptotes.

It will be seen that, according to the definition, hyperboloids of one and two sheets may be similar, as

$$ax^2 + by^2 - cz^2 = 1$$
$$-ax^2 - by^2 + cz^2 = 1,$$

for imaginary radii of one drawn in the same direction as real radii of the other will be in the same ratio.

231. *Sections of the same conicoid by parallel planes are similar and similarly situated conics.*

Sections of similar and similarly situated conicoids by the same plane are similar and similarly situated conics.

These propositions are easily proved by transforming the axes of coordinates, so that the plane of xy is parallel to the cutting plane, when the projection of any section, found by making z constant, will be represented by an equation in x and y, for which the terms of the second degree will the same.

Hence, we can deduce that a plane section of a hyperboloid is a hyperbola if the parallel plane through the centre intersects the conical asymptote in two of its generating lines.

232. It is of great importance to observe that, when two conicoids are similar and similarly situated, the condition that the terms of the second degree are the same in each except for a constant factor, or, in geometrical language, that their real or imaginary asymptotes have their sheets parallel, may be stated as follows: "similar and similarly situated conicoids intersect the plane at infinity in the same real or imaginary conic."

A particular case of this is that "all spheres pass through the same imaginary circle at infinity."

233. *To determine the nature of the section of a conicoid made by any given plane.*

This may of course be done by the substitutions of Art. 151, but for surfaces of the second degree the plane sections will be curves of the second degree, so that simpler methods may advantageously be employed. If it be required only to discover the species of conic to which the section belongs, we may effect this immediately, taking any orthogonal projection of the curve of section, since an ellipse, parabola, or hyperbola, will be projected into a curve of the same species, though in general of different eccentricity. The only exception is when the plane of section is perpendicular to the plane of projection, but as no plane can be perpendicular to all the coordinate planes, there is at least one

of the coordinate planes which may, in any proposed case, be taken as the plane of projection, and which will not be perpendicular to the plane of section.

As an example of this method, we may take the section of the paraboloid $by^2 + cz^2 = x$ made by the plane $lx + my + nz = 0$. The equation of the projection of the curve of section on the plane of yz is $l(by^2 + cz^2) + my + nz = 0$, which is always an ellipse, or always a hyperbola, according as b and c have like or unlike signs. If $l = 0$, the exceptional case above mentioned arises, and taking the projection on zx we have the equation

$$(n^2b + m^2c)\, z^2 = m^2x,$$

or the section is parabolic, unless $n^2b + m^2c = 0$, when it reduces to a straight line, the other straight line completing the curve of intersection being at an infinite distance. Hence, for the paraboloids, all sections parallel to the axis of the principal sections are parabolas, and all other sections ellipses for the elliptic paraboloid, and hyperbolas for the hyperbolic paraboloid.

If, however, a more exact determination is required, it will be convenient to deal with the problem in the manner we propose.

Plane Sections.

234. *To find the locus of the centres of all sections of a central conicoid made by parallel planes.*

Let $ax^2 + by^2 + cz^2 = 1$ be the equation of the surface, and $lx + my + nz = p$ that of one of the parallel planes.

Any straight line drawn in this plane through the centre of the section will be bisected at that point.

Let (ξ, η, ζ) be the centre, r any radius of the section drawn in the direction (λ, μ, ν), therefore the values of r are given by

$$a(\xi + \lambda r)^2 + b(\eta + \mu r)^2 + c(\zeta + \nu r)^2 = 1,$$

and, since the values of r are equal and of opposite signs,

$$a\xi\lambda + b\eta\mu + c\zeta\nu = 0,$$

also, since the direction of r lies in the plane, we have

$$l\lambda + m\mu + n\nu = 0,$$

which equations being true for an infinite number of values of $\lambda : \mu : \nu$, we have $a\xi/l = b\eta/m = c\zeta/n$, (1); therefore the equations of the locus of centres of sections, made by planes whose direction-cosines are l, m, n, are $ax/l = by/m = cz/n$.

235. The equation for determining r, being

$$(a\lambda^2 + b\mu^2 + c\nu^2)\, r^2 = 1 - a\xi^2 - b\eta^2 - c\zeta^2,$$

shews that the parallel plane sections are similar, for, if (λ', μ', ν') be the direction of another radius r', $r^2 : r'^2$ will be independent of p, or constant for all the parallel sections, which are therefore similar and similarly situated.

236. *To find the position of the cutting plane when the curve of intersection becomes a point-ellipse or line-hyperbola.*

The section becomes a point-ellipse, or line-hyperbola, when $r=0$ for all directions which make $a\lambda^2+b\mu^2+c\nu^2$ finite, therefore $a\xi^2+b\eta^2+c\zeta^2=1$; let ϖ be the value of p for the position of the plane required, then by the equations (1)

$$\frac{a\xi^2+b\eta^2+c\zeta^2}{l\xi+m\eta+n\zeta}=\frac{l\xi+m\eta+n\zeta}{l^2/a+m^2/b+n^2/c};$$

$$\therefore\ \varpi^2=l^2/a+m^2/b+n^2/c.$$

The point-ellipse is when the values of λ, μ, ν given by $a\lambda^2+b\mu^2+c\nu^2=0$, and $l\lambda+m\mu+n\nu=0$ are impossible, and the line-hyperbola when they are real. Now, it is not hard to shew that $(ma\lambda-lb\mu)^2+abc\nu^2\varpi^2=0$, hence, the section degenerates to a point-ellipse when abc is positive, or for an ellipsoid and hyperboloid of two sheets, and to a line-hyperbola when abc is negative, or for an hyperboloid of one sheet.

237. *To find the magnitude and direction of the axes of any central plane section of a central conicoid, and the area when the section is elliptic.*

The equations which connect the direction of any radius of the section by a plane, whose equation is $lx+my+nz=0$, are, as in Art. 234,

$$(a\lambda^2+b\mu^2+c\nu^2)\,r^2=1=\lambda^2+\mu^2+\nu^2,\ \text{and}\ l\lambda+m\mu+n\nu=0,\quad(1)$$

$$\therefore\ n^2\left\{(ar^2-1)\,\lambda^2+(br^2-1)\,\mu^2\right\}+(cr^2-1)\,(l\lambda+m\mu)^2=0,\quad(2)$$

is an equation which, for a given length r, gives generally two values of $\lambda:\mu$; but if the given length be that of either semi-axis, the two values of $\lambda:\mu$ will be equal, the condition for which is

$$\left\{(ar^2-1)\,n^2+(cr^2-1)\,l^2\right\}\left\{(br^2-1)\,n^2+(cr^2-1)\,m^2\right\}=(cr^2-1)^2\,l^2m^2,$$

$$\therefore\ l^2\,(br^2-1)\,(cr^2-1)+\ldots=0,$$

$$\text{or}\quad l^2/(ar^2-1)+m^2/(br^2-1)+n^2/(cr^2-1)=0,\quad(3)$$

this quadratic in r^2 gives the squares of the semi-axes of the section.

If 2α, 2β be the axes of the section,

$$\alpha^{-2}\beta^{-2}=l^2bc+m^2ca+n^2ab,$$

$$\text{and}\quad\alpha^{-2}+\beta^{-2}=l^2\,(b+c)+m^2\,(c+a)+n^2\,(a+b).$$

When the section is elliptic its area

$$=\pi\alpha\beta=\pi\,(l^2bc+m^2ca+n^2ab)^{-\frac{1}{2}}.$$

Again, the coefficient of λ^2 in (2) is $n^2\,(ar^2-1)+l^2\,(cr^2-1)$, which, by (3), is easily reduced to $-m^2\,(cr^2-1)\,(ar^2-1)/(br^2-1)$, hence the equation (2) becomes $\{(ar^2-1)\,m\lambda-(br^2-1)\,l\mu\}^2=0$,

$$\therefore\ (ar^2-1)\,\lambda/l=(br^2-1)\,\mu/m=(cr^2-1)\,\nu/n,\quad(4)$$

and, if we write α and β for r, these equations, with (1), will determine completely the directions of the two axes.

The equations (3) and (4) might have been formed by making r^2 a maximum or minimum, but we leave this to the student, the process adopted being more instructive.

If only the area of the section be required, the area of its projection upon one of the coordinate planes, as that of xy, multiplied by n^{-1}, will be the area to be found.

238. *To find the direction of the central plane section whose axes are of a given magnitude.*

The equation giving the axes in terms of the direction of the plane section is $l^2/(ar^2-1)+m^2/(br^2-1)+n^2/(cr^2-1)=0$, whence, if γ, δ be the reciprocals of the squares of the given axes,

$$\begin{aligned} l^2bc \quad + \quad m^2ca \quad + \quad n^2ab \quad &= \gamma\delta, \\ l^2(b+c)+m^2(c+a)+n^2(a+b) &= \gamma+\delta, \\ l^2 \quad + \quad m^2 \quad + \quad n^2 \quad &= 1; \end{aligned}$$

multiplying the second and third equations by $-a$, a^2, and adding, we obtain, for the determination of l, m, and n,

$$l^2(a-b)(a-c)=(a-\gamma)(a-\delta);$$

$$\text{similarly,}\quad m^2(b-c)(b-a)=(b-\gamma)(b-\delta),$$

$$\text{and}\quad n^2(c-a)(c-b)=(c-\gamma)(c-\delta);$$

the second equation shews also that if a, b, c be in order of magnitude, b must be intermediate between γ and δ.

Hence, for a circular section, $\gamma=b=\delta$

$$\therefore\quad m=0,\quad\text{and}\quad l^2/(a-b)=n^2/(b-c)=1/(a-c).$$

239. *To find the angle between the real or imaginary asymptotes of a plane section.*

From the equation $(a\lambda^2+b\mu^2+c\nu^2)r^2=1$, it follows that when $r=\infty$, $a\lambda^2+b\mu^2+c\nu^2=0$, also $l\lambda+m\mu+n\nu=0$, $\therefore$ if ω be the angle between the asymptotes of the section, supposed hyperbolic, it may be shewn by the method of Art. 26 that

$$\cot\omega=\frac{l^2(b+c)+m^2(c+a)+n^2(a+b)}{2\sqrt{\{-(l^2bc+m^2ca+n^2ab)\}}},$$

or it may be obtained directly from the quadratic in r^2, Art. 237, since $\tan^2\frac{1}{2}\omega=-\beta^2/\alpha^2$, $\therefore \cot^2\omega=(\alpha^2+\beta^2)^2/(-4\alpha^2\beta^2)$.

This gives the condition that $\cot\omega$ will be real, infinite, or impossible as $l^2bc+m^2ca+n^2ab$ is negative, zero, or positive, thus determining the condition that the section may be hyperbolic, parabolic, or elliptic.

The nature of the central plane action may be determined from the discriminant of the quadratic (3) Art. 237 by reducing it to the form $\{l^2(b-c)+m^2(c-a)-n^2(a-b)\}^2+4l^2m^2(a-c)(b-c)$; also, by the same quadratic, if the section consists of two parallel generating lines whose distance is $2d$, $d^{-2}=l^2(a-c)b/a+m^2(b-c)a/b$.

240. *To find the area of any elliptic section of a central conicoid made by a plane not passing through the centre.*

Let the equation of the plane be $lx+my+nz=p$; the area of a central elliptic section has been shewn to be

$$\pi\,(l^2bc+m^2ca+n^2ab)^{-\frac{1}{2}},$$

and any radius vector of the section considered is given by

$$(a\lambda^2+b\mu^2+c\nu^2)\,r^2=1-p^2/\varpi^2,$$

where $\varpi^2=l^2/a+m^2/b+n^2/c$, ϖ being the value of p when the section vanishes, and, since $l^2bc+m^2ca+n^2ab$ is positive, ϖ is real only when abc is positive, or for the ellipsoid and hyperboloid of two sheets.

But if we take ϖ for the value of p when the section of the hyperboloid of two sheets, which is conjugate to that of one sheet, vanishes, since in this case a, b, c have their signs changed $\varpi^2=-\,l^2/a-m^2/b-n^2/c$.

Hence, if A be the area of the parallel central section of the ellipsoid and hyperboloid of one sheet, and of the hyperboloid conjugate to the hyperboloid of two sheets; and if A' be the area of the section by the given plane, since they are in the duplicate ratio of homologous lines,

for the ellipsoid $A'=A\,(1-p^2/\varpi^2)$,

for the hyperboloid of two sheets $A'=A\,(p^2/\varpi^2-1)$,

for the hyperboloid of one sheet $A'=A\,(1+p^2/\varpi^2)$.

241. If we take two conjugate hyperboloids $ax^2+by^2+cz^2=\pm 1$, and the asymptotic cone to both, $ax^2+by^2+cz^2=0$, the area of the section of the latter may be found, from those of the former, by making a, b, c infinitely large, preserving their ratios. Hence if A_1, A_2, A_3 be the areas of the sections of the three surfaces made by any plane cutting them all in ellipses, and A that of the parallel central section of the hyperboloid of one sheet, we shall have $A_1=A\,(1+p^2/\varpi^2)$, $A_2=A\,(p^2/\varpi^2-1)$, $A_3=A\,(p^2/\varpi^2)$, whence $A_1+A_2=2A_3$, or the section of the cone is an arithmetic mean between the sections of the two hyperloids.

Also, if V be the volume of the cone cut off by a plane touching the hyperboloid of two sheets, we shall have

$$3V=A_3\varpi=A\varpi,\qquad \therefore\ 3V=\pi/\sqrt{(abc)},$$

which is constant for all positions of the cutting plane.

242. *To find the locus of the centres of all sections of an elliptic or hyperbolic paraboloid made by parallel planes.*

Let $by^2+cz^2=2x$ be the equation of a non-central conicoid, and let the equation of one of the planes be $lx+my+nz=p$, then using the same notation as in the last article, we obtain the equation $b\,(\eta+\mu r)^2+c\,(\zeta+\nu r)^2=2\,(\xi+\lambda r)$, (1), and deduce for an infinite number of values of $\lambda:\mu:\nu$

$$b\eta\mu+c\zeta\nu-\lambda=0 \quad\text{and}\quad m\mu+n\nu+l\lambda=0;$$

$$\therefore\ b\eta/m=c\zeta/n=-\,1/l, \tag{2}$$

thus, the locus of centres of sections made by planes whose direction-cosines are l, m, n is a straight line parallel to the axis of the paraboloid.

243. *To find the position of the plane for which the section is a point-ellipse or line-hyperbola.*

(ξ, η, ζ) being the centre of the section, (1) becomes

$$(b\mu^2 + c\nu^2)\, r^2 = 2\xi - b\eta^2 - c\zeta^2; \text{ also } l\xi + m\eta + n\zeta = p,$$

$$\therefore \text{ by (2) } l\,(l\xi - p) = m^2/b + n^2/c = l^2\,(b\eta^2 + c\zeta^2);$$

$$\therefore\ (b\mu^2 + c\nu^2)\, r^2 = 2p/l + (m^2/b + n^2/c)/l^2 = 2\,(p - \varpi)/l \text{ suppose.}$$

The sections are ellipses, when r cannot be infinite, or when b and c are of the same sign; point-ellipses when $p = \varpi$.

They are hyperbolas where b and c are of opposite signs, and the directions of the asymptotes are given by $b\mu^2 + c\nu^2 = 0$, which shews that the asymptotes are parallel to the same two planes for all values of l, m, and n.

244. *To find the magnitude and direction of the axes of any plane section of a paraboloid.*

The equation of the paraboloid being $by^2 + cz^2 = 2x$, and that of the cutting plane $lx + my + nz = p$, the equation connecting any central radius of the section with its direction (λ, μ, ν) is

$$(b\mu^2 + c\nu^2)\, r^2 = 2\,(p - \varpi)/l = \rho \text{ suppose } = \rho\,(\lambda^2 + \mu^2 + \nu^2),$$

$$\therefore\ l^2\,(b\mu^2 + c\nu^2)\, r^2 = \rho\,\{(m\mu + n\nu)^2 + l^2\,(\mu^2 + \nu^2)\}, \qquad (1)$$

and if r be the length of either of the semi-axes, this equation will give equal values of $\mu : \nu$,

$$\{l^2\,(br^2 - \rho) - m^2\rho\}\ \{l^2\,(cr^2 - \rho) - n^2\rho\} = \rho^2 m^2 n^2,$$

$$\text{or } l^2\,(br^2 - \rho)(cr^2 - \rho) - \rho m^2\,(cr^2 - \rho) - \rho n^2\,(br^2 - \rho) = 0, \quad (2)$$

$$\text{or } l^2bcr^4 - \{l^2\,(b + c) + m^2c + n^2b\}\,\rho r^2 + \rho^2 = 0.$$

This equation gives the magnitude of the axes 2α, 2β, and the area of the section when elliptic

$$= \pi\alpha\beta = \frac{\pi\rho}{l\sqrt{(bc)}} = \frac{2\pi}{l^2\sqrt{(bc)}}\,(p - \varpi).$$

The coefficient of μ^2 in equation (1), is

$$l^2\,(br^2 - \rho) - m^2\rho = \rho n^2\,(br^2 - \rho)/(cr^2 - \rho) \text{ by (2)},$$

and the equation becomes $(br^2 - \rho)\, n\mu = (cr^2 - \rho)\, m\nu$;

$$\therefore\ (br^2 - \rho)\,\mu/m = (cr^2 - \rho)\,\nu/n = -\rho\,\lambda/l,$$

$$\text{since } m\mu + n\nu = -l\lambda \text{ and } m^2/(br^2 - \rho) + n^2/(cr^2 - \rho) = l^2/\rho,$$

which, writing α, β for r, completely determine the directions of the corresponding axes of the section.

245. *To determine the nature of any plane section of a paraboloid.*

The discriminant of the equation (2), viz.

$$\{l^2\,(b + c) + m^2c + n^2b\}^2 - 4l^2bc\,(l^2 + m^2 + n^2), \qquad (1)$$

is reducible to $\quad \{l^2\,(b - c) - m^2c + n^2b\}^2 + 4m^2n^2bc. \qquad (2)$

The two forms (1) and (2) of the discriminant shew that it is positive whether bc be positive or negative, so that the values of $r^2 : \rho$ are always real.

(i) For an elliptic section, bc is positive and ρ positive, therefore $p - \varpi$ has the same sign as l.

(ii) For a hyperbolic section, bc is negative, since the values of r^2 must be of opposite signs.

(iii) For a rectangular hyperbola, bc is negative, and

$$l^2(b+c) + m^2c + n^2b = 0.$$

(iv) For a circular section, bc is positive, and by (2)

$$m = 0 \quad \text{and} \quad l^2/b = n^2/(c-b) = 1/c,$$

$$\text{or} \quad n = 0 \quad \text{and} \quad l^2/c = m^2/(b-c) = 1/b,$$

only one of which gives a real position.

(v) The condition $l = 0$, which makes one value of r^2 infinite and the other finite, corresponds to the case of a parabolic section, since in this case η and ζ in (2), Art. 242, are infinite, and therefore the centre of the section is at an infinite distance.

Cyclic Sections.

246. Although we have already determined the positions of the planes whose intersections with conicoids are circular, by treating such sections as particular cases of ellipses, it will be instructive to consider them from another point of view, since they have an interest peculiar to themselves in the solution of many problems both pure and physical.

247. *To find the cyclic sections of a conicoid central or non-central.*

Let the equation of the conicoid be

$$ax^2 + by^2 + cz^2 + 2dx + e = 0,$$

this may be written in the form

$$b(x^2 + y^2 + z^2) - (b-a)x^2 + (c-b)z^2 + 2dx + e = 0,$$

hence, if the conicoid be cut by a plane whose equation is

$$\sqrt{(b-a)}\,x \pm \sqrt{(c-b)}\,z = p\sqrt{(c-a)},$$

the coordinates of the points of the intersection satisfy the equation

$$b(x^2 + y^2 + z^2) + p\sqrt{(c-a)}\left\{\sqrt{(b-a)}\,x \mp \sqrt{(c-b)}\,z\right\} + 2dx + e = 0,$$

which is the equation of a sphere.

These plane sections will be real, if a, b, c be in order of magnitude when all are positive, if $a > b$ when c is negative, and if $c > b$, without regard to sign, when b and c are both negative. Also if $a = 0$, the sections are real when b and c are both positive, and $c > b$.

Hence, cyclic sections of central surfaces are parallel to the mean axis in the ellipsoid, to the greater transverse axis in the hyperboloid of one sheet, to the greater conjugate axis in the hyperboloid of two sheets.

If α be the inclination to the principal section (a, b),

$$\sin^2\alpha/(b-a) = \cos^2\alpha/(c-b) = 1/(c-a).$$

Since the equations of the planes of the cyclic sections depend only on the differences of a, b, and c, the cyclic planes of a system of central conicoids whose equations are, for all values of k, $(a+k)x^2+(b+k)y^2+(c+k)z^2=1$, are all parallel. Such a system is called a system of *concyclic conicoids.*

For the elliptic paraboloid, putting $a=0$,

$$\sin^2\alpha/b = \cos^2\alpha/(c-b) = 1/c,$$

hence, cyclic sections of the elliptic paraboloid are parallel to the tangent at the vertex of the principal section of greatest latus rectum.

It is obvious that these are the only cyclic sections, since a plane not parallel to one of the axes, as that of y, being of the form $lx+my+nz=p$, could not reduce the expression $(b-a)x^2-(c-b)z^2$ to a linear form, so as to ensure that the points of intersection with the conicoid should lie on a sphere.

248. *Generation of a conicoid by the motion of a variable circle.* From Art. 247 it appears that the central conicoids and the elliptic paraboloid can be generated by the motion of a circle, the plane of which is parallel to either of two fixed planes, and the diameter of which changes so that it is always a chord of the principal section which is perpendicular to the line of intersection of the two fixed planes.

The centre of the circle of course describes in each case the diameter conjugate to the chords which it bisects.

249. DEF. The point-circles in which the variable circle terminates are called *umbilics*; these are real only for the ellipsoid, the hyperboloid of two sheets and the elliptic paraboloid.

Referring to Arts. 234 and 236, the coordinates of the four umbilics of the conicoid $ax^2+by^2+cz^2=1$, satisfy the equations $ax/l=by/m=cz/n=1/\varpi$, where $l=\pm\sqrt{(b-a)}/\sqrt{(c-a)}$, $m=0$, $n=\pm\sqrt{(c-b)}/\sqrt{(c-a)}$, and $\varpi^2=ac/b$; $\therefore\ y=0$ and $\pm ax/\sqrt{(b-a)}=\pm cz/\sqrt{(c-b)}=\sqrt{(ac)}/\sqrt{\{b(c-a)\}}$. Also, for the elliptic paraboloid, $by^2+cz^2=2x$, the coordinates of the two umbilics are given by $by/m=cz/n=-1/l$, Art. 242, where $l=\pm\sqrt{b}/\sqrt{c}$, $m=0$ and $n=\pm\sqrt{(c-b)}/\sqrt{c}$, $\therefore\ y=0$, and $cz=-n/l$ or $z=\mp\sqrt{(c-b)}/c\sqrt{b}$, and $2x=cz^2=b^{-1}-c^{-1}$.

250. *Any two cyclic sections of opposite systems lie on one sphere.* The equations of the planes of two cyclic sections of opposite systems are

$$\{\sqrt{(a-b)}\,x-\sqrt{(b-c)}\,z-k\}\{\sqrt{(a-b)}\,x+\sqrt{(b-c)}\,z-k'\}=0;$$

or, $(a-b)x^2-(b-c)z^2-(k+k')\sqrt{(a-b)}\,x-(k-k')\sqrt{(b-c)}z+kk'=0.$

Hence they intersect the surface in a sphere whose equation is

$$b(x^2+y^2+z^2)-1+(k+k')\sqrt{(a-b)}\,x+(k-k')\sqrt{(b-c)}\,z-kk'=0.$$

251. It is an instructive problem to deduce the positions of the cyclic sections directly from the equation obtained in Art. 235.

This equation may be written $a\lambda^2+b\mu^2+c\nu^2=(\lambda^2+\mu^2+\nu^2)\rho$, and since, for a cyclic section, the values of r, and therefore of ρ, are equal for all values of λ, μ, ν consistent with the equation $l\lambda+m\mu+n\nu=0$, it follows that

$$(\rho-a)(m\mu+n\nu)^2+l^2\{(\rho-b)\mu^2+(\rho-c)\nu^2\}=0$$

is true for an infinite number of values of $\mu:\nu$, the coefficients of μ^2, $\mu\nu$, and ν^2 are therefore each zero; $\therefore\ (\rho-a)mn=0$.

If $\rho=a$, either $l=0$, or $\rho=b=c$, in which latter case the surface is spherical, and the equation is satisfied for any values of l, m, n, *i.e.* for any direction of the plane.

Also if $m=0$, the coefficient of $\mu^2=\rho-b=0$, and similarly, for $n=0$, $\rho=c$.

Hence, if the surface be not spherical, we must have l, m, or $n=0$; suppose $m=0$, then $\rho=b$, and the coefficient of $\nu^2=(\rho-a)n^2+(\rho-c)l^2=0$;

$$\therefore\ l^2/(b-a)=n^2(c-b)=1/(c-a),$$

which give real values for l and n only under the same circumstances as are already stated in Art. 247.

The corresponding process for non-central surfaces can be followed out by the student.

252. *Geometrical investigation of the direction of a cyclic section of an ellipsoid.*

In the ellipsoid let OA, OB, OC be the semi-axes in order of magnitude, and if possible let a central circular section not pass through B, but cut AB and BC in P, Q respectively, $OP=OQ$, O being the centre of the section; but OP is intermediate between OA and OB in magnitude, and OQ between OB and OC, which is absurd; hence, the central circular section must contain the mean axis.

The inclination of the plane to OAB is the same as the angle ROA, OR being that radius vector of the section AC which $=OB$.

Hence, if α be the inclination to OA of a radius of the ellipse (a, c) whose length is b, $b^2\cos^2\alpha/a^2+b^2\sin^2\alpha/c^2=1$, this equation gives the position of the circular sections.

A similar investigation will give the positions of the cyclic sections of the hyperboloid of one sheet and a somewhat similar one those of the hyperboloid of two sheets.

XIX.

(1) All spheres, which intersect a given conicoid in plane sections and pass through a fixed point on it, pass also through one of two fixed circles.

(2) Prove that through any point on an ellipsoid two planes of circular section can be drawn; but that when the circles are equal, the points must lie on one of the principal planes passing through the mean axis.

(3) A plane drawn through the origin perpendicular to any generating line of the cone $x^2(a^2-d^2)+y^2(b^2-d^2)+z^2(c^2-d^2)=0$, will intersect the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ in a section of constant area.

(4) The locus of centres of all plane sections of a given conicoid drawn through a given point is a similar and similarly situated conicoid, of which the given point and the centre of the given surface are extremities of a diameter.

(5) The hyperboloid $x^2+y^2-t^2z^2=a^2$ is built up of thin circular disks of cardboard, strung on a straight wire passing through their centres, prove that, if the wire be turned about the origin into the direction (l, m, n) the planes of the disks remaining unaltered, the equation of the surface will be

$$(nx-lz)^2+(ny-mz)^2=n^2(t^2z^2+a^2).$$

(6) The product of the radii of two circular sections of an elliptic paraboloid of opposite systems is constant, shew that the locus of the intersection of their planes is a hyperbolic cylinder, the asymptotes of the principal sections of which are parallel to the circular sections.

(7) In a paraboloid of revolution, the eccentricity of any section is the cosine of the inclination of the plane to the axis of the surface, and the foci of the section are the points of contact with spheres inscribed in the surface.

(8) Find the equation of an ellipsoid referred to the planes of its central circular sections and a central plane at right angles to them. When these are rectangular axes, prove that the squares of the axes are in harmonical progression, and that the equation takes the form

$$c(x+z)^2+a(x-z)^2+(a+c)y^2=2.$$

(9) The section of the surface $yz+zx+xy=a^2$, by the plane $lx+my+nz=p$, will be a parabola if $l^{\frac{1}{2}}+m^{\frac{1}{2}}+n^{\frac{1}{2}}=0$; and that of the surface

$$x^2+y^2+z^2-2yz-2zx-2xy=a^2$$

will be a parabola, if $mn+nl+lm=0$.

XX.

(1) The sphere $x^2+y^2+z^2+a^2-b^2-c^2=2x(a^2-b^2)^{\frac{1}{2}}(a^2-c^2)^{\frac{1}{2}}/a$ meets the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ only at umbilics.

(2) In the hyperboloid $x^2/a^2+(y^2-z^2)/c^2=1$, $(a>c)$, the spheres, of which parallel circular sections are great circles, will have a common radical plane.

(3) If two circular sections of different systems be such that the sphere on which both lie is of constant radius mb, the locus of the centre of the sphere is the hyperbola $x^2/(a^2-b^2)-z^2/(b^2-c^2)=1-m^2$, $y=0$; a, b, c being in descending order of magnitude.

(4) A sphere is described, having for a great circle a plane section of a given conicoid; prove that the plane of the circle in which it again meets the conicoid intersects the plane of the former circle in a straight line which lies in one of two fixed planes.

(5) Prove that the difference of the reciprocals of the squares of the axes of a central section is proportional to the product of the sines of the angles which it makes with the planes of circular section.

(6) Shew that, if elliptic paraboloids have one of their cyclic sections coincident with a central cyclic section of $ax^2 + by^2 + cz^2 = 1$, a, b, c being in order of magnitude, and axes parallel to that of x, the locus of their vertices will have the equations $y = 0$, and $bz^2 \pm 2bxz \sqrt{\{(b-a)/(c-b)\}} + (b-a)/(c-a) = 0$. Also, that the equation of the plane of the other cyclic section common to the conicoid and one of the paraboloids will be $x + bl/a = \pm z \sqrt{\{(c-b)/(b-a)\}}$, where l is the latus rectum of any section parallel to the plane of xy.

(7) On a central circular section of the ellipsoid $ax^2 + by^2 + cz^2 = 1$ a right circular cylinder is constructed, shew that if b be an arithmetic mean between a and c, the cylinder will again intersect the ellipsoid in an ellipse, the plane of which will be given by $(3a-c)x \pm (3c-a)z = 0$, and that the area of the ellipse will be $\pi\{2(c^2 + a^2) - 3b^2\}^{\frac{1}{2}}/b^2$.

(8) The semi-axes of a central section of the surface $ayz + bzx + cxy$ made by a plane, whose direction is (l, m, n), are given by the equation

$$r^4(2bcmn + \ldots - a^2l^2 - \ldots) - 4abcr^2(amn + \ldots) + 4a^2b^2c^2 = 0.$$

(9) The locus of the axes of sections of the surface $ax^2 + by^2 + cz^2 = 1$, which contain the line $x/\alpha = y/\beta = z/\gamma$, is the cone

$$(b-c)yz(\beta z - \gamma y) + (c-a)zx(\gamma x - \alpha z) + (a-b)xy(\alpha y - \beta x) = 0.$$

(10) Prove that the section of the surface

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + 2a''x + 2b''y + 2c''z + d = 0,$$

by the plane $lx + my + nz = 0$, will be a rectangular hyperbola, if

$$l^2(b+c) + m^2(c+a) + n^2(a+b) = 2a'mn + 2b'nl + 2c'lm,$$

and a parabola, if $\quad l^2(bc - a'^2) + \ldots + 2mn(b'c' - aa') + \ldots.$

If $a'b'c' = aa'^2 = bb'^2 = cc'^2$, the last equation is identically true, explain why this must be the case.

(11) If sections of an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ be made by planes passing through the centre and through another given point $(x'y'z')$, the sections of greatest and least area will be at right angles to each other, and the areas will be $\pi abc/\varpi_1$, $\pi abc/\varpi_2$, ϖ_1, ϖ_2 being the semi-axes of the section made by the plane $xx'/a + yy'/b + zz'/c = 0$. Shew that the product of the areas will be constant if the point lie on the curve of intersection of the ellipsoid and a concentric sphere.

(12) The radius r of the central circular sections of the surface $ayz + bzx + cxy = 1$ is given by the equation $abcr^6 + (a^2 + b^2 + c^2)r^4 = 4$, and the direction cosines of the sections by the equations

$$(m^2 + n^2)l/a = (n^2 + l^2)m/b = (l^2 + m^2)n/c = -lmnr^2.$$

(13) Shew that the foci of all parabolic sections of the surface $by^2 + cz^2 = 2x$, lie on the surface $bc(2x - by^2 - cz^2)(by^2 + cz^2) = b^2y^2 + c^2z^2$.

CHAPTER XIII.

TANGENTS. CONICAL AND CYLINDRICAL ENVELOPES. NORMALS. CONJUGATE DIAMETERS.

253. ON many accounts it is desirable that the student should be early acquainted with the chief properties connected with tangent lines and tangent planes to conicoids, before he is led to consider more general surfaces. We shall therefore give in this chapter some of the principal propositions relating to tangency in the case of the conicoids as represented by their equations in the simplest form. We shall also explain the properties of conjugate diameters and diametral planes.

Tangent Lines and Planes.

254. *To find the condition that a straight line shall touch a given conicoid, at a given point.*

Let the equation of the conicoid be $ax^2+by^2+cz^2=1$, the equations of a straight line drawn in a direction (λ, μ, ν) through the given point P, (f, g, h), are

$$(x-f)/\lambda=(y-g)/\mu=(z-h)/\nu=r, \qquad (1)$$

the values of r at the points P, Q, where it meets the conicoid, are given by $a(f+\lambda r)^2+b(g+\mu r)^2+c(h+\nu r)^2=1$.

If the direction be such that Q coincides with P, the straight line will become a tangent, and in this case the two values of r will be zero; therefore $af^2+bg^2+ch^2=1$, which only implies that P is on the conicoid, and $af\lambda+bg\mu+ch\nu=0$, (2), the condition of contact at the point (f, g, h).

255. *To find the equation of a tangent plane at a given point of a conicoid.*

The locus of all the tangent lines which can be drawn through the point (f, g, h) is found by eliminating λ, μ, ν between the equations (1) and (2) of the last article, giving

$$af(x-f)+bg(y-g)+ch(z-h)=0,$$

$$\text{or } afx+bgy+chz=1.$$

The locus is therefore a plane, and this plane is called the *tangent plane* to the surface.

If p be the perpendicular from the centre upon the tangent plane, $p^{-2}=a^2f^2+b^2g^2+c^2h^2$, Art. 71.

COR. 1. The generating lines of a hyperboloid of one sheet through the point (f, g, h) being two of the tangent lines, the tangent plane contains these lines, which together form what we have called the line-hyperbola in Art. 236.

COR. 2. Since any generating line is intersected at every point by some line of the opposite system, no two of which lie in the same plane, it follows that the tangent plane to the hyperboloid at any point in a generating line changes its position as the point moves along the line.

256. *To find the equation of a tangent plane to a conicoid drawn in a given direction.*

Let (l, m, n) be the given direction of the normal to the tangent plane, so that its equation is $lx+my+nz=p$; comparing with the equation $afx+bgy+chz=1$, $af/l=bg/m=ch/n=1/p$, (1), and, since $af^2+bg^2+ch^2=1$, $p^2=l^2/a+m^2/b+n^2/c$, thus the equations of the two tangent planes in the given direction are determined.

257. The equation of a tangent plane to a cone $ax^2+by^2+cz^2=0$ can be deduced from the preceding, in either of the forms, (i) $a\lambda x+b\mu y+c\nu z=0$, (λ, μ, ν) the direction of the line of contact taking the place of (f, g, h) the point of contact, and being determined by the equations $a\lambda/l=b\mu/m=c\nu/n$; or (ii) $lx+my+nz=0$, l, m, and n being subject to the condition $l^2/a+m^2/b+n^2/c=0$.

258. *To find the equations of an asymptote to a central conicoid.*

Let the equation of the conicoid be $ax^2+by^2+cz^2=1$, and let (ξ, η, ζ) be any point in the asymptote whose equations are $(x-\xi)/\lambda=(y-\eta)/\mu=(z-\zeta)/\nu=r$, then the two values of r are infinite in the equation $a(\xi+\lambda r)^2+b(\eta+\mu r)^2+c(\zeta+\nu r)^2-1=0$;

$$\therefore\ a\lambda^2+b\mu^2+c\nu^2=0, \text{ and } a\xi\lambda+b\eta\mu+c\zeta\nu=0,$$

and, if $a\xi^2+b\eta^2+c\zeta^2-1$ be not finite, the straight line lies entirely in the conicoid.

Hence, every straight line drawn in a tangent plane to the cone $ax^2+by^2+cz^2=0$, parallel to the line of contact, is an asymptote, including the generating lines in which it may intersect the conicoid.

259. *To find the nature of the intersection of a central conicoid with the tangent plane at a given point.*

Let the equation of the conicoid be $ax^2+by^2+cz^2=1$, that of the tangent plane at (f, g, h) is $afx+bgy+chz=1$, we have also $af^2+bg^2+ch^2=1$.

At the points of intersection

$$(af^2+bg^2)(ax^2+by^2)-(afx+bgy)^2=(1-ch^2)(1-cz^2)-(1-chz)^2;$$
$$\therefore\ ab(gx-fy)^2+c(z-h)^2=0.$$

For the ellipsoid and hyperboloid of two sheets the only solution is $x/f = y/g = z/h = 1$, since ab, and c are of the same sign; for the hyperboloid of one sheet the section is two lines, since ab, and c are of contrary signs.

260. *To find the magnitude and direction of the axes of the section of a central conicoid made by a given plane through the centre.*

Let the equations of the conicoid and plane be

$$ax^2 + by^2 + cz^2 = 1, \quad \text{and} \quad lx + my + nz = 0.$$

The equation of a sphere of radius r is $x^2 + y^2 + z^2 = r^2$; therefore the cone $(ar^2 - 1)x^2 + (br^2 - 1)y^2 + (cr^2 - 1)z^2 = 0$ is the locus of all diameters of the conicoid which are of equal length $2r$; the cone, therefore, intersects the given plane in two lines which are the direction of equal diameters of the central section, and if r be chosen so that these directions coincide, the given plane will be a tangent plane to the cone, and the line of contact will be an axis of the section; therefore, by (ii) Art. 257,

$$l^2/(ar^2 - 1) + m^2/(br^2 - 1) + n^2/(cr^2 - 1) = 0,$$

which is the quadratic giving the lengths of the semi-axes as in Art. 234. And, by (i) Art. 257, if (λ, μ, ν) be the direction of the axis $2r$,

$$(ar^2 - 1)\lambda/l = (br^2 - 1)\mu/m = (cr^2 - 1)\nu/n.$$

261. An important application of this proposition is to the investigation of the equation of the *wave surface*, which may be constructed as follows:

Take any central section of an ellipsoid and through the centre draw a perpendicular to the plane of the section, along which measure lengths equal to the semi-axes of the section, the locus of their extremities is the wave surface.

Let (x, y, z) be the extremity of the length r equal to one of the axes, measured in the direction (l, m, n); $\therefore$ $x/l = y/m = z/n$, and if the equation of the ellipsoid be $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, since the equation which determines the axes is $a^2l^2/(r^2 - a^2) + \ldots = 0$, the equation of the wave surface is

$$a^2x^2/(r^2 - a^2) + b^2y^2/(r^2 - b^2) + c^2z^2/(r^2 - c^2) = 0.$$

262. *To find the locus of the points of contact of all tangent planes which pass through a given point external to a given conicoid.*

Let (f, g, h) be the given point, $ax^2 + by^2 + cz^2 = 1$, the equation of the conicoid.

The equation of a tangent plane at any point (ξ, η, ζ) on the conicoid is $a\xi x + b\eta y + c\zeta z = 1$, and if it pass through the given point $af\xi + bg\eta + ch\zeta = 1$.

The tangent planes at every point of the conicoid whose coordinates satisfy this equation pass through the given point, the locus required is therefore the section of the conicoid by the plane whose equation is $afx + bgy + chz = 1$.

263. DEF. The plane containing the points of contact of all tangents from any point to a conicoid is the *Polar Plane* of the point, and the point is the *Pole* of the plane, with respect to the conicoid.

This will be a definition whether the point be external or internal, if we consider that imaginary tangent planes have a real plane containing the imaginary curve of contact.

Another definition will be given which does not involve the consideration of tangency.

One of the most important propositions connecting the pole and polar plane is the following.

264. *If U be the polar plane of any point P with respect to a conicoid, the polar plane of any point Q in the plane U will pass through P.*

For, if $ax^2+by^2+cz^2=1$ be the equation of the conicoid, and (f, g, h) be the point P, the equation of its polar plane U will be

$$afx+bgy+chz=1,$$

and, if (f', g', h') be any point Q in U, $aff'+bgg'+chh'=1$; but the equation of the polar plane of Q is $af'x+bg'y+ch'z=1$, which by the last equation contains (f, g, h), hence the polar plane of Q passes through P.

Conical and Cylindrical Envelopes.

265. *To find the conical envelope of a conicoid the vertex of the cone being a given point.*

If (f, g, h) be the given vertex, and (l, m, n) the direction of any generating line of the cone, the equation

$$a(f+lr)^2+b(g+mr)^2+c(h+nr)^2=1$$

must give equal values of r; therefore

$$(af^2+bg^2+ch^2-1)(al^2+bm^2+cn^2)=(afl+bgm+chn)^2.$$

If (x, y, z) be any point in the generating line,

$$(x-f)/l=(y-g)/m=(z-h)/n,$$

hence the equation in l, m, n being homogeneous,

$$(af^2+bg^2+ch^2-1)\{a(x-f)^2+b(y-g)^2+c(z-h)^2\}$$
$$=\{af(x-f)+bg(y-g)+ch(z-h)\}^2$$

is the equation of the conical envelope. This is readily reducible to the form

$$(af^2+bg^2+ch^2-1)(ax^2+by^2+cz^2-1)=(afx+bgy+chz-1)^2.$$

For, writing u, v, u_0 for $ax^2+by^2+cz^2-1$, $afx+bgy+chz-1$, and $af^2+bg^2+ch^2-1$ respectively, $u_0(u-2v+u_0)=(v-u_0)^2$; therefore $u_0u=v^2$.

266. This latter form may be obtained directly by Art. 262, since it is a surface of the second degree which passes through (f, g, h) and touches the conicoid where the plane $afx+bgy+chz=1$ cuts it.

For $Au=v^2$ is the equation of a surface which touches the conicoid $u=0$ where $v=0$, and, being satisfied by (f, g, h), we obtain, by substituting $A=u_0$, since v becomes u_0.

Similar considerations shew that the equation of two tangent planes to a cone $ax^2+by^2+cz^2=0$ which intersect in a line whose direction is (λ, μ, ν) is

$$(a\lambda^2+b\mu^2+c\nu^2)(ax^2+by^2+cz^2)=(a\lambda x+b\mu y+c\nu z)^2.$$

267. *To find the equation of a cylinder, which envelopes a given central conicoid, and has its generating lines in a given direction.*

Let (λ, μ, ν) be the direction of the generating lines of the cylinder, and $ax^2+by^2+cz^2=1$ the equation of the conicoid.

The equations of a generating line through any point (ξ, η, ζ) of the cylinder are $(x-\xi)/\lambda=(y-\eta)/\mu=(z-\zeta)/\nu=r$, and where this line touches the conicoid the values of r, which are equal, are given by $a(\xi+\lambda r)^2+b(\eta+\mu r)^2+c(\zeta+\nu r)^2=1$;

$$\therefore\ (a\lambda^2+b\mu^2+c\nu^2)(a\xi^2+b\eta^2+c\zeta^2-1)=(a\lambda\xi+b\mu\eta+c\nu\zeta)^2;$$

and, since (ξ, η, ζ) is any point on the cylinder, this is the equation of the enveloping cylinder.

This equation may also be deduced from that of the conical envelope by making (f, g, h) pass off to infinity in the direction (λ, μ, ν), so that $f:g:h=\lambda:\mu:\nu$.

Non-central Surfaces.

268. The corresponding propositions in the case of the non-central surfaces whose equations are of the form $by^2+cz^2=2x$ should be obtained by the student.

i. Condition for the tangency of $(x-f)/l=(y-g)/m=(z-h)/n$,

$$bgm+chn-l=0.$$

ii. Equation of the tangent plane at (f, g, h),

$$bgy+chz=x+f.$$

iii. Equation of the tangent plane in direction (l, m, n) is

$$lx+my+nz=-(m^2/b+n^2/c)/2l.$$

iv. Equation of the enveloping cone, vertex (f, g, h),

$$(bg^2+ch^2-2f)\{b(y-g)^2+c(z-h)^2\}=\{bg(y-g)+ch(z-h)-(x-f)\}^2,$$

or $(bg^2+ch^2-2f)(by^2+cz^2-2x)=(bgy+chz-x-f)^2$.

v. Equation of the enveloping cylinder, direction (λ, μ, ν),

$$(b\mu^2+c\nu^2)(by^2+cz^2-2x)=(b\mu y+c\nu z-\lambda)^2.$$

Normals.

269. *To find the equations of the normal to a conicoid at any point.*

DEF. A *normal* at a point is the straight line drawn perpendicular to the tangent plane at that point.

If (f, g, h) be the point, the equation of the tangent plane is $afx + bgy + chz = 1$; therefore the equations of the normal will be

$$(x-f)/af = (y-g)/bg = (z-h)/ch = \pm rp,$$

if r be the distance between (x, y, z) and (f, g, h), and p be the perpendicular on the tangent plane from the centre.

270. *To shew that six normals can be drawn from a given point to a central conicoid.*

Let the equation of the conicoid be $ax^2 + by^2 + cz^2 = 1$.

The equations of a normal at a point (x, y, z) are

$$(\xi - x)/ax = (\eta - y)/by = (\zeta - z)/cz = \rho,$$

if this pass through a given point (f, g, h)

$$f = x(\rho a + 1), \quad g = y(\rho b + 1), \quad h = z(\rho c + 1);$$

$$\therefore \; af^2/(\rho a + 1)^2 + bg^2/(\rho b + 1)^2 + ch^2/(\rho c + 1)^2 = 1,$$

which gives generally six values of ρ determining the feet of six normals from the given point.

271. *To shew that the locus of a point, from which three normals can be drawn to a central conicoid, which have their feet in a given plane section of the conicoid, is a straight line, and to find the condition to which the given plane section must be subject.**

Let the equation of the conicoid be $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, and that of the given intersecting plane $lx/a + my/b + nz/c = 1$, (1), and let (ξ, η, ζ) be a point from which if six normals be drawn the feet of three of them will lie on the given plane section, the other three must then lie on some other plane section given by $l'x/a + m'y/b + n'z/c = d$, (2); hence the six feet lie on the surface

$$(lx/a + my/b + nz/c - 1)(l'x/a + m'y/b + n'z/c - d) - x^2/a^2 - y^2/b^2 - z^2/c^2 + 1 = 0, \; (3);$$

the six feet also lie on each of the three surfaces

$$U \equiv (b^2 - c^2)\,yz - b^2\eta z + c^2\zeta y = 0,$$
$$V \equiv (c^2 - a^2)\,zx - c^2\zeta x + a^2\xi z = 0,$$
$$W \equiv (a^2 - b^2)\,xy - a^2\xi y + b^2\eta x = 0,$$

and therefore on the surface $\lambda U + \mu V + \nu W = 0$, (4).

Now we can make (3) and (4) identical by writing for the equation (2) $x/la + y/mb + z/nc + 1 = 0$, and equating the remainder of the coefficients, so that

$$\lambda(b^2 - c^2) = (m/n + n/m)/bc, \text{ \&c.,} \quad \nu b^2\eta - \mu c^2\zeta = (l - l^{-1})\,a^{-1}, \text{ \&c.,} \quad (5);$$

hence it follows that, when the plane (1) is given, the locus of the point (ξ, η, ζ) is a straight line, since equations (5) are equivalent to two equations in ξ, η, ζ and a relation between l, m, n, which must be satisfied in order that normals at some three points of the plane section may meet in a point.

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* *Quarterly Journal* vol. VIII, p. 69.

This equation of condition may be written

$$(m^2n^2 - l^2)(b^2 - c^2)^2 + (n^2l^2 - m^2)(c^2 - a)^2 + (l^2m^2 - n^2)(a^2 - b^2)^2 = 0.$$

as in Wolstenholme's *Problems*.

Since l/a, m/b, n/c, are the Boothian coordinates of the given plane, this gives the tangential equation of a surface fixed with reference to the conicoid, to which all the planes which satisfy the required condition must be tangents.

Conjugate Diameters.

272. *To find the locus of the middle points of a system of parallel chords of a given central conicoid.*

Let the equation of the surface be $ax^2 + by^2 + cz^2 = 1$, (λ, μ, ν) the direction of each of a series of parallel chords, and (ξ, η, ζ) the middle point of any one of them, whose equation will be

$$(x - \xi)/\lambda = (y - \eta)/\mu = (z - \zeta)/\nu = r,$$

and we shall have, for the points in which it meets the surface, the equation $a(\xi + \lambda r)^2 + b(\eta + \mu r)^2 + c(\zeta + \nu r)^2 = 1$.

But, since (ξ, η, ζ) is the middle point of the chord, the values of r obtained from this equation will be equal and of opposite signs, and therefore the equation $a\lambda\xi + b\mu\eta + c\nu\zeta = 0$ will give the locus required. The form of this equation shews that it passes through the centre, as it manifestly ought to do.

DEF. The locus of the middle points of a system of parallel chords of a conicoid is a plane, which is called a *diametral plane.*

We shall have, conversely, that any central plane whose equation is $lx + my + nz = 0$, will bisect a series of chords parallel to the straight line $ax/l = by/m = cz/n$, which is called the diameter *conjugate* to the plane. It appears from Art. 234 that the locus of the centres of a series of sections of the surface parallel to a given central plane is the diameter conjugate to that plane.

273. DEF. Three diameters are *conjugate*, when each diameter is conjugate to the plane containing the other two, and three diametral planes are *conjugate* when each is conjugate to the intersection of the other two.

Let OP be any radius of a central conicoid, QOR the diametral plane conjugate to OP, and OQ, OR any conjugate semi-diameters of the section QR; OQ bisects all chords of the section PQ parallel to OP, therefore OP is conjugate to OQ and bisects all chords parallel to OQ; also, OR bisects all chords parallel to OQ, therefore POR is the diametral plane conjugate to OQ. Similarly QOP is the diametral plane parallel to OR.

Hence, there are an infinite number of systems of three conjugate diameters. One can be chosen arbitrarily, and the remaining two are any conjugate diameters of the section made by the diametral plane conjugate to the first.

It should be observed that a diametral plane is the polar plane of the point of the conjugate diameter which is at an infinite distance. Also, that when the conicoid degenerates into a cone, *any* point in one of three conjugate diameters is the pole with respect to the cone of the plane containing the other two.

274. If a conicoid be referred to a diametral plane, as that of xy, and the corresponding conjugate diameter as the axis of z, since every straight line parallel to Oz will be bisected by the plane of xy, the equation of the surface can only contain even powers of z. Hence, since we can find three planes such that the intersection of any two is conjugate to the third, the equation of the surface referred to these planes will be of the form $ax^2+by^2+cz^2=1$.

275. *To find the conditions that each of three central planes of a central conicoid may be conjugate to the intersection of the other two.*

Let the direction-cosines of normals to the planes be (l_1, m_1, n_1), (l_2, m_2, n_2), and (l_3, m_3, n_3).

The equations of the diameter conjugate to the first will be $ax/l_1 = by/m_1 = cz/n_1$, and if this be the intersection of the other two planes, and therefore lie in each of them, we shall have

$$l_2l_1/a + m_2m_1/b + n_2n_1/c = 0, \text{ and } l_3l_1/a + m_3m_1/b + n_3n_1/c;$$

hence, if the three conditions

$$l_2l_3/a+m_2m_3/c+n_2n_3/c=l_3l_1/a+m_3m_1/b+n_3n_1/c=l_1l_2/a+m_1m_2/b+n_1n_2/c=0$$

be satisfied, the planes will be such as required.

276. *To find the relations between the coordinates of the extremities of a system of conjugate diameters of a central conicoid.*

The equation of the surface being $ax^2+by^2+cz^2=1$, and the coordinates of the extremities of the semi-diameters r_1, r_2, r_3 being (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) we shall have, since the points lie on the surface,

$$ax_1^2+by_1^2+cz_1^2 = ax_2^2+by_2^2+cz_2^2 = ax_3^2+by_3^2+cz_3^2 = 1, \quad (1)$$

and, since the diameters through the points are conjugate,

$$\begin{aligned} ax_2x_3+by_2y_3+cz_2z_3 &= ax_3x_1+by_3y_1+cz_3z_1 \\ &= ax_1x_2+by_1y_2+cz_1z_2 = 0. \quad (2) \end{aligned}$$

The systems (1) and (2) shew that

$$x_1\sqrt{a},\ y_1\sqrt{b},\ z_1\sqrt{c};\quad x_2\sqrt{a},\ y_2\sqrt{b},\ z_2\sqrt{c};\quad x_3\sqrt{a},\ y_3\sqrt{b},\ z_3\sqrt{c};$$

are the direction-cosines of three straight lines at right angles to each other, and we know therefore, Art. 145, that they are equivalent to the systems

$$ax_1^2+ax_2^2+ax_3^2 = by_1^2+by_2^2+by_3^2 = cz_1^2+cz_2^2+cz_3^2 = 1,$$

$$y_1z_1+y_2z_2+y_3z_3 = z_1x_1+z_2x_2+z_3x_3 = x_1y_1+x_2y_2+x_3y_3 = 0. \quad (3)$$

Hence, in the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, we shall have

$$x_1^2 + x_2^2 + x_3^2 = a^2, \quad y_1^2 + y_2^2 + y_3^2 = b^2, \quad z_1^2 + z_2^2 + z_3^2 = c^2, \quad (4)$$

or the sum of the squares of the projections of three conjugate diameters on one of the axes is equal to the square of that axis; also,

$$r_1^2 + r_2^2 + r_3^2 = a^2 + b^2 + c^2.$$

If (l, m, n) be the direction of any line, by (3) and (4)

$$(lx_1 + my_1 + nz_1)^2 + (lx_2 + my_2 + nz_2)^2 + (lx_3 + my_3 + nz_3)^2$$
$$= l^2a^2 + m^2b^2 + n^2c^2 = p^2,$$

$$\therefore r_1^2 - (lx_1 + my_1 + nz_1)^2 + \ldots = a^2 + b^2 + c^2 - p^2;$$

but $(lx_1 + my_1 + nz_1)^2$ and $r_1^2 - (lx_1 + my_1 + nz_1)^2$ are respectively squares of the projections of r_1 upon a line and a plane whose directions are given by (l, m, n), hence it follows that

The sum of the squares of the projections of three conjugate diameters on any line or any plane is constant.

277. *To find the relations which exist between the lengths of a system of conjugate diameters of a central conicoid and the angles between them.*

Let the equation of the surface referred to its principal axes be $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, and, referred to a system of conjugate diameters inclined at angles α, β, γ, $x^2/a'^2 + y^2/b'^2 + z^2/c'^2 = 1$. The invariants derived from $h(x^2 + y^2 + z^2) - x^2/a^2 - y^2/b^2 - z^2/c^2$, and the transformed expression, see Art. 157, give the equations

$$a'^2 + b'^2 + c'^2 = a^2 + b^2 + c^2, \quad (1)$$

$$b'^2c'^2 \sin^2\alpha + c'^2a'^2 \sin^2\beta + a'^2b'^2 \sin^2\gamma = b^2c^2 + c^2a^2 + a^2b^2, \quad (2)$$

and

$$a'^2b'^2c'^2 (1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma) = a^2b^2c^2. \quad (3)$$

When the surface is an ellipsoid, all these lengths are real, and we see from (1) that the sum of the squares of three conjugate radii is constant; from (2) that the sum of the squares of the faces of a parallelepiped having three conjugate radii as conterminous edges is constant; and from (3) that the volume of such a parallelepiped is constant.

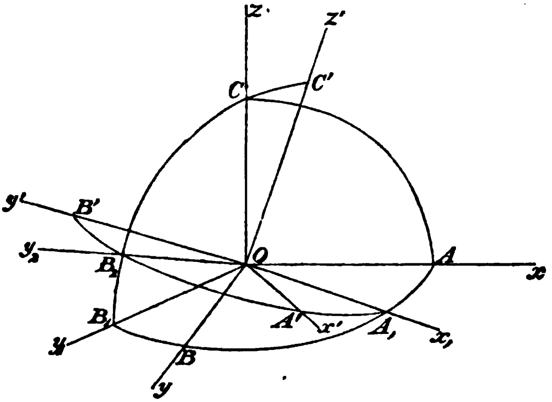
In the hyperboloid of one sheet, since $a^2b^2c^2$ is negative, and $1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma$ is always positive, $a'^2b'^2c'^2$ must be negative, but a'^2, b'^2, and c'^2 cannot all be negative, hence one and only one is negative; that is, in a hyperboloid of one sheet, two of a system of conjugate diameters meet the surface in real points, and the third does not.

In the hyperboloid of two sheets, $a^2b^2c^2$, and therefore $a'^2b'^2c'^2$ is positive, hence two, or none, of the three a'^2, b'^2, c'^2 are negative. If none be negative, writing $-b^2$, $-c^2$ for b^2, c^2 respectively, we must have both $a^2 - b^2 - c^2$ and $b^2c^2 - a^2(b^2 + c^2)$ positive, which are

easily shewn to be inconsistent. Hence two must be negative; or, in the hyperboloid of two sheets, one and only one of a system of conjugate diameters meets the surface in real points.

278. The relations (1) and (3) may be obtained geometrically. We will give the proof in the case of the ellipsoid, and leave the other two cases as exercises for the student.

Let Ox, Oy, Oz be the directions of the axes of the surface, Ox', Oy', Oz' those of a system of conjugate diameters; A, B, C, A', B', C', the extremities of the semi-diameters along those axes; a, b, c, a', b', c' their lengths. Also, let the sections AB, $A'B'$ intersect in A_1, let OB_2 be the semi-diameter conjugate to OA_1 in the section $A'B'$, and let CB_2 meet AB in B_1; $OA_1 = a_1$, $OB_1 = b_1$, $OB_2 = b_2$.



Then, the plane $A'B'$ being conjugate to OC', OA_1, OB_2, OC' will be a system of conjugate radii, or OA_1 will be conjugate to the plane $C'B_2$, and since OA_1 lies in a principal plane, $C'OB_2$ will be perpendicular to that plane, and will therefore contain OC; and OC, OB_1 being the principal semi-axes of the section B_1C', OA_1, OB_1, OC will be a system of conjugate diameters, and OA_1, OB_1 will be conjugate in the section AB.

Hence we have the equations

$$a_1^2 + b_2^2 = a'^2 + b'^2, \quad b_1^2 + c^2 = b_2^2 + c'^2, \quad a^2 + b^2 = a_1^2 + b_1^2,$$

and from these we obtain the relation $a'^2 + b'^2 + c'^2 = a^2 + b^2 + c^2$.

Also, since the parallelogram of which OA', OB' are conterminous sides, is equal to that of which OA_1, OB_2 are conterminous sides, and similarly for the section B_1C', AB,

$$\text{vol}\,(a', b', c') = \text{vol}\,(a_1, b_2, c') = \text{vol}\,(a_1, b_1, c) = \text{vol}\,(a, b, c);$$

denoting any parallelepiped by three conterminous edges.

279. *To find the diametral plane bisecting a given system of parallel chords, in the case of non-central conicoids.*

Let the equation of the surface be $by^2 + cz^2 = 2x$, and let (λ, μ, ν) be the direction of the chords, the equation of the diametral plane will be $\mu by + \nu cz = \lambda$, shewing that all the diametral planes are parallel to the common axis of the principal parabolic sections; a fact which might have been anticipated from the consideration that these surfaces have their centre on that axis at an infinite distance.

280. We cannot in these surfaces, as in the central conicoid, have a system of three conjugate planes at a finite distance, but we can find an infinite number, such that, for two of them, each bisects the chords parallel to the other and to a third plane. By taking the origin where the intersection of these two meets the paraboloid, and referring to these three planes, the equation of the surface will assume the form $b'y^2 + c'z^2 = 2x$.

Let the equations of the two diametral planes be

$$m_1 y + n_1 z = 1, \quad (1), \qquad m_2 y + n_2 z = 1, \quad (2),$$

and let the direction of the third plane be (l_3, m_3, n_3). The direction-cosines of the chords bisected by (1) are in the ratios

$$1 : m_1/b : n_1/c,$$

and if these be parallel to (2) and the third plane, we shall have

$$m_1 m_2/b + n_1 n_2/c = 0, \quad l_3 + m_3 m_1/b + n_3 n_1/c = 0.$$

Similarly, in order that (2) may be conjugate to the intersection of the other two, we shall have

$$m_1 m_2/b + n_1 n_2/c = 0, \quad l_3 + m_2 m_3/b + n_2 n_3/c = 0.$$

One of these is coincident with one of the former, and, there being thus only three relations necessary between the four quantities determining the planes, an infinite number of such systems can be determined.

Polar Plane.

281. *Through a fixed point a straight line is drawn meeting a central conicoid, and on this line a point is taken, such that its distance from the fixed point is a harmonic mean between the segments of the line made by the conicoid; to find the locus of the point.*

Let the equation of the conicoid be $ax^2 + by^2 + cz^2 = 1$, and let the equations of the straight line through the fixed point (f, g, h) be $(x-f)/l = (y-g)/m = (z-h)/n = r$.

The values of r at the points of intersection are given by the equation $a(f+lr)^2 + b(g+mr)^2 + c(h+nr)^2 = 1$.

If r_1, r_2 be the roots of this equation, and r be taken for the distance from (f, g, h) of the point whose locus is required,

$$2r^{-1} = r_1^{-1} + r_2^{-1} = -2(afl + bgm + chn)/(af^2 + bg^2 + ch^2 - 1),$$

$\therefore\ af^2+bg^2+ch^2-1+(afl+bgm+chn)\,r$ and $x=f+lr$, &c.;

$\therefore\ afx+bgy+chz=1$ is the equation required.

This theorem gives rise to the definition of the polar plane of a point alluded to in Art. 263, viz.:

DEF. The *polar plane* of any fixed point, with respect to a given conicoid, is a plane, which, with the conicoid, divides harmonically all straight lines passing through the fixed point.

282. The corresponding locus for an arithmetic mean is a conicoid similar to the given one, of which a diameter is the line joining the given point and the centre.

For a geometric mean, the locus is a similar conicoid, which meets the given conicoid in the polar plane of the given point.

XXI.

(1) The tangent planes to an ellipsoid at points lying on a plane section will intersect any fixed plane in straight lines which touch a conic section.

(2) Tangent planes are drawn to an ellipsoid $x^2/a^2+\ldots=1$, and are such that their intersections with the plane zx are parallel to the lines

$$cx\sqrt{(b^2-c^2)}\pm az\sqrt{(a^2-b^2)}=0,$$

shew that the points of contact all lie on circular sections.

(3) The locus of the centres of sections of $ax^2+by^2+cz^2=1$ by planes which touch $\alpha x^2+\beta y^2+\gamma z^2=1$, is

$$a^2x^2/\alpha+b^2y^2/\beta+c^2z^2/\gamma=(ax^2+by^2+cz^2)^2.$$

(4) Find the equation of the tangent plane to a conicoid upon the principle that no other plane can pass between it and the surface in the neighbourhood of the point through which it is drawn.

(5) Prove that the tangent planes of the cone

$$x^2/(b+c)+y^2/(c+a)+z^2/(a+b)=0,$$

cut the surface $ax^2+by^2+cz^2=1$ in rectangular hyperbolas.

(6) If the area of the central curve in which a cylinder touches an ellipsoid be equal to that of the section containing the greatest and least axes, prove that the axis of the cylinder will lie on one of two planes.

(7) Find the locus of straight lines which meet the two lines $x=a$, $y=0$, and $x=-a$, $z=0$, and touch the sphere $x^2+y^2+z^2=c^2$; shew that when $c=a$ the locus reduces to two hyperboloids.

(8) The normal at any point P of an ellipsoid meets the principal planes in G_1, G_2, G_3; prove that $PG_1 \,.\, PG_2 \,.\, PG_3$ varies as the cube of the area of the central section made by a plane conjugate to the diameter through P.

(9) If an ellipsoid be placed on a horizontal plane with an axis $2c$ vertical, shew that the tangent of the altitude of a star which will cast on the plane a circular shadow is $2c/d$, where d is the distance of the foci of the horizontal principal section.

(10) If r be measured inwards along the normals to an ellipsoid so that $pr = m^2$ a constant, p being the perpendicular from the centre on the tangent plane, prove that the locus of the point thus obtained will be

$$a^2x^2/(a^2 - m^2)^2 + b^2y^2/(b^2 - m^2) + c^2z^2/(c^2 - m^2)^2 = 1.$$

What does the locus become when $m = b$?

(11) If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the extremities of three conjugate diameters of the ellipsoid $ax^2 + by^2 + cz^2 = 1$, the equation of the plane passing through these points will be

$$ax(x_1 + x_2 + x_3) + by(y_1 + y_2 + y_3) + cz(z_1 + z_2 + z_3) = 1.$$

If one of the ends (x_1, y_1, z_1) be fixed, shew that the perpendicular from the centre on this plane will describe the cone

$$x^2/a + y^2/b + z^2/c = 3(xx_1 + yy_1 + zz_1)^2.$$

(12) The locus of the centre of gravity of the triangle formed by joining the extremities of three conjugate diameters is $ax^2 + by^2 + cz^2 = \frac{1}{3}$, and the locus of the intersection of tangent planes drawn through their extremities is $ax^2 + by^2 + cz^2 = 3$.

(13) Prove that the sum of the products of the perpendiculars from the two extremities of each of three conjugate diameters of a conicoid upon any tangent plane is equal to twice the square of the perpendicular from the centre.

(14) The locus of points, from which rectilinear asymptotes can be drawn to the conicoid $ax^2 + by^2 + cz^2 = 1$, at right angles to each other, is the cone $a^2(b + c)x^2 + b^2(c + a)y^2 + c^2(a + b)z^2 = 0$.

(15) The six normals to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ from the point (f, g, h) lie on the cone $(b^2 - c^2)f/(x - f) + \ldots = 0$.

(16) The locus of the middle points of a chord of a conicoid drawn in a given direction, and such that the normals at its ends intersect, is a straight line.

XXII.

(1) Find the locus of the feet of the perpendiculars from a point (α, β, γ) on the tangent planes to the cone $ax^2 + by^2 + cz^2 = 0$; and prove that if the locus be a plane curve it will be a circle, and that, if $b > a$ and c negative, the point must lie on one of the lines $y = 0$, $ax^2/(b - a) = cz^2/(c - b)$.

(2) Two similar and similarly situated concentric ellipsoids have their axes in the ratio of $1 : n$, $n > 1$; from any point on the exterior as vertex a cone is drawn enveloping the interior, shew that the plane of its curve of intersection with the exterior ellipsoid touches another similar ellipsoid whose axes are to those of the interior as $n^2 \sim 2 : n$.

(3) The sines of the angles at which a straight line cuts an ellipsoid are proportional to the perpendiculars from the centre on the tangent planes at the points of intersection.

(4) Two planes are drawn through the six feet of the normals drawn to an ellipsoid from a given point, each plane containing three; prove that, if (α, β, γ), $(\alpha', \beta', \gamma')$ be the poles of these planes with reference to the ellipsoid

$$\alpha\alpha' + a^2 = \beta\beta' + b^2 = \gamma\gamma' + c^2 = 0.$$

(5) Q is any point from which six normals are drawn to an ellipsoid, centre O, and semi-axes a, b, c; and $N_1, N_2, \ldots N_6$ are projections upon OQ of $P_1, P_2, \ldots P_6$ which are the feet of the normals; shew that

$$\Sigma(OP^2) - OQ\,\Sigma(ON) = 2(a^2 + b^2 + c^2).$$

(6) Prove that, if n be the length of a normal chord to a surface $ax^2 + by^2 + cz^2 = 1$ at a point for which r is the distance from the centre, and p the perpendicular on the tangent plane

$$2/n - (a + b + c)\, p + (ab + bc + ca - abcr^2)\, p^3 = 0.$$

(7) Any three equal conjugate diameters of an ellipsoid lie on the cone

$$(2a^2 - b^2 - c^2)\, x^2/a^2 + (2b^2 - c^2 - a^2)\, y^2/b^2 + (2c^2 - a^2 - b^2)\, z^2/c^2 = 0;$$

and the planes containing two of three equal conjugate diameters touch the cone $x^2/\alpha + y^2/\beta + z^2/\gamma = 0$, where $\alpha = a^2 (2a^2 - b^2 - c^2)$ &c.

(8) The locus of the intersection of three tangent planes to the conicoids $x^2/a + y^2/b + z^2/c = 1$ and $y^2/b + z^2/c = 2x$, the planes being mutually at right angles, are respectively $x^2 + y^2 + z^2 = a + b + c$ and $2x + b + c = 0$.

(9) The locus of the intersection of three tangent lines to the conicoid $x^2/a + y^2/b + z^2/c = 1$, mutually at right angles, is

$$(b + c)\, x^2 + (c + a)\, y^2 + (a + b)\, z^2 = bc + ca + ab.$$

(10) Prove that a normal to the hyperboloid of one sheet $ax^2 + by^2 + cz^2 = 1$ at a point (f, g, h), at which the generating lines are at right angles, meets the hyperboloid again at a point (f', g', h'), where $f' : f = b + c - a : b + c + a$, &c.

(11) If three of the generating lines of the enveloping cone of a paraboloid be mutually at right angles, shew that the vertex will be on a paraboloid, and that the polar plane of the vertex will touch another paraboloid.

(12) The points on a conicoid, the normals at which intersect the normal at a given point, all lie on a cone of the second degree having its vertex at the given point.

(13) The locus of the intersection of two tangent planes to the cone $x^2/a + y^2/b + z^2/c = 0$, which are at right angles, is the cone

$$(b + c)\, x^2 + (c + a)\, y^2 + (a + b)\, z^2 = 0.$$

(14) The enveloping cones which have as vertices two points on the same diameter of a conicoid intersect in two parallel planes between whose distances from the centre that of the tangent plane at the end of the diameter is a mean proportional.

XXIII.

(1) A, B, C, D are the feet of the normals drawn from any point to the cone $ax^2 + by^2 + cz^2 = 0$, prove that the perpendiculars from the origin on the faces of the tetrahedron $ABCD$ are generators of the cone

$$a (b - c)^2 x^2 + b (c - a)^2 y^2 + c (a - b)^2 z^2 = 0.$$

(2) Two conicoids A and B touch each other along a plane curve, a plane touching B in P meets A in a conic S. Shew that the generators of B passing through P are tangents to S.

(3) If the plane $lx + my + nz = p$ cut the hyperboloid $ax^2 + by^2 + cz^2 = 1$ in a parabola, prove that $l^2a^{-1} + m^2b^{-1} + n^2c^{-1} = 0$, and that the vertex will lie in the plane $ax\,(b^{-1} - c^{-1})/l + \ldots = 0$.

(4) The normals to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at points on the planes $x/a + y/b + z/c = \pm 1$ all intersect the straight line

$$ax\,(b^2 - c^2) = by\,(c^2 - a^2) = cz\,(a^2 - b^2).$$

(5) The normals to the paraboloid $y^2/b + z^2/c = 2x$, at points on the plane $px + qy + rz = 1$, will all meet one straight line if

$$p^3 (b - c)^2 + 2p\,(q^2b - r^2c)(b - c) = 2\,(q^2b + r^2c).$$

(6) Shew that the cone, whose vertex is (f, g, h), which envelopes the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, will be cut by the plane of xy in a rectangular hyperbola, if the vertex lie on the spheroid $(x^2 + y^2)/(a^2 + b^2) + z^2/c^2 = 1$. Also, if P be the centre of this section when V is the vertex, shew that the locus of P will be the inverse of the projection on the plane of xy of whatever curve V is made to describe upon the spheroid.

(7) If two planes be drawn at right angles to each other touching the central conicoid $ax^2 + by^2 + cz^2 = 1$, and having their line of intersection in a given direction (l, m, n), shew that the locus of their line of intersection will be the right circular cylinder

$$x^2 + y^2 + z^2 = (lx + my + nz)^2 + (m^2 + n^2)/a + (n^2 + l^2)/b + (l^2 + m^2)/c.$$

(8) If the non-central conicoid $y^2/b + z^2/c = 2x$, be taken in the last problem, the locus will be $2l(lx + my + nz) - 2x = b(n^2 + l^2) + c(l^2 + m^2)$.

(9) Prove that a tangent plane to the cone $2x^2/(b-c) + y^2/b - z^2/c = 0$ will meet the paraboloid $y^2/b + z^2/c = 2x$ in points, the normals at which will all intersect the same straight line, and the surface generated by the straight line will have for its equation

$$2(b-c)\{x(by^2 - cz^2) + bc(y^2 - z^2)\}^2 = (by^2 - cz^2)(by^2 + cz^2)^2.$$

(10) The generators at a point P of a hyperboloid of one sheet meet the generators at a fixed point O in points D and E, so that the area of the triangle ODE is constant, prove that P must lie on a cone of the second degree, whose vertex is at O, such that the tangent planes to it through the line joining O to the centre touch it in lines lying in the tangent plane to the hyperboloid at O.

(11) The tangent plane at an umbilicus meets any enveloping cone in a conic of which the umbilicus is a focus, and the intersection of the plane of contact and the tangent plane the corresponding directrix.

(12) A cone whose vertex is any point of the hyperbola $x = 0$, $kz^2 - hy^2 = 1$, envelopes the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, whose least semi-axis is c; and h and k satisfy the relation $b^2 - c^2 = hb^2(a^2 - b^2) + kc^2(a^2 - c^2)$; shew that the directrices of all the sections of the ellipsoid made by the planes of contact lie in one or other of two fixed planes.

CHAPTER XIV.

CONFOCAL CONICOIDS. ELLIPTIC COORDINATES. FOCAL CONICS. BIFOCAL CHORDS. CORRESPONDING POINTS.

283. In the preceding chapters we have considered the intersections of planes and straight lines with conicoids; in this chapter we shall discuss the mutual relations of conicoids grouped in a particular manner and called confocal conicoids, and prove certain theorems relating to their intersections, which will be useful hereafter when we treat of the curvature of surfaces and geodesic lines.

A knowledge of the whole theory of confocal surfaces is essential for the solution of many important problems in Physics; in fact, it was in the study of the attraction of ellipsoids that Maclaurin was first led to consider the properties of this class of surfaces.

The theory may be said to have been completed by Chasles,* although many valuable propositions are due to M'Cullagh.†

284. Def. Two conicoids are confocal, when the foci, real or imaginary, of their principal sections coincide; or, when the directions of their principal axes coincide, and their squares differ by a constant quantity.

Another definition will be afterwards given, but for our present purpose this definition has the advantage of greater simplicity.

If $x^2/a^2+y^2/b^2+z^2/c^2=1$, and $x^2/a'^2+y^2/b'^2+z^2/c'^2=1$ be the equations of two confocal surfaces, $a^2-a'^2=b^2-b'^2=c^2-c'^2$, or $a^2-b^2=a'^2-b'^2=\beta^2$ and $a^2-c^2=a'^2-c'^2=\gamma^2$. These relations have given rise to two methods of stating the equation of a group of confocal conicoids, called, for the sake of brevity, *confocals*.

In one method a is called the *primary* semi-axis, and the equation of the group is written

$$x^2/a^2+y^2/(a^2-\beta^2)+z^2/(a^2-\gamma^2)=1,$$

β and γ being constant quantities, viz. half the distance between the foci of the principal sections containing the primary axes, the individuals of the group being determined by assigning particular values to the primary axis.

* Briot and Bouquet, *Géometrié Analytiqué*.
Aperçu Historique, L'Acad. Brux. 1837.

† With reference to the relative claims of Chasles and M'Cullagh in priority in certain investigations, see Liouville's *Journal*, vol. XI., p. 120, and *Proceedings of the Irish Academy*, vol. II., p. 501.

In the other method the equation

$$x^2/(a+k)+y^2/(b+k)+z^2/(c+k)=1$$

represents all conicoids confocal with $x^2/a+y^2/b+z^2/c=1$, by assigning arbitrary constant values to the parameter k.

285. *To shew that three conicoids can be drawn through a given point, confocal with a given central conicoid, and that these three will be an ellipsoid and the two hyperboloids.**

Let $x^2/a+y^2/b+z^2/c=1$ be the given conicoid in which $a>b>c$, and (f, g, h) the given point. Any confocal which contains (f, g, h) has its parameter k determined by the equation

$$f^2/(a+k)+g^2/(b+k)+h^2/(c+k)=1,$$

or $$(k+a)(k+b)(k+c)-f^2(k+b)(k+c)$$
$$-g^2(k+c)(k+a)-h^2(k+a)(k+b)=0.$$

If now we write for k in the left side of the equation $\infty, -c, -b, -a$ successively, the signs of the result will be $+, -, +, -$, hence the equation has three real roots separated by these quantities.

Also, the quantities $a+k$, $b+k$, $c+k$, will have the following signs, corresponding to the values of k:

$$\begin{array}{ll} k>-c, & +,\ +,\ +, \\ -b<k<-c, & +,\ +,\ -, \\ -a<k<-b, & +,\ -,\ -, \end{array}$$

which proves the proposition.

COR. *Two confocal ellipsoids or two confocal hyperboloids of the same kind cannot intersect.*

Elliptic coordinates.

286. The position of a point on an ellipsoid is determined, when the octant on which it lies is known, by the primary axes of the confocal hyperboloids which pass through it; these semi-axes were taken by Liouville as *elliptic coordinates.*

If this description be taken, the Cartesian coordinates of any point on the ellipsoid can be expressed in terms of the elliptic coordinates as follows.

Let the equation of the ellipsoid be

$$x^2/a^2+y^2/(a^2-\beta^2)+z^2(a^2-\gamma^2)=1,$$

and let ρ be the primary axis of a confocal hyperboloid through any point (x, y, z);

$$\therefore x^2/\rho^2+y^2/(\rho^2-\beta^2)+z^2/(\rho^2-\gamma^2)=1;$$

* *Aper. Hist.* (30), p. 392; *Proc. Ir. Acad.* vol. II., 496.

subtracting, and dividing by $\rho^2 - a^2$, since ρ^2 is not equal to a^2,

$$(\rho^2-\beta^2)(\rho^2-\gamma^2)\,x^2/a^2+\rho^2(\rho^2-\gamma^2)\,y^2/(a^2-\beta^2)$$
$$+\rho^2(\rho^2-\beta^2)\,z^2/(a^2-\gamma^2)=0,$$
$$\text{or}\quad \rho^4+A\rho^2+\beta^2\gamma^2x^2/a^2=0,$$

and if a''^2, a'''^2 be the two values of ρ^2, then

$$\beta^2\gamma^2x^2=a^2a''^2a'''^2 \text{ or } (a^2-b^2)(a^2-c^2)\,x^2=a^2a''^2a'''^2.$$

Similar expressions hold for y^2 and z^2; and if b''^2, $-c''^2$ and $-b'''^2$, $-c'''^2$ belong to the two confocals,

$$(a^2-b^2)(b^2-c^2)\,y^2=b^2b''^2b'''^2,\quad (a^2-c^2)(b^2-c^2)\,z^2=c^2c''^2c'''^2.$$

287. Another definition of elliptic coordinates used by Cayley,* also derived from the two confocal hyperboloids through the point, is more symmetrical.

If the equation of the ellipsoid be in the form

$$x^2/a+y^2/b+z^2/c=1,$$

that of a confocal through (x, y, z) being

$$x^2/(a+k)+y^2/(b+k)+z^2/(c+k)=1,$$

we obtain as before

$$(b+k)(c+k)\,x^2/a+(c+k)(a+k)\,y^2/b+(a+k)(b+k)\,z^2/c=0,$$

μ, ν the values of k for the two hyperboloids are the elliptic coordinates of any point (x, y, z) on the ellipsoid.

Solving with respect to $a+k$ as in the last article,

$$x^2(a-b)(a-c)=a(a+\mu)(a+\nu),$$
$$y^2(b-c)(b-a)=b(b+\mu)(b+\nu),$$
$$\text{and}\quad z^2(c-a)(c-b)=c(c+\mu)(c+\nu),$$

where, since $b+\nu$, $c+\mu$, and $c+\nu$ are negative, y and z are real.

Cayley writes α, β, γ for $b-c$, $c-a$, $a-b$, which are constant for all confocals and satisfy the equation $\alpha+\beta+\gamma=0$.

288. These systems of coordinates are chiefly employed in investigating the forms of certain curves on ellipsoids, but if λ, μ, ν be the roots of the equation $x^2/(a+k)+y^2/(b+k)+z^2/(c+k)=1$, these may be called the *elliptic coordinates* of any point in space with reference to the fundamental conicoid.

The equations, in elliptic coordinates, of the two curves of intersection of the confocal hyperboloids with the ellipsoid are, by the first definition, a' and $a''=$ constant, by the second μ and ν $=$ constant.

289. From the cubic equation in k, $\xi^2/(a+k)+\ldots=1$, the roots of which λ, μ, ν are the elliptic coordinates of (ξ, η, ζ), we have

$$\lambda+\mu+\nu=\xi^2+\eta^2+\zeta^2-a-b-c,\quad (1)$$

* *Proc. of London Math. Soc.*, vol. IV.

$$\mu\nu + \nu\lambda + \lambda\mu = bc + ca + ab - (b + c)\,\xi^2 - \ldots, \quad (2)$$
$$\lambda\mu\nu = abc\,(\xi^2/a + \eta^2/b + \zeta^2/c - 1); \quad (3)$$

by (1) the distance of the point (λ, μ, ν) from the centre of the system of confocals is given;

by (2) $a\xi^2 + b\eta^2 + c\zeta^2 = (\lambda + \mu + \nu + a + b + c)(a + b + c) + \mu\nu + \nu\lambda + \lambda\mu - bc - \ldots;$

by (3) $\qquad \xi^2/a + \eta^2/b + \zeta^2/c = \lambda\mu\nu/abc + 1;$

whence the equations in elliptic coordinates of any conicoids similar to and co-axial with the fundamental conicoid, or of its reciprocal with respect to the centre can be found.

290. *When three confocals pass through a point each of the normals to the confocals at that point is perpendicular to the other two.**

It will be sufficient to prove that at every point in the curve of intersection of two confocals the normals are at right angles; thus, if (ξ, η, ζ) be any point common to the confocals $x^2/a + \ldots = 1$ and $x^2/(a+k) + \ldots = 1$, it follows by subtraction that

$$\xi^2/\{a\,(a + k)\} + \ldots = 0,$$

therefore the normals are at right angles.

291. *To find the lengths of the perpendiculars from the centre upon the tangent planes at a given point to the three confocals which pass through that point in terms of the elliptic coordinates.*

Proceeding as in Art. 287, we have the equation

$$(b + k)(c + k)f^2/a + (c + k)(a + k)g^2/b + (a + k)(b + k)h^2/c = 0,$$

or, since $p^{-2} = f^2/a^2 + \ldots,$

$$k^2 + Ak + abc/p^2 = 0;$$

$$\therefore\ abc/p^2 = \mu\nu = (a - a')(a - a''),$$

where a', a'' are the squares of the primary semi-axes of the confocals, similarly for p' and p'', the other perpendiculars.

292. From the similarity of the expressions for the coordinates, viz. $aa'a''/x^2 = (a - b)(a - c)$, &c., Chasles† has given the following construction:

If with the normals at the point of intersection of three confocals as axes, three new confocals be constructed, of which the squares of the semi-axes are respectively a, a', a''; b, b', b''; and c, c', c''; the coordinates of their points of intersection will be p, p', p''. Also, the perpendiculars on the three tangent planes to the second confocals will be the coordinates of the points of intersection of the first. Hence, the second confocals intersect in the common centre of the first, and the principal planes of the first are tangent planes to the second.

293. *If two parallel tangent planes be drawn to two confocals, the difference of the squares of the perpendiculars from the common centre on these planes will be constant.*‡

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* *Aper. Hist.* (30), p. 392.
† *Aper. Hist.* (36), p. 393.
‡ *Aper. Hist.* (37), p. 393; *Proc. Ir. Acad.* vol. II., 491.

Let p, p' be the perpendiculars on the tangent planes to the confocals $x^2/a+...=1$ and $x^2/(a+k)+...=1$, (l, m, n) their common direction; $p^2=l^2a+m^2b+n^2c$ and $p'^2=l^2(a+k)+m^2(b+k)+n^2(c+k)$, $\therefore p'^2-p^2=k$.

Cor. *If an ellipsoid and hyperboloid be confocal, all tangent planes to the ellipsoid drawn parallel to tangent planes to the conical asymptote of the hyperboloid will be at the same distance from the centre.*

294. *The poles of a given plane, taken with reference to each of a series of confocals, lie on a straight line perpendicular to this plane.**

Let $x^2/(a+k)+y^2(b+k)+z^2/(c+k)=1$ be the equation of any one of the confocals, $\lambda x+\mu y+\nu z=1$ that of the given plane, and let (ξ, η, ζ) be its pole with respect to this confocal, therefore the equation of the plane must be the same as

$$\xi x/(a+k)+\eta y/(b+k)+\zeta z/(c+k)=1,$$

$$\therefore\ \xi/\lambda=a+k, \text{ \&c. } \quad \text{and} \quad \xi/\lambda-a=\eta/\mu-b=\zeta/\nu-c;$$

these are the equations of the locus, which is evidently perpendicular to the given plane; and, since the point of contact of the particular confocal which touches the given plane is the pole with reference to that confocal, the locus is the normal at the point of contact to the confocal to which the given plane is a tangent.

295. *To find the length of the portion of a normal to a conicoid cut off by the polar with respect to any confocal of the point from which the normal is drawn.*

Let the normal be drawn from the point (f, g, h) of the conicoid $x^2/a+y^2/b+z^2/c=1$, and let (ξ, η, ζ) be the point in which it meets the polar of (f, g, h) with respect to the confocal $x^2/(a+k)+...=1$, by the last article it follows that $\xi/(a+k)=\lambda=f/a$,

$$\therefore\ \xi-f=kf/(a), \text{ \&c.}, \quad \therefore\ (\xi-f)^2+...=k^2(f^2/a^2+...)=k^2/p^2;$$

hence N the length required is k/p, or $Np=\pm k=$ the difference of the squares of the primary semi-axes, p being the perpendicular from the centre on the tangent plane at (f, g, h).

296. *Three confocals, A, A', A'' pass through a point P, and a central section of A is made by a plane parallel to the tangent plane to it at P. To shew that the axes of this section are parallel to the normals at P to A' and A'', also that the squares of the semi-axes of the section are equal to the difference of the squares of the primary semi-axes of A, A' and A, A'' respectively.*†

Let $x^2/a+...=1$ and $x^2/(a+k)+...=1$ represent the three confocals through $P(f, g, h)$, by giving to k the two values k', k'' derived from the quadratic $f^2/\{a(a+k)\}+...=0$. (1)

* *Aper. Hist.* (50), p. 397. † *Proc. Ir. Acad.*, vol. II. p. 499.

If (l, m, n), (l', m', n') be the directions of the normals at P to A and A', then $al/f = bm/g = cn/g$, and

$$(a+k')\,l'/f = (b+k')\,m'/g = (c+k')\,n'/h;$$

hence, by (1),

$$al^2/(a+k') + bm^2/(b+k') + cn^2/(c+k') = 0,$$

$$\text{and} \quad (a+k')\,l'/al = (b+k')\,m'/bm = (c+k')\,n'/cn;$$

therefore, by Art. 287, $-k'$ or $a-(a+k')$ is the square of the semi-axis of the section of A by the plane $lx+my+nz=0$, and its direction is (l', m', n') parallel to the normal to A', the proposition is therefore true for A', and similarly for A''.

When A is the ellipsoid, the major axis is parallel to the normal to the hyperboloid of two sheets.

COR. *When two confocals intersect, and a diameter of one is drawn parallel to a normal to the other, at any point P of the curve of intersection, this diameter is of constant length, and is one of the axes of the central section by a plane parallel to the tangent plane at P.**

This leads to the following important proposition.

297. *If p be the perpendicular on the tangent plane to an ellipsoid at any point of the curve of intersection with a confocal hyperboloid, and D be the central radius parallel to the tangent to the curve at that point, pD will be constant for all points of the curve.*

For if D' be the central radius of the ellipsoid parallel to the normal to the hyperboloid at the point considered, D, D' will be the semi-axes of the section by a plane parallel to the tangent plane; therefore pDD' is constant, Art. 276 or 277, and, by the corollary of the last article, D' is constant for all points of the curve, and therefore also pD.

298. *The normals to the confocals to an ellipsoid which pass through any external point meet the polar plane with respect to the ellipsoid in three points which, with the external point, determine a tetrahedron self conjugate with respect to the ellipsoid.*

This can be shewn by means of the theorem of Art. 294.

Let N_1, N_2, N_3 be the points of intersection of the normals at P with the polar plane of P, PN_2N_3 is a tangent plane to a confocal, and its pole with regard to the ellipsoid is in the normal PN_1, it lies also in the polar plane of P, Art. 264, therefore N_1 is the pole of PN_2N_3.

299. *To shew that the principal axes of a cone, which envelopes a given central conicoid, are normals to the three confocals which pass through the vertex.*

Let $x^2/a + y^2/b + z^2/c = 1$ be the equation of the given conicoid, (f, g, h) the vertex of the cone, then, writing u_0 for $f^2/a+g^2/b+h^2/c-1$,

* *Proc. Ir. Acad.* vol. II. p. 499.

the equation of the enveloping cone referred to parallel axes through the vertex is $u_0(x^2/a+y^2/b+z^2/c)=(fx/a+gy/b+hz/c)^2$, Art. 265.

Let $x^2/(a+k)+y^2/(b+k)+z^2/(c+k)=1$ be any confocal through (f, g, h), and $lx+my+nz=p$ the tangent plane at that point, so that $lf+mg+nh=p$, and $l(a+k)=pf$, &c.

The centre (ξ, η, ζ) of a section parallel to the tangent plane is given, as in (1) Art. 234, by

$$(u_0\xi-fv)/la=(u_0\eta-gv)/mb=(u_0\zeta-hv)/nc=-v/p,$$

writing v for $f\xi/a+g\eta/b+h\zeta/c$,

$$\therefore\ pu_0\xi=vl(a+k)-vla=vkl,$$

$$\therefore\ \xi/l=\eta/m=\zeta/n,$$

which proves the theorem, since the normals to the three confocals are shewn to be perpendicular conjugate diameters.

COR. *All cones with a fixed vertex enveloping a family of confocals are co-axial.*

300. The theorem of the last article can be shewn by that of Art. 298, for, using the same notation, $N_1N_2N_3$ is a self-conjugate triangle with respect to the section of the cone by the polar plane of P, therefore PN_1, PN_2, PN_3 are a system of conjugate diameters of the cone, and, being a rectangular system, they are the axes of the cone.

301. *To find the equation of the enveloping cone referred to the normals to the confocals through the vertex as axes.*

Since these are principal axes the equation of the cone is of the form $Ax^2+By^2+Cz^2=0$, and, if p_1, p_2, p_3 be the perpendiculars from the centre of the conicoid on the tangent planes to the three confocals, the equations of the line joining the centre and vertex of the cone, referred to the same axes, will be $x/p_1=y/p_2=z/p_3$, and since this line passes through the centre of the curve of contact, the plane of contact will be parallel to the plane conjugate to this line, whose equation is $Ap_1x+Bp_2y+Cp_3z=0$, Art. 272, but the equation of the plane of contact will be $x/\lambda+y/\mu+z/\nu=1$, if λ, μ, ν be the intercepts on the three normals,

$$\therefore\ Ap_1\lambda=Bp_2\mu=Cp_3\nu;$$

hence, if $a+k_1$, $a+k_2$, $a+k_3$ be the squares of the primary semi-axes of the three confocals, since $p_1\lambda=k_1$ &c., Art. 295, the equation required will be $x^2/k_1+y^2/k_2+z^2/k_3=0.$*

COR. *All cones having the same vertex and enveloping confocals are confocal, as well as co-axial.*

302. *To find the equation of the enveloped conicoid referred to the three normals through the vertex.*

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* Scott, *Quarterly Journal*, vol. VI., p. 258.

Since the equation of the plane of contact is $p_1x/k_1 + p_2y/k_2 + p_3z/k_3 = 1$, that of the conicoid is of the form

$$\rho(x^2/k_1 + y^2/k_2 + z^2/k_3) = (p_1x/k_1 + p_2y/k_2 + p_3z/k_3 - 1)^2;$$

if, therefore, we transform the origin to the centre $(-p_1, -p_2, -p_3)$, by writing $x - p_1$ for x, &c., the coefficients of x, y, z being equated to zero, we shall have $\rho = p_1^2/k_1 + p_2^2/k_2 + p_3^2/k_3 + 1$, the equation required will therefore be

$$(p_1^2/k_1 + p_2^2/k_2 + p_3^2/k_3 + 1)(x^2/k_1 + y^2/k_2 + z^2/k_3) = (p_1x/k_1 + \ldots - 1)^2.$$

303. *If from any point of a central conicoid a line be drawn touching two given confocals, the portion of this line intercepted between the point and the plane through the centre, parallel to the tangent plane at the point, will be constant.**

If, with P the given point as vertex, two cones be described enveloping the two confocals, the line under consideration will be one of their common sides.

Let a_1, a_2, a_3 be the squares of the primary axes of the given conicoid, and the two confocals through the point P, $a_1 + k$ that of either of the confocals to which the line through P is a tangent, whose direction referred to the normals to the confocals through P is (l, m, n).

Then $\quad l^2/k + m^2/(k + a_1 - a_2) + n^2/(k + a_1 - a_3) = 0,\quad$ Art. 301

whence $\quad k^2 + Ak + (a_1 - a_2)(a_1 - a_3)l^2 = 0; \quad (1)$

if k_1, k_2 be the two values of k, $k_1k_2 = (a_1 - a_2)(a_1 - a_3)l^2$.

But, if r be the intercepted portion, $rl = p_1$, where

$$p_1^2(a_1 - a_2)(a_1 - a_3) = a_1b_1c_1, \quad \text{Art. 291},$$

$$\therefore\ k_1k_2r^2 = a_1b_1c_1, \quad \text{or} \quad (a_1 - a)(a_1 - a')r^2 = a_1b_1c_1,$$

where a, a' are the squares of the primary axes of the two given confocals; r is therefore constant.

304. *Two conicoids can be drawn confocal with a given conicoid and touching a given straight line, and the normals at the points of contact will be at right angles.*

The first part follows from the equation (1) of the last article, which gives only two values for k.

The second is shewn by considering that if (f, g, h) and (f', g', h') be the points of contact with the given straight line which is a common generator of the cones $x^2/k_1 + \ldots = 0$ and $x^2/k_2 + \ldots = 0$, (f', g', h') is a point in the tangent plane to the first, and (f, g, h) in the tangent plane to the second;

$$\therefore\ ff'/k_1 + \ldots = 0 \quad \text{and} \quad ff'/k_2 + \ldots = 0,$$

and, by subtracting,

$$\therefore\ ff'/k_1k_2 + gg'/(k_1 + a_1 - a_2)(k_2 + a_1 - a_2) + hh'/(k_1 + a_1 - a_3)(k_2 + a_1 - a_3) = 0,$$

which represents that the normals are perpendicular.

305. *If a chord of a given central conicoid touch two other surfaces confocal with it, the length of the chord will be proportional to the square of the diameter of the first surface parallel to it.*

Let a central section be taken containing the chord PP'; draw CQ a radius of this section parallel to $P'P$, and produce it to meet the tangent at P in T; let CN bisect PP', and PM, parallel to NC, meet CQ in M, then $CM.CT = CQ^2$; hence, since CT is constant by Art. 303, $PP' = 2CM \propto CQ^2$.

306. *When two confocals are viewed by an eye in any position, their apparent boundaries cut one another at right angles wherever they appear to intersect.*†

The boundaries will appear to intersect in any line drawn from the eye so as to touch both surfaces.

* *Proc. Ir. Acad.*, vol. II., p. 498.

† *Aper. Hist.*, (88), p. 892. *Proc. Ir. Acad.*, vol. II., p. 504.

There may be four, two, or no apparent points of intersection, and, when they exist, they will be in the direction of the common generating lines of the two enveloping cones of which the eye is the common vertex, and the proposition follows from Art. 304.

307. The following method of dealing with tangents to confocal surfaces is due to Gilbert;* it enabled him to solve with great facility many of the problems in this subject, and we give some of them which have been proved above.

He shews that, if P, P' be points on two confocals $x^2/a + y^2/b + z^2/c = 1$ and $x^2/a' + y^2/b' + z^2/c' = 1$, and if ψ, ψ', θ be the angles which the normals at P, P' make with PP', $P'P$, and one another, and p, p' the central perpendiculars on the tangent planes at P and P', then, the normals being measured in the directions of p and p', $(a' - a)\cos\theta = PP'\,(p'\cos\psi - p\cos\psi')$.

Let (f, g, h) and (f', g', h') be the points P, P', $PP' = r$,

$$\therefore\ \cos\psi = (f' - f)/r \times pf/a + \ldots;$$

$$\therefore\ r\cos\psi = p\,(f'f/a + g'g/b + h'h/c - 1).$$

$$\text{Similarly}\quad r\cos\psi' = p'\,(f'f/a' + g'g/b' + h'h/c' - 1);$$

$$\therefore\ r\,(p'\cos\psi - p\cos\psi') = (a' - a)\,(pf/a \times p'f'/a' + \ldots) = (a' - a)\cos\theta.$$

COR. If PP' be a tangent at P', $\cos\psi' = 0$,

$$\therefore\ (a' - a)\cos\theta = rp'\cos\psi.$$

308. *If two confocals touch the same straight line the normals at the points of contact will be at right angles.*

$$\text{For,}\quad \cos\psi = \cos\psi' = 0,\quad \therefore\ \cos\theta = 0.$$

309. *To find the equation of the cone enveloping a given central conicoid, referred to the normals to the three confocals which pass through the vertex*

Let k_1, k_2, k_3 be the values of k for the confocals through P, ψ_1, ψ_2, ψ_3 the angles between any side PQ of the cone and the three normals at P, θ_1, θ_2, θ_3 those between the normal at Q and the three normals.

$$\text{Since}\ \cos\psi = \cos\theta_1\cos\psi_1 + \cos\theta_2\cos\psi_2 + \cos\theta_3\cos\psi_3 = 0;$$

and by Art. 307, Cor., $k_1\cos\theta_1 = rp\cos\psi_1$, &c.;

$$\therefore\ \cos^2\psi_1/k_1 + \cos^2\psi_2/k_2 + \cos^2\psi_3/k_3 = 0;$$

hence, the equation required is $x^2/k_1 + y^2/k_2 + z^2/k_3 = 0$.

310. *If any point P be taken in a fixed plane U, and on the normals to the three confocals passing through P lengths equal to the primary semi-axes be set off, the sum of the squares of the projections of these lengths on a normal to the plane U will be constant for all positions of P in that plane, viz. the square of the primary semi-axis of the confocal touching U.*†

Let a be the square of the primary semi-axis of the confocal touching U, $a + k_1$, $a + k_2$, $a + k_3$ those of the three confocals,

$$\cos\theta_1\cos\psi_1 + \cos\theta_2\cos\psi_2 + \cos\theta_3\cos\psi_3 = 0;$$

$$\therefore\ k_1\cos^2\theta_1 + k_2\cos^2\theta_2 + k_3\cos^2\theta_3 = 0\quad\text{or}\quad (a + k_1)\cos^2\theta_1 + \ldots = a.$$

Confocal Paraboloids.

311. The propositions relating to paraboloids can be deduced from those relating to central conicoids by removing the origin to a vertex, as in Art. 201, but it is well to prove them independently, although the equations of the paraboloids being

* *Nouv. Annales*, vol. VI., p. 529. † *Com. Rend.*, vol. XXII., p. 67.

unsymmetrical, the proofs of the theorems corresponding to those already given for central confocals are not so simple; the following are the principal results.

312. The equation of any confocal to $y^2/b+z^2/c=4x$ is

$$y^2/(b-k)+z^2/(c-k)=4(x-k), \quad (1)$$

since the foci coincide.

313. The three real values of k for confocals passing through (f, g, h) are given by

$$4(k-f)(k-b)(k-c)-g^2(k-c)-h^2(k-b)=0,$$

∞, b, c, $-\infty$ separating the roots. The confocals are therefore two elliptic and one hyperbolic.

314. The direction-cosines of the normals at (f, g, h) are as $-2 : g/(b-k) : h/(c-k)$, and if k_1, k_2 be values of k for two of the confocals, by subtracting the equations derived from (1),

$$g^2/(b-k_1)(b-k_2)+h^2/(c-k_1)(c-k_2)+4=0,$$

therefore the three normals are at right angles.

315. For Art. 294. Since the polar plane is

$$\eta y/(b-k)+\zeta z/(c-k)-2x-2(\xi-2k)=0,$$

and the given plane $\qquad \lambda x+\mu y+\nu z=1,$

we obtain the equations of the line

$$\eta/\mu+2b/\lambda=\zeta/\nu+2c/\lambda=\xi/\lambda+1/\lambda^2.$$

316. For Art. 295. Take k' the value of k for the confocal touching the given plane at (f, g, h), the normal has equations

$$x-f=2N\rho, \quad y-g=-gN\rho/(b-k'), \text{ \&c},$$

$$\text{where} \quad \rho^{-2}=4+g^2/(b-k')^2+h^2/(c-k')^2;$$

if N be the portion of the normal cut off by the polar with respect to any confocal (k), shew that $N\rho=k-k'$.

317. For Art. 299. The conical envelope of $y^2/b+z^2/c=4x$ is

$$u_0(y^2/b+z^2/c)=(gy/b+hz/c-2x)^2,$$

$$\text{if} \quad u_0=g^2/b+h^2/c-4f.$$

Let (l, m, n) be the direction of a normal to the confocal through (f, g, h), so that

$$l/(-2)=m(b-k)/g=n(c-k)/h=(2lf+mg+nh)/(-4k), \quad (1)$$

(ξ, η, ζ) the centre of the section made by the plane $lx+my+nz=q$, if $v\equiv g\eta/b+h\zeta/c-2\xi$, is given by

$$2v/l=(u_0\eta-gv)/bm=(u_0\zeta-hv)/cn=2u_0\xi/(2lf+mg+nh);$$

$$\therefore\ \eta/m = v/u_0(2b/l+g/m) = v/u_0 . 2k/l, \text{ by (1)},$$
$$\text{also } v = u_0\xi/2k;\ \therefore\ \xi/l = \eta/m, \text{ and similarly } = \zeta/n.$$

318. For Art. 301. Let $Ax^2+By^2+Cz^2=0$ be the equation of the cone, the line through the vertex and the centre of the curve of contact is parallel to the original axis of x, and the direction-cosines of this line are $-2\rho_1$, $-2\rho_2$, $-2\rho_3$, where

$$\rho_1^{-2} = 4 + g^2/(b-k_1)^2 + h^2/(c-k_1)^2, \text{ \&c.},$$

the plane conjugate to this line is $A\rho_1 x + B\rho_2 y + C\rho_3 z = 0$, and the intercepts of the axes made by the plane of contact will be N_1, N_2, N_3, Art. 316; $\therefore AN_1\rho_1 = BN_2\rho_2 = CN_3\rho_3$, hence the equation of the cone is $x^2/k_1 + y^2/k_2 + z^2/k_3 = 0$.

Focal Conics.

319. Among the surfaces of the system of confocals to an ellipsoid, obtained by giving all values to the parameter k in the equation

$$x^2/(a^2-k) + y^2/(b^2-k) + z^2/(c^2-k) = 1,$$

there are two which have a particular interest.

If $a>b>c$, suppose k to increase from zero, the surfaces will change from ellipsoids to hyperboloids of one sheet as k passes through c^2, and from hyperboloids of one sheet to those of two sheets, as it passes through b^2.

When k is a little less than c^2, the confocal is a very flat ellipsoid, and when a little greater, a very flat hyperboloid of one sheet, the boundary of both being the ellipse

$$x^2/(a^2-c^2) + y^2/(b^2-c^2) = 1;\ z=0;$$

when $k=c^2$, $z^2=0$, representing two planes coincident with that of xy.

In the same manner the hyperbola

$$x^2/(a^2-b^2) - z^2/(b^2-c^2) = 1,\ y=0,$$

is the boundary of two flat hyperboloids of one and two sheets, for which k is a little less or greater than b^2.

These conics are called the *focal ellipse* and *hyperbola* of any of the confocals; they pass through the foci of the two principal sections containing respectively the least and the mean axes of the ellipsoids of the system.

The focal hyperbola also passes through the umbilics of the ellipsoid, for which

$$x^2/(a^2-b^2)\,a^2 = z^2/(b^2-c^2)\,c^2 = 1/(a^2-c^2).$$

But there are other properties which make the term *focal conics* peculiarly appropriate, which will be discussed in the next chapter.

320. We may observe here that as these *focal conics* belong to the group of confocals, many of the propositions given above can be applied to them. For example, a cone on a focal conic as base corresponds to an enveloping cone, since the focal conic is in this case the curve of contact of a flat ellipsoid enveloped by the cone; and the normals to the confocals through the vertex are axes of the cone.

321. *To find the confocal hyperboloids which pass through a point in the principal section of an ellipsoid which contains the greatest and least axes.*

Let the equation of the ellipsoid be $x^2/a^2 + y^2/(a^2 - \beta^2) + z^2 (a^2 - \gamma^2) = 1$, and $(f, 0, h)$ the point in the principal section, so that $f^2/a^2 + h^2/(a^2 - \gamma^2) = 1$.

If a' be the primary semi-axis of a confocal hyperboloid,

$$f^2/a^2a'^2 + 0/(a^2 - \beta^2)(a'^2 - \beta^2) + h^2/(a^2 - \gamma^2)(a'^2 - \gamma^2) = 0;$$

$\therefore$ $a' = \beta$ or $f\gamma/a$, the first solution gives the focal hyperbola, which must be considered as a flat hyperboloid of one or two sheets according to the position of the point; the other solution gives the hyperboloid

$$x^2/f^2\gamma^2 + y^2/(f^2\gamma^2 - a^2\beta^2) - z^2/(a^2 - f^2)\gamma^2 = a^{-2},$$

which is of one or two sheets, as $f\gamma >$ or $< a\beta$, *i.e.* as the point is on one side or the other of the focal hyperbola.

322. *To find the locus of the vertices of all circular cones which envelope a given conicoid.*

Since the positions of the principal axes of such cones, which are perpendicular to their axes of revolution, are indeterminate, we must consider three confocals through the vertex of some enveloping cone for which the directions of the normals to two of them will be indeterminate. It is evident that if we draw normals to a conicoid of which one of the axes is infinitely small, these normals will be parallel to that axis, unless the points at which they are drawn are indefinitely near the edge, and in passing round this edge from one side to the other the normals will assume every direction in a plane perpendicular to the tangent to the bounding focal conic, and this tangent being the normal to the third confocal will be the axis of a right cone. Hence, the vertices of right cones must lie in one of the focal conics.

This locus and that of the next article may be found by making the equation of the conical envelope coincide with that of the circular cone found in Art. 60.

323. *The locus of the vertices of right cones on a given elliptic base is a hyperbola in a plane perpendicular to its plane, and vice versâ.*

For any ellipse $x^2/a^2 + y^2/b^2 = 1$ may be looked upon as the focal ellipse of an ellipsoid of which the focal hyperbola is $x^2/a^2 - z^2/\gamma^2 = 1$, if $a^2 - a^2 = b^2 = \gamma^2$, and the vertex must therefore lie on the focal hyperbola; hence the equations of the locus of the vertices are $x^2/(a^2 - b^2) - z^2/b^2 = 1$, and $y = 0$.

324. *To find the vertical angle of the circular cone enveloping an ellipsoid, the vertex being a given point on the focal hyperbola.*

The squares of the primary semi-axes of the confocals through a point $(f, 0, h)$ of the focal hyperbola, which is the boundary of two flat hyperboloids,

are $a^2-b^2+\lambda$, a^2-b^2, a^2-b^2, where

$$f^2/(a^2-b^2)-h^2/(b^2-c^2)=1 \quad \text{and} \quad f^2/(a^2-b^2+\lambda)+h^2/(c^2-b^2+\lambda)=1,$$

whence $\lambda=f^2(b^2-c^2)/(a^2-b^2)+h^2(a^2-b^2)/(b^2-c^2)=(a^2-b^2)(b^2-c^2)/p_1^2$,

p_1 being the perpendicular from the centre on the tangent to the hyperbola.

But the equation of the enveloping cone referred to the three normals to the confocals is $x^2/(\lambda-b^2)=(y^2+z^2)/b^2$; hence, if 2α be the vertical angle of the cone, $b^2=(\lambda-b^2)\tan^2\alpha$ and $\lambda=b^2\operatorname{cosec}^2\alpha$,

$$\therefore \ p_1 b=\sqrt{(a^2-b^2)}\sqrt{(b^2-c^2)}\sin\alpha.$$

Bifocal Chords.

325. Def. A *bifocal chord* of a conicoid is a chord which intersects two focal conics of the conicoid.

326. *If P be one extremity of a bifocal chord of a conicoid, the portion of the chord intercepted between P and a plane through the centre, parallel to the tangent plane at P, will be equal to the primary semi-axis of the conicoid.*

The focal conics being limits of confocals of the conicoid, this is a particular case of the theorem of Art. 303, in which $k_1=b_1$, $k_2=c_1$.

327. The direction-cosines of the bifocal chords drawn through any point P of an ellipsoid and its focal ellipse and hyperbola, referred to the normals to the three confocals through P, are $\pm p_1/a_1$, $\pm p_2/a_2$, and $\pm p_3/a_3$, where a_1, a_2, a_3 are the primary semi-axes, and p_1, p_2, p_3 the perpendiculars from the centre on the tangent planes at P.

328. *The length of the bifocal chord of a conicoid is a third proportional to the primary axis and a diameter parallel to the chord.*

This is a particular case of Art. 305.

329. *To shew that the four bifocal chords through any point P of an ellipsoid lie in two planes passing through the normal at P, and intersecting the primary axis of the ellipsoid in the feet of the normals at the umbilics.*

Since, by Art. 327, the equations of the four bifocal chords, referred to the three normals to the confocals through P, are $xa_1/p_1=\pm ya_2/p_2=\pm za_3/p_3$, they lie in pairs in the two planes $ya_2/p_2\pm za_3/p_3=0$, intersecting in the normal to the ellipsoid.

The coordinates of the feet of the normals at the umbilics, referred to axes through P, (f, g, h), parallel to the axes of the ellipsoid, are $\pm\beta\gamma/a_1-f$, $-g$, $-h$; therefore, at the feet of the normals at the umbilics,

$$y=\left(\pm\frac{\beta\gamma}{a_1}-f\right)\frac{p_2 f}{a_2^2}-g\frac{p_2 g}{b_2^2}-h\frac{p_2 h}{c_2^2}=\left(\pm\frac{\beta\gamma f}{a_1 a_2^2}-1\right)p_2;$$

$$\therefore \ y/p_2=\pm a_3/a_2-1, \quad \text{Art. 286.}$$

Similarly, $z/p_3=\pm a_2/a_3-1$, and these points lie one in each of the two planes given above.

330. *If a tangent plane be drawn perpendicular to a bifocal chord, the distance from the centre of the ellipsoid to the point where the chord meets this plane will be equal to the primary semi-axis.*

Let $x^2/a^2+y^2/b^2+z^2/c^2=1$ be the equation of the ellipsoid, (λ, μ, ν) the direction of the bifocal chord through (f, g, h). Then, by the intersection of the two cones standing on the focal conics,

$$\frac{(f\nu-h\lambda)^2}{a^2-c^2}+\frac{(h\mu-g\nu)^2}{b^2-c^2}=\nu^2, \quad \text{Art. 175, Cor. 1.}$$

$$\text{and} \quad \frac{(g\lambda-f\mu)^2}{a^2-b^2}-\frac{(h\mu-g\nu)^2}{b^2-c^2}=\mu^2:$$

and multiplying by a^2-c^2 and a^2-b^2, and adding,

$$(f\nu-h\lambda)^2+(g\lambda-f\mu)^2+(h\mu-g\nu)^2=a^2-(\lambda^2a^2+\mu^2b^2+\nu^2c^2);$$

therefore, if CQ be perpendicular from the centre on the chord, R the point where the chord meets the tangent plane, $CQ^2=a^2-QR^2$, $\therefore$ $CR=a$.

Corresponding Points.

331. If a, b, c and a', b', c' be the semi-axes of two ellipsoids, any two points P and P', whose coordinates are x, y, z and x', y', z' referred to the axes of the ellipsoids, are said to be *corresponding points*, if the following relations hold, $x/a=x'/a'$, $y/b=y'/b'$, and $z/c=z'/c'$.

It is plain that if P be on the first ellipsoid the corresponding point P' will be on the second.

Ivory first made use of points so connected in order to establish a relation between the attractions of an ellipsoid on an external and on an internal point, proving the following proposition:

332. *If P, Q be two points on an ellipsoid, and P', Q' the corresponding points on a confocal ellipsoid, $PQ'=P'Q$.*

Let (x, y, z), (ξ, η, ζ) be the points P, Q on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, and let (x', y', z'), (ξ', η', ζ') be corresponding points P', Q' on the confocal $x^2/a'^2+y^2/b'^2+z^2/c'^2=1$.

Since $x/a=x'/a'$ and $\xi/a=\xi'/a'$, we have $(x-\xi')^2-(\xi-x')^2$

$$=(x-\xi a'/a)^2-(\xi-xa'/a)^2=(a^2-a'^2)(x^2/a^2-\xi^2/a^2),$$

and similarly for the other coordinates;

$$\therefore\ (x-\xi')^2+(y-\eta')^2+(z-\zeta')^2-\{(\xi-x')^2+(\eta-y')^2+(\zeta-z')^2\}=0,$$

$$\text{or } PQ'=P'Q.$$

333. Since $x^2-x'^2=(a^2-a'^2)x^2/a^2$, &c., if O be the centre of the ellipsoid, then $OP^2-OP'^2=a^2-a'^2$.

334. *If a concentric and co-axial ellipsoid be drawn through the vertex of a cone enveloping a given ellipsoid, the tangent plane at the point corresponding to the vertex will meet the ellipsoid through the vertex in an ellipse, every point of which will correspond to a point in the plane of contact; and, if the ellipsoids be confocal, the lengths of the tangents from the vertex will be equal to the corresponding radii of that ellipse.*

Let (f, g, h) be the vertex of the cone enveloping the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, (1), and let $x^2/a'^2+y^2/b'^2+z^2/c'^2=1$, (2), be an ellipsoid on which the vertex lies.

The plane of contact of the cone is $fx/a^2+gy/b^2+hz/c^2=1$, in which let (ξ, η, ζ) be any point, then, if (ξ', η', ζ') be the corresponding point on (2), and (f', g', h') on (1) correspond to the vertex, we have $f\xi=f'\xi'$;

$$\therefore\ f'\xi'/a^2+g'\eta'/b^2+h'\zeta'/c^2=1,$$

or (ξ', η', ζ') is on the plane which touches (1) at (f', g', h').

Also, if the ellipsoids be confocal, the latter part of the proposition is obvious by Ivory's theorem.

335. *If three points on an ellipsoid be the extremities of three conjugate diameters, the three corresponding points on any other ellipsoid will be also at the extremities of conjugate diameters.*

For the corresponding points on a concentric sphere are the same for both ellipsoids, and these are obviously at the extremities of three perpendicular radii.

336. *Confocal ellipsoids are cut by a fixed confocal hyperboloid; to shew that if any point be taken on the curve of intersection of one of the ellipsoids, the corresponding point on any other will lie on its curve of intersection.*

If (x, y, z) be a point on the intersection of $x^2/a^2+y^2/b^2+z^2/c^2=1$ with the hyperboloid $x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2=1$ confocal with it,

$$(a^2-\alpha^2)(x^2/a^2\alpha^2+y^2/b^2\beta^2+z^2/c^2\gamma^2)=0,$$

and if (x', y', z') be the point corresponding to (x, y, z) on the ellipsoid $x^2/a'^2+y^2/b'^2+z^2/c'^2=1$, $x'^2/a'^2\alpha^2+y'^2/b'^2\beta^2+z'^2/c'^2\gamma^2=0$, and multiplying by $a'^2-\alpha^2$, we obtain

$$x'^2/\alpha^2+y'^2/\beta^2+z'^2/\gamma^2=x'^2/a'^2+y'^2/b'^2+z'^2/c'^2=1,$$

therefore the point (x', y', z') lies also on the hyperboloid.

COR. *The point on any ellipsoid which corresponds to an umbilic on a confocal ellipsoid will be itself an umbilic.* If the hyperboloid which cuts the confocal ellipsoids be either of the flat hyperboloids, whose common edge is the confocal hyperbola, all points on this edge will be corresponding points.

337. One of the series of ellipsoids in Art. 336 is the flat surface bounded by the focal ellipse, and the corresponding curve of intersection is the principal section of the hyperboloid, which is an ellipse or hyperbola as the hyperboloid is of one or two sheets, being confocal with the principal sections of the ellipsoids.

If hyperboloids of one and two sheets be confocal with the series of ellipsoids, a, a' their primary semi-axes will be elliptic coordinates of the points on any of the ellipsoids in which the curves of intersection with the hyperboloids intersect; and if r, r' be the distances of the corresponding point on the flat ellipsoid from the nearer and farther foci of the principal section of the ellipsoid, $r'+r=2a$ and $r'-r=2a'$, whence the plane curve corresponding to any curve on the ellipsoid, given in elliptic coordinates, can be found, or *vice versâ*. As an example, take the following:

A curve is drawn on an ellipsoid such that, if a central section be taken parallel to the tangent plane at any point, the distance of the foci of the section will be constant, to find the corresponding curve on the plane of the focal ellipse.

If a, a' be the elliptic coordinates of a point on the curve, the squares on the semi-axes of the section corresponding to this point will be a^2-a^2 and $a^2-a'^2$, hence $a^2-a'^2$ is constant and the curve required will be $rr'=$ constant; if the given curve pass through the extremity of the least axis, the corresponding curve will be a lemniscate.

XXIV.

(1) Prove that the locus of the points of intersection of tangent planes to three confocals, which are perpendicular to each other, is a sphere.

(2) If normals be drawn from a fixed point to each of a series of confocals, shew that they will form a cone of the second degree.

(3) If a, a', a'', a''', be the transverse semi-axes of an ellipsoid, and the three confocals which can be drawn through a given point, and if $a'^2 + a''^2 + a'''^2 = 3a^2$, then three tangent planes mutually at right angles can be drawn from the given point to the given ellipsoid.

(4) Through a straight line in one of the principal planes tangent planes are drawn to a series of confocal ellipsoids; prove that the points of contact lie on a plane. If a plane be drawn cutting the three principal planes, and through each of the lines of section tangent planes be drawn to the series of conicoids, prove that the three planes which are the loci of the points of contact will intersect in a straight line perpendicular to the cutting plane.

(5) Prove that the polar of the foot of a normal to an ellipsoid with respect to the focal ellipse is the polar of the foot of the ordinate with respect to the principal section of the ellipsoid; also that the line joining the two feet is a normal to an ellipse similar to the principal section.

(6) P, Q are two points on a generator of a hyperboloid, P', Q' corresponding points on a confocal hyperboloid; shew that $P'Q'$ is a generator of the latter, and that $P'Q = PQ'$

(7) Shew that the locus of the point corresponding to a given point of an ellipsoid, on a system of confocal ellipsoids, is the intersection of two hyperbolic cylinders.

(8) The curve on a sphere corresponding to the curve of intersection of an ellipsoid and a confocal hyperboloid lies on the asymptotic cone of the hyperboloid.

XXV.

(1) If $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ be the equation of an ellipsoid, and (f, g, h) be any point, a', b', c', a'', b'', c'', a''', b''', c''' the semi-axes of the confocals through that point, then

$$\frac{f^2}{a^2a'^2a''^2a'''^2} + \frac{g^2}{b^2b'^2b''^2b'''^2} + \frac{h^2}{c^2c'^2c''^2c'''^2} = \frac{1}{a^2b^2c^2}.$$

(2) The points on a series of confocals, at which the normals are parallel, lie on an equilateral hyperbola of which one asymptote is parallel to the normals.

(3) If two cylinders with parallel generators circumscribe confocal conicoids, the sections of the cylinders by any plane perpendicular to the axes of both will be confocal.

(4) When the two confocal hyperboloids through a point degenerate into flat surfaces bounded by the focal hyperbola, explain the perpendicularity of the three normals at that point.

(5) Find the three confocals of an ellipsoid through a point in one of the focal conics. Shew that the ellipsoid which passes through a point $(f, 0, h)$ on the focal hyperbola of the ellipsoid $x^2/a + y^2/b + z^2/c = 1$ is

$$x^2/(b-c)f^2 + y^2(a-c)/\{(b-c)^2f^2 + (a-b)^2h^2\} + z^2/(a-b)h^2 = (a-c)/(a-b)(b-c).$$

(6) The rectangle contained by the side of a cone of revolution enveloping an ellipsoid, intercepted between the vertex and point of contact, and the perpendicular from the centre upon the tangent plane at that point, is constant.

(7) PQ is a tangent at Q to a conicoid; λ, μ, ν are the primary semi-axes of the confocals through P; l, m, n the direction-cosines of PQ referred to the normals to the confocals at P; l', m', n' those of the perpendicular p from the centre on the tangent plane at Q; prove that, if $PQ = r$,

$$\lambda^2 ll' + \mu^2 mm' + \nu^2 nn' + pr = 0.$$

(8) Prove that the points on the plane of the focal ellipse of an ellipsoid, which correspond to those of a circular section, lie also in a circle, whose area is to that of the section as $b^2 - c^2 : b^2$.

(9) A cone is described whose base is a given conic, and one of whose axes passes through a fixed point in the plane of the conic, prove that the locus of the vertex is a circle.

XXVI.

(1) Find the eccentricities of the focal conics of $yz + \sqrt{3}\,(zx - xy) = 1$.

(2) If ϖ be the perpendicular from the origin on the tangent plane at (ξ, η, ζ) to the conicoid $\xi^2/(\alpha+\lambda) + \eta^2/(\beta+\lambda) + \zeta^2/(\gamma+\lambda) = 1$, prove that

$$\frac{d^2\lambda}{d\xi^2} + \frac{d^2\lambda}{d\eta^2} + \frac{d^2\lambda}{d\zeta^2} = 2\varpi^2\left\{\frac{1}{\alpha+\lambda} + \frac{1}{\beta+\lambda} + \frac{1}{\gamma+\lambda}\right\}.$$

(3) Prove that the equation in elliptic coordinates of any circular section of an ellipsoid is $a'^2 + a''^2 - 2a'a''d/d_0 = (a^2 - c^2)\,r^2/b^2$, where r is the radius of the section, and d, d_0 are the distances of its centre and of the umbilics from the centre of the ellipsoid.

(4) If a series of confocal paraboloids be touched by parallel planes, the points of contact will all lie in a bifocal line.

(5) The locus of a point, where a tangent plane to a paraboloid is met by a bifocal line to which it is perpendicular, is a plane touching the paraboloid at the vertex.

(6) If four curves on an ellipsoid, which are the intersections with four confocal hyperboloids, form a small rectangle of sides ds, ds', shew that there will be a corresponding rectangle on a sphere, whose sides $d\sigma$, $d\sigma'$ will be connected with ds, ds' by the relations $rds = \lambda' d\sigma$ and $rds' = \lambda d\sigma'$, λ^2 and λ'^2 being the differences of the squares of the semi-axes of the ellipsoid and the hyperboloids which intersect in ds and ds' respectively.

(7) If λ be the length of a bifocal chord of the paraboloid $y^2/b + z^2/c = x$, which makes, with the axes of y and z, angles whose cosines are m and n,

$$\lambda^{-1} = m^2/b + n^2/c.$$

(8) The foci of a series of parallel sections of an ellipsoid, perpendicular to the plane of a focal conic, lie on an ellipse which touches the trace on that plane and the focal conic.

(9) Any point in the plane of the focal ellipse of an ellipsoid will be the focus of two plane sections perpendicular to that plane, which will be real only when the point lies within the trace on that plane and without the focal conic.

CHAPTER XV.

MODULAR AND UMBILICAL GENERATION OF CONICOIDS. PROPERTIES OF CONES AND SPHERO-CONICS.

338. THE modular and umbilical methods of generating conicoids, invented by MacCullagh and Salmon respectively, may be stated as follows:

For the *modular* method, "The locus of a point whose distance from a fixed point is in a constant ratio to its distance from a fixed straight line, *measured parallel to a fixed plane*, is a surface of the second degree."

The fixed point is called a *modular focus*, the fixed line a *directrix*, the constant ratio the *modulus*, and the plane the *directing plane*.

339. Since this locus contains *ten* disposable constants, viz. *three* dependent on the position of the fixed point, *four* on that of the fixed straight line, and *two* on the direction of the fixed plane, and *one* more, namely the constant ratio, the locus may, *in general*, be made to coincide with any surface which can be represented by an equation of the second degree in an infinite number of ways, since there will be only nine equations connecting the ten disposable constants.

If all but the three coordinates of the focus be eliminated, there will result two final equations determining a curve locus of such points; such curves are called *focal conics*, being the same as the limits of the confocals discussed in the last chapter.

Again, if all but the four constants which determine the position of the directrix be eliminated, there will be three final equations which, with the equations of the straight line, will determine a ruled surface, called a *dirigent cylinder*, the trace of which on the plane of the focal conic is called a *dirigent conic*.

340. For the *umbilical* method, "The locus of a point, the square of whose distance from a fixed point bears a constant ratio to the rectangle under its perpendicular distances from two directing planes, is a surface of the second degree."

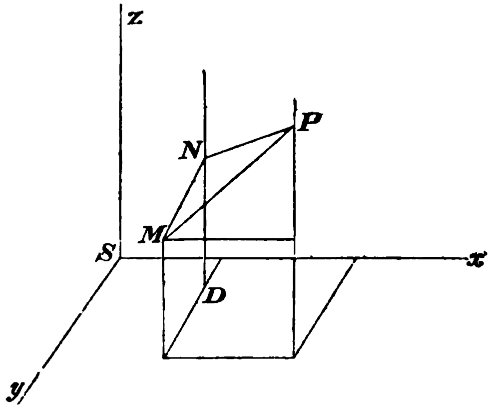
This locus contains ten disposable constants; *three* dependent on the position of the fixed point, *three* on the position of each of the directing planes, and *one* more, namely the constant ratio.

The fixed point, therefore, will not generally be unique, but may be any point of a curve locus.

The fixed point is called an *umbilical focus*, the intersection of the planes a *directrix*, the constant ratio the *umbilical modulus*, and the locus of the focus the *umbilical focal conic*, the trace of the dirigent cylinder on the principal plane being the *umbilical dirigent conic.*

341. *To find the locus of a point whose distance from a focus is in a constant ratio to its distance from a directrix, measured parallel to a given directing plane.*

Let S the focus be taken for origin, Sz, Sy parallel to the directrix DN and the directing plane respectively, and let α, β be the coordinates of D in xy, e the modulus, and ω the angle of inclination of the directing plane to the plane of xy.



Let PN be drawn from the point (x, y, z) to the directrix parallel to the directing plane, NM parallel to Sy, and PM perpendicular to NM; then PMN will be parallel to the directing plane. Hence we shall have $MN = y - \beta$, and $PM = (x - \alpha) \sec\omega$; and, P being a point in the locus, $SP = e.PN$;

$$\therefore\ x^2 + y^2 + z^2 = e^2 \{(x-\alpha)^2 \sec^2\omega + (y-\beta)^2\};$$

this is the equation of the locus required, which is a surface of the second order.

Since $z = x \tan\omega + h$ is the equation of any plane parallel to the directing plane, we have, at the points of intersection with the surface, $x^2 \sec^2\omega + y^2 = x^2 + y^2 + (z-h)^2$, which, combined with the equation of the locus, shews that the curve of intersection lies on a sphere, except when $e = 1$, in which case it lies on another plane; hence, all sections parallel to the directing plane are circles, or, when $e = 1$, straight lines.

342. That the section by a plane through S parallel to the directing plane is a circle is obvious geometrically, for, if this plane cut the directrix in H, the section is the locus of a point whose distances from S and H are in a constant ratio, and is therefore a circle, unless $e=1$, in which case it is a straight line, and the surface is a hyperbolic paraboloid.

343. *To find the locus of a point, the square of whose distance from a focus is in a constant ratio to the rectangle under its distances from two fixed directing planes.*

Let the focus S be taken for the origin, the planes bisecting the angles between the directing planes being parallel to the planes of xy, yz.

Let also ω be the inclination of the directing planes to the plane of xy, α, γ the coordinates in the plane of zx of any point in the directrix, and e the constant ratio.

From any point P, let PQ, PR be drawn perpendicular to the directing planes; $\therefore SP^2 = ePQ.PR$; therefore since the equations of the directing planes are $(x-\alpha)\sin\omega \pm (z-\gamma)\cos\omega = 0$; if x, y, z be the coordinates of P,

$$x^2 + y^2 + z^2 = e\{(x-\alpha)^2 \sin^2\omega - (z-\gamma)^2 \cos^2\omega\}$$

will be the equation of the locus, which is of the second degree.

If the surface be cut by a plane, parallel to either directing plane, whose equation is $(x-\alpha)\sin\omega \pm (z-\gamma)\cos\omega = p$, the curve of intersection will obviously lie on a sphere, and will therefore be a circle.

344. We have seen in both modes of generation, and it is also evident from the consideration of the number of constants, that the equation of a conicoid can be put into the form $S = UV$, where $U=0$ and $V=0$ are the equations of two real or imaginary planes, and $S=0$ is the equation of a point sphere, or the imaginary cone having its vertex at a point which we have called a focus, and passing through the circle at infinity which is common to all spheres, Art. 232.

The conicoid and cone intersect in two plane curves crossing one another in two points P, Q which lie in the line of intersection of the planes U, V, called the directrix; a plane containing the tangent lines to the two curves at P will be a tangent plane to both conicoid and cone at P, and will therefore contain a tangent to the circle at infinity, which lies on the cone.

The generating line SP of the cone, whose vertex is the focus S, will be the intersection of two consecutive tangent planes to both conicoid and cone, each of which tangent planes contains a tangent line to the circle at infinity, and since the same argument holds for Q, SQ will be another such generating line.

If, therefore, a series of planes be drawn which touch both the conicoid and circle at infinity, these planes will envelope a torse, and a focus will be a point on the torse in which two of its generating lines, which are not consecutive, intersect.

The locus of the foci will therefore lie on a double curve on the torse, and this curve will be the same for all conicoids enveloped by the same torse, touching also the circle at infinity.

Chasles suggested the following definition of confocals.

DEF. Conicoids are confocal when they are capable of being enveloped by the same developable surface described so as to touch the imaginary circle at infinity.

345. *To find the focal and dirigent conics in the case of central conicoids.*

Let (ξ, η, ζ) be a focus, and $(\xi', \eta', 0)$ be the foot of the corresponding directrix supposed parallel to the axis of z.

The equation of the conicoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, whether generated by the modular or umbilical method, must coincide with the equation

$$(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 = \lambda(x-\xi')^2 + \mu(y-\eta')^2,$$

λ, μ being of the same or opposite signs. Comparing these equations, we have $\xi = \lambda\xi'$, $\eta = \mu\eta'$, $\zeta = 0$, and

$$(1-\lambda)a^2 = (1-\mu)b^2 = c^2 = \lambda\xi'^2 + \mu\eta'^2 - \xi^2 - \eta^2;$$

$$\therefore\ \lambda = 1 - c^2/a^2,\quad \mu = 1 - c^2/b^2,$$

$$\text{and } \lambda\xi'^2 - \xi^2 = \{a^2/(a^2-c^2) - 1\}\,\xi^2 = c^2/(a^2-c^2)\,\xi^2;$$

$$\therefore\ \xi^2/(a^2-c^2) + \eta^2/(b^2-c^2) = 1.$$

The focal conic is therefore confocal with the conicoid, and lies in the principal plane perpendicular to the directrix. Again, since

$$\xi^2 = (a^2-c^2)\,\xi'^2/a^2, \text{ and } \eta^2 = (b^2-c^2)\,\eta'^2/b^2,$$

the equation of the dirigent cylinder is

$$(a^2-c^2)\,\xi^2/a^4 + (b^2-c^2)\,\eta^2/b^4 = 1.$$

346. *The focal and dirigent conics are reciprocals of each other with respect to the principal section in the plane of which they lie, and the line joining the foot of any directrix with the corresponding focus is a normal to the focal conic.*

Since $x^2/(a^2-c^2) + y^2/(b^2-c^2) = 1$ is the equation of a focal conic, the equation of the tangent at (ξ, η) is

$$x\xi/(a^2-c^2) + y\eta/(b^2-c^2) = 1, \text{ or } x\xi'/a^2 + y\eta'/b^2 = 1;$$

whence it is the polar of (ξ', η'), the foot of the corresponding directrix, with respect to the section in xy.

Also, since $a^2(\xi'-\xi) = c^2\xi'$, and $b^2(\eta'-\eta) = c^2\eta'$, the equation of the tangent may be written $x(\xi'-\xi) + y(\eta'-\eta) = c^2$; it is therefore perpendicular to the line joining (ξ, η) and (ξ', η'), whence the second part of the proposition.

347. *If a section of a conicoid be made by a plane perpendicular to that of a focal conic, so that it contains a directrix, to shew that the distance of any point of the section from the directrix will have a constant ratio to the distance from the corresponding focus.*

For if $(\xi, \eta, 0)$ be the focus corresponding to the directrix (ξ', η'), the equation of the conicoid may be put into the form

$$(x-\xi)^2+(y-\eta)^2+z^2=\lambda(x-\xi')^2+\mu(y-\eta')^2,$$

and, if the equation of the plane be $(x-\xi')/l=(y-\eta')/m=r$, then for any point P of the section $(x-\xi)^2+(y-\eta)^2+z^2=(\lambda l^2+\mu m^2)r^2$; therefore $SP \propto PQ$, if PQ be the perpendicular on the directrix.

348. COR. If the plane containing a directrix be perpendicular to the focal conic, the corresponding focus S will be a point in the plane, Art. 346, and will therefore be a focus of the section; hence, *Every point of a focal conic of a conicoid is a focus of the section made by a plane perpendicular to the focal conic at that point.*

349. *To find where a conicoid is intersected by its focal conics.*

If the directrix be parallel to Oz, the equations of the focal conic of the conicoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, will be $x^2/(a^2-c^2)+y^2/(b^2-c^2)=1$, $z=0$, and it will intersect the conicoid if $x^2/(a^2-c^2)\,a^2+y^2/(b^2-c^2)\,b^2=0$ give $x^2:y^2$ positive.

Hence, if the focal conic and the corresponding principal section be both ellipses or both hyperbolas they do not intersect; but, if they be not of the same kind, they will intersect in the umbilics of the conicoid; whence the name umbilical focal conic.

Now, referring to the equations of Art. 341 and 343, we see that in the modular method of generation the focus cannot lie on the conicoid, but may do so in the umbilical method, thus the umbilical focal conics correspond to the umbilical method of generation, and the other focal conics to the modular method.

350. *To find the focal and dirigent conics for non-central surfaces.*

For the paraboloids, comparing the equation

$$(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2=\lambda(x-\xi')^2+\mu(y-\eta')^2$$

with $y^2/b+z^2/c=2x$, we have $\lambda=1$, $b(1-\mu)=c=\xi-\xi'$,

$$\eta=\mu\eta',\ \zeta=0,\ \text{and}\ \xi^2+\eta^2=\xi'^2+\mu\eta'^2;$$

$$\therefore\ \mu=1-c/b,\ \text{and}\ c(\xi+\xi')=\eta^2\{b/(b-c)-1\};$$

$$\therefore\ \eta^2=2(b-c)(\xi-\tfrac{1}{2}c).$$

The focal conic is therefore a parabola, which has its vertex at the focus of the parabolic section parallel to the directrix, and is confocal with the section in its plane, since the abscissa of its focus is $\frac{1}{2}c+\frac{1}{2}(b-c)=\frac{1}{2}b$.

Also, the equation of the dirigent conic is $(b-c)\eta'^2=2b^2(\xi'+\frac{1}{2}c)$, which is the reciprocal of the focal parabola with respect to the section $y^2=2bx$.

It will be found that the focal conic of an elliptic or hyperbolic cylinder is the two straight lines containing the foci of the principal sections; and that that of a parabolic cylinder is two straight lines,

one of which is at an infinite distance and the other contains the foci of the principal sections.

351. In order to apply the modular method of generation to the investigation of properties of conicoids, the modulus and directing plane must be real, as well as the focal and dirigent conics, and, referring to Arts. 331 and 335, we obtain the following conditions:

i. For the ellipsoid $x^2/a + y^2/b + z^2/c = 1$, $a > b > c$,

$$(1 - e^2 \sec^2\omega)\, a^2 = (1 - e^2)\, b^2 = c^2,$$

the only focal conic which is applicable being $x^2/(a^2 - c^2) + y^2/(b^2 - c^2) = 1$.

For an oblate spheroid $a = b$ and $\omega = 0$. The prolate spheroid, for which $b = c$, cannot be generated by the modular method.

ii. For the hyperboloid of one sheet $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$, $b > a$, the directing plane is parallel to Oy, and both the focal conics $x^2/(a^2 + c^2) + y^2/(b^2 + c^2) = 1$ and $y^2/(b^2 - a^2) - z^2/(c^2 + a^2) = 1$ are applicable, the corresponding moduli e, e' being given by $(e^2 - 1)\, b^2 = c^2$ and $(1 - e'^2)\, b^2 = a^2$, where $\cos^2\omega/e^2 + \sin^2\omega/e'^2 = 1$, so that the hyperboloid of one sheet can be generated by means of foci lying in a focal ellipse or a focal hyperbola, the greater modulus corresponding to the ellipse.

iii. For the hyperboloid of two sheets $x^2/a^2 - y^2/b^2 - z^2/c^2 = 1$, $b > c$, the directing plane is parallel to Oy, and the focal conic is $x^2/(a^2 + c^2) - y^2/(b^2 - c^2) = 1$, the modulus being given by $(1 - e^2)\, b^2 = c^2$.

The hyperboloid of revolution of two sheets, where $b = c$, cannot be constructed by the modular method.

iv. For the elliptic paraboloid $y^2/b + z^2/c = 2x$, $b > c$,

$$e = \cos\omega, \quad b\,(1 - e^2) = c = b\sin^2\omega,$$

the focal conic is $y^2 = 2\,(b - c)\,(x - \tfrac{1}{2}c)$.

v. For the hyperbolic paraboloid $y^2/b - z^2/c = 2x$,

$$e = 1, \quad b\,(1 - \sec^2\omega) = -c, \quad \text{or} \quad b\tan^2\omega = c;$$

the focal parabolas are $y^2 = (b + c)\,(2x + c)$ and $z^2 = -(b + c)\,(2x - b)$, each of which satisfies the modular method.

352. *To trace the changes of the surfaces and real focal conics corresponding to changes of the modulus from* 0 *to* ∞.

If we transfer the origin used in the equation of Art. 331 to the centre, and have regard to the sign of the constant term, we shall obtain the following results:

$e < \cos\omega$,	Surface an ellipsoid, including an oblate spheroid.
	Focal conic an ellipse.
$e = \cos\omega$,	Surface an elliptic paraboloid.
	Focal conic a parabola.
$e > \cos\omega$ and < 1,	Surface at first a hyperboloid of two sheets, passing through a cone, to a hyperboloid of one sheet, conjugate axis perpendicular to the directrix.
	Focal conic at first a hyperbola, transverse axis perpendicular to the directive axis, passing through the asymptotic limit, viz. two straight lines, to a hyperbola, transverse axis parallel to the directive axis.
$e = 1$,	Surface a hyperbolic paraboloid.
	Focal conics two parabolas.
$e > 1$,	Surface a hyperboloid of one sheet, conjugate axis parallel to the directrix, including a hyperboloid of revolution.
	Focal conic an ellipse, transverse axis parallel to the directive axis.

The hyperboloid of revolution of two sheets is lost between $e = 1$ and $e = \cos\omega$.

353. The directrix in the umbilical method of generation being parallel to the intersections of the two series of circular sections, the plane of the focal conic is known, and by Art. 343 the umbilical modulus can be found in the same manner as in the modular method, and it will be seen that all surfaces can be generated, except the hyperboloid of one sheet, the hyperbolic paraboloid, and the oblate spheroid.

Properties of Conicoids deduced by the modular and umbilical methods.

354. *Every plane section of a conicoid, which is normal to a focal conic at any point, has that point for a focus.*

For, if S be the point through which the plane section passes, the corresponding directrix also will be in the plane; let PQ be perpendicular to this directrix from a point P on the section.

If the focal conic be modular, let PR parallel to a directing plane meet the directrix in R, then the ratios $SP : PR$ and $PR : PQ$ will be constant for every point in the section; and, therefore, $SP : PQ$ will be a constant ratio.

If the focal conic be umbilical, let PM, PN be perpendiculars on the planes through the directrix parallel to cyclic sections; then $SP^2 \propto PM . PN$, and for all points of the section $PM : PQ$ and $PN : PQ$ will be constant ratios, therefore $SP \propto PQ$.

355. *If a section of a conicoid be made by a plane perpendicular to the plane of a focal conic, it will contain two directrices; to shew that the sum or difference of the distances of any point of the section from the two corresponding foci will be constant.*

Let QD, $Q'D'$ be the two directrices, and S, S' the corresponding foci, which in this case will not be necessarily in the plane of the section; draw through any point P of the section QPQ' perpendicular to the directrices.

If the focal conic be modular, draw RPR' parallel to a directing plane meeting the directrices in R and R'.

Since the modulus is the same for both foci,

$$SP : PR :: S'P : PR';$$

$$\therefore\ SP : S'P :: PR : PR' :: PQ : PQ',$$

$$\text{and } SP \pm S'P : PQ \pm PQ' :: SP : PQ.$$

Now $PQ + PQ'$ or $PQ \sim PQ'$ is constant, according as P is or is not between the directrices, and $SP : PQ$ is constant, since $PR : PQ$ is so; therefore $SP \pm S'P$ is constant.

If the focal curve be umbilical, draw PM, PN perpendicular to the planes through QD parallel to the cyclic sections, and let PM', PN' be corresponding perpendiculars for $Q'D'$; then $PM : PQ :: PM' : PQ'$ and $PN : PQ :: PN' : PQ'$,

$$\text{also } SP^2 = e . PM . PN,\ \ S'P^2 = e . PM' . QN';$$

$$\therefore\ SP : S'P :: PQ : PQ',$$

and the argument proceeds as before.

356. *If a chord of a conicoid meet a directrix, the line joining the point in the directrix with the corresponding focus will bisect the angle between the focal distances of the extremities of the chord or its supplement.*

Let the chord PP' meet a directrix in Q, and let S be the corresponding focus, then $SP : PQ :: SP' : P'Q$; $\therefore$ $SP : SP' :: PQ : P'Q$, which proves the proposition.

357. COR. If PQ be a tangent to a conicoid at P, meeting a directrix in Q, and let S be the corresponding focus, the angle PSQ will be a right angle.

358. *A straight line touching a conicoid makes equal angles with the lines drawn from the point of contact to the foci which correspond to the directrices which the line intersects.*

For, if P be the point of contact, Q, Q' the points in which the tangent meets the directrices, S, S' the foci, since the modulus is the same for both foci, we shall have $SP : PQ :: S'P : PQ'$.

Also the angles PSQ, $PS'Q'$ are right angles, therefore the triangles are similar, and the angles QPS, $Q'PS'$ are equal.

359. *If a cone, having its vertex in any directrix, envelope a conicoid, the plane of contact will pass through the corresponding focus, and be perpendicular to the line joining the focus with the vertex.*

If V be the vertex and S the focus, and VP be any side of the cone touching the surface in P, PSV will be a right angle. Hence the locus of P, which will be the curve of contact, will be in a plane through S perpendicular to VS.

360. *If the vertex of a cone be any point in a focal curve of a conicoid, and the base be any plane section of the conicoid, the line joining the vertex with the point in which the corresponding directrix meets the plane of section will be an axis of the cone.*

Let S be the vertex, and let the plane section cut the directrix in E, and EP, EP' be tangents to the section at P, P', then SP, SP' will be perpendicular to SE, the intersection of two tangent planes to the cone through SP, SP'; therefore SE will be an axis.

Cor. 1. The second plane of section of the cone and conicoid will intersect the corresponding directrix in the same point as the first plane.

Cor. 2. If the first plane of section pass through the directrix, the second will do so also, and in this case, since there will be an infinite number of axes of the cone, it will be one of revolution.

361. *If the vertex of an enveloping cone of a conicoid be a point on a focal conic of the conicoid, the cone will be one of revolution, and its internal axis will be the tangent to the focal curve at the vertex.*

Let V be the vertex of the cone, VP, VP' the tangents to the trace of the conicoid on the plane of the focal curve, then PP' will be a tangent to the dirigent conic at the foot of the corresponding directrix, Art. 346; and since the plane of contact is perpendicular to the plane of the focal curve, it will contain the corresponding directrix, the cone therefore will be one of revolution, Art. 360, Cor. 2.

Also, since the tangent at V to the focal conic is perpendicular to the directrix, and to the line joining V and the foot of the directrix, Art. 346, it will be perpendicular to the plane of circular section, and will be the internal axis of the cone.

Cones and Sphero-Conics.

362. The properties of cones of the second degree, and of their intersections with a sphere whose centre is at the vertex, called *sphero-conics*, have been discussed in an elaborate manner in two memoirs by Chasles.* In these investigations he has made use of certain reciprocal properties of the cyclic sections and focal lines, by which any theorem relating to cyclic sections involves a corresponding theorem concerning focal lines.

We can only make a selection of some of the innumerable propositions given by Chasles, in the proof of which we shall generally employ the properties of focal lines, in place of the

* *Nouv. Mem. de l'Acad. Roy. de Bruxelles*, vol. VI.

reciprocal properties of the cyclic sections, employed with so much skill in those memoirs, for which we refer the student to a valuable translation by Graves.

363. *Focal conics of cones.*

Since a cone may be considered as the limit of either of the hyperboloids when the axes are made indefinitely small, if a, b, c be finite quantities proportional to the principal semi-axes of a hyperboloid, supposed indefinitely diminished, we obtain the equations of the cone $x^2/a^2+y^2/b^2-z^2/c^2=0$, $a>b$, and the corresponding focal conics, viz.

$$x^2/(a^2+c^2)+y^2/(b^2+c^2)=0,$$
$$y^2/(a^2-b^2)+z^2/(a^2+c^2)=0,$$
$$x^2/(a^2-b^2)-z^2/(b^2+c^2)=0.$$

The same consideration shews that *the line joining any focus with the foot of the corresponding directrix is perpendicular to the focal line containing the focus.*

The student should obtain these results by a direct comparison of the equation of the cone with such an equation as

$$(x-\xi)^2+y^2+(z-\zeta)^2=\lambda(x-\xi')^2+\nu(z-\zeta')^2,$$

which will give the focal and dirigent lines

$$\xi^2/(a^2-b^2)-\zeta^2/(b^2+c^2)=0, \text{ and } (a^2-b^2)\,\xi^2/a^4-(b^2+c^2)\,\zeta^2/c^4=0.$$

If he compare with the equation

$$(x-\xi)^2+(y-\eta)^2+z^2=\lambda'(x-\xi')^2+\mu(y-\eta')^2,$$

he will obtain the equation $\xi^2/(a^2+c^2)+\eta^2/(b^2+c^2)=0$, hence, when the directrix is in the axis, the vertex is a modular focus, λ and λ' are the squares of the moduli in the two cases, and are equal to $1-b^2/a^2$ and $1+c^2/a^2$.

The cone has therefore the property that all the three focal conics are real, having a common point in the vertex, two of them being ellipses evanescent in the transition between real and imaginary existence, and the third the limit of a hyperbola consisting of two right lines intersecting in the vertex.

The vertex is therefore not only modular, but *doubly* modular, since it is a point in two modular focal curves, and it is also an umbilical focus, as we see from the fact that the cone is the limit of two hyperboloids, for both of which the real focal hyperbola is modular, and for one the real focal ellipse is modular, while for the other it is umbilical.

Of the two moduli in the modular generation of the cone, the less modulus belongs to the focal lines, and is called by MacCullagh the *linear modulus*, while the other, to which only a single focus corresponds, is called the *singular modulus.*

364. *The focal lines of the cone enveloping a central conicoid are generators of the confocal hyperboloid of one sheet which passes through the vertex.*[*]

Let a, a_1, a_2, a_3 be the squares of the primary semi-axes of the enveloped conicoid and of the three confocals through the vertex. The equation, referred to the three normals, of the cone is $x^2/(a_1-a)+y^2/(a_2-a)+z^2/(a_3-a)=0$, and of its focal lines $x^2/(a_1-a_2)=z^2/(a_2-a_3)$, and a_1-a_2, a_2-a_3 are the squares of the semi-axes of the section of the hyperbolic section of the hyperboloid of one sheet by a plane through the centre parallel to zx, the focal lines are therefore parallel to the asymptotes of this section, and therefore are the generators through the vertex, which form a line-hyperbola similar to the hyperbolic section.

365. In the same way it can be shewn that *the focal lines of the cone enveloping a paraboloid are generators of the confocal hyperbolic paraboloid which passes through the vertex.*

366. *Cyclic sections of a cone.*

The equation of a cone $x^2/a^2+y^2/b^2=z^2/c^2$ may be written $x^2+y^2+z^2=(1+a^2/c^2)z^2-(a^2/b^2-1)y^2$; therefore, any plane section which is parallel to one of the planes $(1+a^2/c^2)z^2=(a^2/b^2-1)y^2$, lies on a sphere and is circular.

The planes through the vertex, to which circular sections are parallel, are called *cyclic planes.*

367. *A sphere, which passes through the vertex of a cone and any circular section touches the cyclic plane of the opposite system.*

A sphere can be described through any two circular sections parallel respectively to the two cyclic planes; let the plane of one of the circles approach indefinitely near to the vertex, in which case the circle degenerates into a point-circle lying on a cyclic plane, which is therefore a tangent plane to the sphere.

Conjugate Diameters of a Cone.

368. Take any line VA through the vertex of a cone, let VBC be its polar plane, and VAC any plane through VA intersecting VBC in VC, then VB the polar line of VAC will lie in VBC; also VC will be the polar line of the plane through VA, VB. Thus, if *any* plane cut VA, VB, VC, in A, B, and C, the triangle ABC will be self-conjugate with respect to the section by the plane. If then a section be made by a plane parallel to VBC, the polar of the point in which this plane will cut VA will be at infinity, and the point will be the centre of the section. VA is therefore the locus of the centres of all sections by planes parallel to VBC; and VB, VC have the same relation to VAC, VAB respectively. VA, VB, VC, therefore, form a system of conjugate diameters of the cone.

COR. If a plane cut a system of conjugate diametral planes of a cone, the triangle formed by the lines of intersection is self-conjugate with respect to the section of the cone by the plane.

[*] Jacobi, *Crelle*, vol. XII. p. 137. McCullagh, *Collected Works*, p. 311.

Reciprocal Cones.

369. *If a cone be constructed whose sides are perpendicular to the tangent planes of any given cone, the tangent planes to it will be perpendicular to the sides of the given cone.*

Let any two tangent planes be drawn to a cone A, then two corresponding sides of the other cone B, perpendicular to those tangent planes, will be perpendicular to their line of intersection; the line of intersection of the tangent planes to A is, therefore, perpendicular to the plane containing the corresponding sides of B.

Proceeding to the limit, the line of intersection becomes ultimately a side of the cone A, and the plane containing the sides of B a tangent plane to B; whence the truth of the proposition.

From this reciprocal property the cones are called *reciprocal cones.*

If $x^2/a^2 + y^2/b^2 - z^2/c^2 = 0$ be the equation of a cone, $a^2x^2 + b^2y^2 - c^2z^2 = 0$ will be that of the reciprocal cone.

370. Any plane through the common vertex, having relations to one of the cones, has perpendicular to it a line which has reciprocal relations to the other cone, and the plane and line are said to *correspond.*

If two lines correspond respectively to two planes, they will each be perpendicular to the line of intersection of the planes, and the plane containing the two lines will correspond to the line of intersection of the two planes; also the angle between the planes will be equal to the angle between the corresponding lines.

371. The student will have no difficulty in establishing the following theorems:

To a line through the vertex of a cone and its polar plane with reference to the cone, correspond a plane and its polar line with reference to the reciprocal cone.

To three conjugate diameters of a cone correspond three conjugate diametral planes of the reciprocal cone.

372. *The cyclic planes of a cone correspond to the focal lines of the reciprocal cone.*

The equation of the cyclic planes of the cone $a^2x^2 + b^2y^2 = c^2z^2$ is $(a^2 - b^2)x^2 = (b^2 + c^2)z^2$, and that of the focal lines of the reciprocal cone $x^2/a^2 + y^2/b^2 = z^2/c^2$ is $x^2/(a^2 - b^2) = z^2/(b^2 + c^2)$; these focal lines are therefore perpendicular to the cyclic planes of the reciprocal cone.

The relation between the focal lines of one cone and the cyclic planes of the reciprocal cone is deduced geometrically thus:

To a cyclic plane corresponds a line VS perpendicular to it; any two conjugate axes in the cyclic plane are at right angles; therefore any two conjugate diametral planes of the reciprocal cone through VS are at right angles.

Let a plane be drawn perpendicular to VS through any point S, this plane will meet the two diametral planes in two perpendicular lines, and, by Art. 368, Cor., the pole of one of these lines with respect to the section of the cone will lie on the other line; therefore this pole is on the directrix, and S is the focus of the conic section; VS is therefore a focal line, having the focal property proved for any conicoid in Arts. 354 and 348.

The locus of these directrices is called a *dirigent plane* by McCullagh and a *director plane* by Chasles, and this plane, with the perpendicular planes through the focal line, forms a system of conjugate planes; it corresponds, therefore, with the third axis, conjugate to two perpendicular lines in a cyclic section, which contains the centres of circular sections parallel to that cyclic section.

373. *Equation of the cone reciprocal to a cone enveloping a central conicoid.*

Let (f, g, h) be the vertex, $lx + my + nz = p$ the equation of a tangent plane to the enveloping cone, which also touches the conicoid $x^2/a + y^2/b + z^2/c = 1$;

$$\therefore \; lf + mg + nh = p, \text{ and } al^2 + bm^2 + cn^2 = p^2,$$

and (l, m, n) is the direction of a generating line of the reciprocal cone, therefore its equation referred to axes through the vertex parallel to the axes of the conicoid is $ax^2 + by^2 + cz^2 = (fx + gy + hz)^2$.

374. The planes of the two cyclic sections of this cone can be obtained by combining this equation with that of a sphere, therefore, for some value of σ,

$$ax^2 + by^2 + cz^2 - (fx + gy + hz)^2 + \sigma(x^2 + y^2 + z^2),$$

is the product of two linear factors, and those factors equated to zero are the equations of the planes of the cyclic sections.

The corresponding cone for any confocal $x^2/(a + \lambda) + \ldots = 1$ gives the same planes, viz. those obtained from the same product

$$(a + \lambda)x^2 + \ldots - (fx + \ldots)^2 + (\sigma - \lambda)(x^2 + y^2 + z^2);$$

hence, *if cones, having a common vertex, envelope a series of confocals, the reciprocal cones are concyclic and co-axial.*

Hence the cones themselves are confocal and co-axial.

375. *The normals to the confocals which pass through the common vertex are the principal axes of the enveloping cones.*

The enveloping cone to any confocal which passes through the vertex becomes coincident with the tangent plane, and the reciprocal cone then becomes an indefinitely thin cone, one of whose axes is the normal, hence all the reciprocal cones being co-axial, this normal is one of the three principal axes, and since cones and their reciprocals are co-axial, the proposition is proved. In Art. 299 the proposition was proved independently and the co-axiality deduced from it.

376. *Equation of a cone enveloping the conicoid $x^2/a + \ldots = 1$, referred to the normals to the confocals through the vertex.*

The equation of the cone reciprocal to the envelope, referred to the normals to the confocals through the vertex, is of the form $\alpha x^2 + \beta y^2 + \gamma z^2 = 0$, by Art. 299 or the last article, and that of the cone corresponding to any confocal $x^2/(a - k) + \ldots = 1$ will therefore be

$$\alpha x^2 + \beta y^2 + \gamma z^2 - k(x^2 + y^2 + z^2) = 0.$$

Let this confocal be one of the confocals passing through th vertex, say that the normal to which is the axis of x, and for whic $k = k_1$, the reciprocal cone then becomes coincident with the normal $\therefore \alpha = k_1$, similarly $\beta = k_2$, $\gamma = k_3$, and the equation of the reciproca cone is $k_1x^2 + k_2y^2 + k_3z^2 = 0$, and therefore that of the envelopin cone is $x^2/k_1 + y^2/k_2 + z^2/k_3 = 0$, as in Art. 301.

377. The method of dealing with propositions connected wit focal lines, by the modular and umbilical methods, may be seen b the following, in which we shall state the reciprocal theorems.

Properties of Cones of the Second Degree.

378. *The sines of the angles, which any side of a cone makes with a foc line and the corresponding dirigent plane, are in a constant ratio.*

Let a plane pass through any directrix DQ and the corresponding focus S and let P be any point in the section of the cone made by this plane; V th vertex of the cone.

Draw PR, PQ perpendicular to the dirigent plane and directrix.

Then $SP : PQ$ and $PR : PQ$, and therefore $SP : PR$ are constant ratios and DS being perpendicular to VS, VS is perpendicular to the plane of sectior and PSV is a right angle.

Hence, the ratio of the sines proposed is $PS/PV : PR/PV$, and i therefore constant.

Reciprocal theorem. The ratio of the sines of the angles made by tangen planes with a cyclic plane and with the polar line of this cyclic plane is constan

379. *The product of the sines of the angles which any side of a cone make with the directing or cyclic planes is constant.*

If V be the vertex of a cone, P any point on the cone, PL, PL' perpen dicular on the directing planes through V, then by the umbilical generatio of the cone, Art. 343, PV^2 is proportional to $PL.PL'$; or $PL/PV.PL'/P$ is constant, which is the property enunciated.

Reciprocal theorem. The product of the sines of the angles which eac tangent plane to a cone makes with the two focal lines is constant.

380. *A tangent plane to a cone makes equal angles with the planes throug the side of contact and each of the focal lines.*

For, let the tangent QPQ' perpendicular to the side VP meet the dirigen planes in the points Q, Q', and take S, S' the foci corresponding to the direc trices through Q, Q'; then SQ is perpendicular to VS, and also to PS, an therefore to the plane VPS; also VP is perpendicular to SQ and PQ, an therefore to SP; hence SPQ is the inclination of the planes VPS, VPQ and being equal to $S'PQ'$, Art. 358, the proposition is proved.

Reciprocal theorem. A tangent plane to a cone intersects the two cycli planes in two straight lines, which make equal angles with the side of the con along which it is touched by the tangent plane.

381. The last theorems are particular cases of the two following:

The planes passing through the two focal lines of a cone, and through th intersection of two tangent planes to the cone, make equal angles with thes tangent planes.

And the reciprocal theorem:

A plane containing two sides of a cone intersects the cyclic planes in tw straight lines, which respectively make equal angles with the two sides.

We give Chasles' proof of the reciprocal theorem as a good example of th geometrical treatment of problems connected with cyclic planes.

Take two circular sections of opposite systems of the cone, the plane of the two sides cuts the planes of the two circles in two chords, which, with the portions of the sides of the cone intercepted, form a quadrilateral inscribed in the circle in which the sphere containing the circular sections is cut by the plane of the two sides; two opposite angles of this quadrilateral are supplementary, hence the chords make equal angles with the sides of the cone, and, since they are parallel to the sections by the cyclic planes, the theorem is proved.

382. Simple propositions for the circle can be transformed into others relating to the cone with the same facility as in plane geometry properties of conics are obtained.

This is effected by considering the lines and points in the circle as the intersections of planes and straight lines, passing through the vertex of a cone, with the plane which cuts the cone in this circle.

It will be sufficient to give two examples of this transformation.

383. Two tangents to a circle make equal angles with the chord which joins the two points of contact, hence

Two tangent planes to a cone and the plane of the two sides of contact intersect a cyclic plane in three straight lines, the third of which makes equal angles with the other two.

The reciprocal theorem is,

If planes be drawn through a focal line of a cone, and two sides of the cone, and through the line of intersection of two planes touching the cone along these sides, the third plane will bisect the angle between the first two.

384. Two tangents to a circle make equal angles with the line joining their point of intersection with the centre of the circle, hence

Two tangent planes to a cone, and the plane passing through their line of intersection, and through the conjugate of a cyclic plane, meet that cyclic plane in three lines, one of which bisects the angle between the other two.

The reciprocal theorem is,

The planes passing through a focal line of a cone and two sides of the cone make equal angles with the plane passing through the same focal line and the straight line in which the plane containing the two sides intersects the dirigent plane.

Sphero-conics.

385. If a cone of the second degree be cut by a sphere whose centre is at the vertex of the cone, the complete curve of intersection will be two closed curves, which will be plane curves if the cone be one of revolution.

Chasles observes that we obtain three distinct curves if we consider the portions of the complete curve of intersection contained on the three hemispheres cut off by the three principal planes of the cone.

First, consider the hemisphere whose base is perpendicular to the interior or principal axis of the cone, the figure is then a closed curve, and may be called a *spherical ellipse*, the foci of which are the points where the focal lines cut the hemisphere, having, it will be seen, properties in all respects corresponding to the foci of a plane ellipse.

Secondly, consider the hemisphere whose base is the other principal plane perpendicular to that containing the focal lines the figure is then composed of two halves of spherical ellipses which may together be called a *spherical hyperbola* whose foci lie within the concave portions, and it will be seen that sections of the sphere by the cyclic planes have properties similar to those of asymptotes.

Thirdly, consider the hemisphere whose base is the plane containing the focal lines, the figure is then formed by two halves of spherical ellipses and has four foci and a centre where the minor axis of the cone meets the hemisphere.

We shall consider a sphero-conic to be one of the first two of these curves, viz. the spherical ellipse or hyperbola.

The curves in which a sphere cuts two reciprocal cones, of which its centre is the common vertex, are called *reciprocal sphero-conics.*

The principal reciprocal property connecting the two may be stated thus:

Every point of a sphero-conic is the pole of a great circle which touches the reciprocal sphero-conic.

386. *The intersection of a central conicoid with a concentric sphere is a sphero-conic.*

For, if $ax^2+by^2+cz^2=1$ and $x^2+y^2+z^2=r^2$ be their equations, the curve of intersection lies on the cone

$$(ar^2-1)\,x^2+(br^2-1)\,y^2+(cr^2-1)\,z^2=0\,;$$

this cone is evidently concyclic with the conicoid.

387. We give below two or three of the numerous properties of sphero-conics, which are the counterparts of properties of plane conics, each of which has its duplicate obtained by forming the reciprocal proposition. The proofs of these can be gathered from the previous articles; but as exercises in spherical trigonometry the student may take almost any ordinary property in plane conics relating to foci and directrices, and to asymptotes of hyperbolas, which correspond to the cyclic arcs, and find analogues to them in sphero-conics; he may also find equations corresponding to the polar equation of a conic or of a tangent to a conic, or of the auxiliary circle, or of the locus of the intersection of perpendicular tangents. Many properties of sphero-conics are given by Routh in his *Dynamics*, vol. II. p. 108, ed. 1884.

A tangent to a sphero-conic makes equal angles with the radii vectores drawn from the foci to the point of contact, Art. 380.

An arc of a great circle which touches a sphero-conic and is cut off by the cyclic arcs is bisected at the point of contact.

The sum or difference of two radii drawn from the foci to any point of a sphero-conic is constant.

The sum or difference of the angles which a tangent to a sphero-conic makes with the cyclic arcs is constant.

The first theorem is proved by limits as in plane conics.

The product of the sines of arcs drawn perpendicular to the cyclic arcs from any point of a sphero-conic is constant, Art. 379.

The product of the sines of arcs drawn from the foci at right angles to a tangent to a sphero-conic is constant.

We give the following as an example of the mode of applying spherical trigonometry.

388. *The locus of the intersection of perpendicular tangents to a sphero-conic is another sphero-conic for which the product of the cosines of the distances from the foci of the first sphero-conic is constant.*

Let tangents at P, P' intersect at right angles in Q, $2a$ the major axis, 2γ the distance of the foci S, S', $SQ=\rho$, $S'Q=\rho'$, $\angle SQP=\angle S'QP'=\psi$, then

$$\cos 2\gamma = \cos\rho\cos\rho' + \sin\rho\sin\rho'\sin 2\psi,$$

and if p, p' be perpendiculars on PQ from S, S',

$$\sin p = \sin\psi\sin\rho, \quad \sin p' = \cos\psi\sin\rho',$$

and $\sin p \sin p'$, being constant, is equal to $\sin(a-\gamma)\sin(a+\gamma)$;

$$\therefore \cos\rho\cos\rho' = \cos 2\gamma - 2\sin(a-\gamma)\sin(a+\gamma) = \cos 2a,$$

from which the equation of the cone determining the sphero-conic may be easily deduced, viz. $(c^2-b^2)x^2+(c^2-a^2)y^2-(a^2+b^2)z^2=0$.

This equation may be obtained as follows:

The equation of two tangent planes to a cone $ax^2+by^2+cz^2=0$, drawn through a point (ξ, η, ζ) is

$$(a\xi^2+b\eta^2+c\zeta^2)(ax^2+by^2+cz^2)-(a\xi x+b\eta y+c\zeta z)^2=0,$$

$$\text{or } (lx+my+nz)(l'x+m'y+n'z)=0;$$

$$\therefore a(b\eta^2+c\zeta^2)/ll' = b(c\zeta^2+a\xi^2)/mm' = c(a\xi^2+b\eta^2)/nn',$$

and, when the tangent planes are at right angles,

$$a(b\eta^2+c\zeta^2)+b(c\zeta^2+a\xi^2)+c(a\xi^2+b\eta^2)=0;$$

$$\text{whence } \xi^2/a+\eta^2/b+\zeta^2/c=(a^{-1}+b^{-1}+c^{-1})(\xi^2+\eta^2+\zeta^2).$$

XXVII.

(1) In every hyperboloid of one sheet two circular cylinders can be inscribed.

(2) Two tangent planes to a cone intersect its two cyclic planes in four straight lines which are sides of the same cone of revolution, whose axis is perpendicular to the plane of the two sides of contact.

(3) If e, e' be the eccentricities of the principal sections (a, b) and (a, c) of an ellipsoid, shew that the distance of two points S, S' on the focal conics in these planes, whose distances from the section (b, c) are x', x'', will be $(e'^2x''-e^2x')/ee'$, and that the shortest distance of the corresponding directrices will vary as SS'.

(4) If from a point upon a focal line of a cone, perpendiculars be let fall upon the tangent planes to this cone, their feet will be upon a circle, the plane of which will be perpendicular to the other focal line.

(5) The focal ellipse of an ellipsoid corresponds on the flat confocal ellipsoid to the principal section in its plane, and the focus of the principal section corresponds to the umbilic.

(6) If PG be a normal to an ellipsoid, G the foot of the normal on the plane of the focal ellipse, P' the point of the flat confocal, bounded by the focal ellipse, which corresponds to P, G' the point on the ellipsoid corresponding to G on the flat confocal, $P'G'$ will be perpendicular to the plane of the focal ellipse and be equal to PG.

(7) A spherical triangle has a given area and two sides lie on two fixed circles, prove that its base touches a sphero-conic, and is bisected by the point of contact.

XXVIII.

(1) The plane of any cyclic section of an ellipsoid will intersect the dirigent cylinder in an ellipse similar to the principal section in which the focal conic lies; if the plane touch at an umbilic, the umbilic will be a focus of the section of the cylinder.

(2) If two conicoids have a common focus S, and a common directrix, and if a tangent to one of the surfaces at P meet the other surface in Q, Q', and the directrix in R, SP will bisect the interior or exterior angle QSQ'.

(3) The square of the distance between a focus and the corresponding directrix of the section of an ellipsoid, made by the plane of contact with any enveloping cone of revolution, is $b^4/\{(a^2-b^2)(b^2-c^2)\}$.

(4) In any hyperboloid there are two diameters, such that any two conjugate planes passing through either of them are at right angles, and these diameters are the focal lines of the asymptotic cone of the hyperboloid.

(5) Shew that the equation of the cone containing the locus of the foot of the perpendicular from a focus of a sphero-conic upon a tangent is $(a^2+c^2)\,x^2+(b^2+c^2)\,y^2=a^2\,(x^2+y^2+z^2)$; and that its cyclic sections are the same as those of the cone containing the locus of the intersection of perpendicular tangents.

(6) Two fixed tangents to a sphero-conic are intersected by any third tangent; shew that the arcs joining the focus and the two points of intersection include a constant angle. Shew also that this angle will be a right angle if the fixed tangents intersect on the directrix arc of the sphero-conic.

State the reciprocal theorem.

(7) If the arc joining two points of a sphero-conic pass through a focus, the sum of the cotangents of the arcs between the focus and the two points will be constant.

State the reciprocal theorem.

CHAPTER XVI.

GENERAL EQUATION OF THE SECOND DEGREE.

389. OUR object in this chapter is to investigate the position of the origin, and the directions of the axes (which we shall suppose to be a rectangular system) by transformation to which any proposed equation of the second degree will assume its simplest form; and also to find the relations among the coefficients of the general equation which discriminate the various kinds of surfaces capable of being represented by the equation.

390. The general equation of the second degree will be written

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy$$
$$+ 2a''x + 2b''y + 2c''z + d = 0$$

or $u \equiv u_2 + u_1 + d = 0$;

this equation will sometimes be made homogeneous by the introduction of w for the unit-length, which will enable us to employ the known properties of homogeneous functions; in this case we shall have

$$u \equiv ax^2 + by^2 + cz^2 + dw^2 + 2a'yz + 2b'zx + 2c'xy + 2a''xw$$
$$+ 2b''yw + 2c''zw.$$

391. *Discriminant of u_2.*

The discriminant of u_2 is $\begin{vmatrix} a, & c', & b' \\ c', & b, & a' \\ b', & a', & c \end{vmatrix}$; it will be convenient to denote it by $H(u_2)$ or Δ, and its minors with reference to a, a', viz. $bc - a'^2$, $b'c' - aa'$, ..., by A, A', B, B', C, C'.

It is easily shewn that $B'C' - AA' \equiv a'\Delta$ and $BC - A'^2 \equiv a\Delta$; and it follows that, when $\Delta = 0$, $B'C' = AA'$, $BC = A'^2$, &c., also that A, B, C are all of the same sign.

392. *Discriminant of u.*

The discriminant of u, viz. $\begin{vmatrix} a, & c', & b', & a'' \\ c', & b, & a', & b'' \\ b', & a', & c, & c'' \\ a'', & b'', & c'', & d \end{vmatrix}$, will be denoted by $H(u)$; expanded it becomes

$$-(Aa''^2 + Bb''^2 + Cc''^2 + 2A'b''c'' + 2B'c''a'' + 2C'a''b'') + \Delta d;$$

and if $\Delta=0$, by the last article, since $A''^2=BC$, &c., $H(u)$ is a complete square, viz.

$$-(Aa''+C'b''+B'c'')^2/A, \text{ or } -(a''\sqrt{A}+b''\sqrt{B}+c''\sqrt{C})^2,$$

where $\sqrt{A}$, $\sqrt{B}$, $\sqrt{C}$ must have all the same signs as A, C', B', or all different.

393. Before we examine the relations among the constants, which correspond to the different forms which the locus of the general equation can assume, it will be useful to state the most important propositions relating to the tangency of lines and planes, while the equation retains its most general form as given in Art. 390.

The proofs of these propositions will be given concisely, the processes as well as the notation being much the same as those used in Chapter XIII.

394. *Condition that a straight line may touch a conicoid at a given point* (f, g, h).

The line being $(x-f)/\lambda=...=r$, meets the surface at points given by $F(f+\lambda r, g+\mu r, h+\nu r)=0$, and since the two values of r are zero, the coefficient of r must vanish, and the condition is

$$\left(\lambda\frac{d}{df}+\mu\frac{d}{dg}+\nu\frac{d}{dh}\right)F(f, g, h)=0. \qquad (1)$$

395. *Equation of the tangent plane at* (f, g, h).

The locus of the tangent lines is, from the condition (1),

$$(x-f)\frac{dF}{df}+(y-g)\frac{dF}{dg}+(z-h)\frac{dF}{dh}=0,$$

this is the equation of the tangent plane which, written full, is

$$(af+c'g+b'h+a'')x+(c'f+bg+a'h+b'')y$$
$$+(b'f+a'g+ch+c'')z+a''f+b''g+c''h+d=0, \qquad (2)$$

since f, g, h satisfy the equation $F(x, y, z)=0$.

396. *Condition that the general equation may represent a cone.*

The vertex of a cone is a point at which the tangent plane is indeterminate. Therefore, if the surface be a cone, there must be a point (f, g, h) upon it, for which the coefficients in the equation (2) vanish simultaneously, hence the condition $H(u)=0$.

397. *The discriminant of* u *is an invariant.*

This is shewn in Salmon's *Higher Algebra*, p. 20, but it can be shewn from the property of the vertex of a cone as follows.

Let ρ have such a value that $u+\rho(x^2+y^2+z^2+1)=0$ represents a cone, and, after a transformation of coordinates to any new set of rectangular axes with any origin such that

$$x=l(x'+\xi)+m(y'+\eta)+n(z'+\zeta),\ y=l'(x'+\xi)+...,\ z=l''(x'+\xi)+...,$$

let u become $ax'^2+\ldots+2a'y'z'+\ldots+2a''x'+\ldots+\delta$ or u', then $u+\rho(x^2+y^2+z^2+1)$ becomes

$$u'+\rho\{(x'+\xi)^2+(y'+\eta)^2+(z'+\zeta)^2+1\}.$$

The discriminants of each of these quadrics vanish with the same four values of ρ, and, since the coefficient of ρ^4 is unity in each, the coefficients of ρ^3, ρ^2, ρ, and the last terms are invariants, that is $H(u)=H(u')$.

COR. If u be reduced by transformation to $ax^2+\beta y^2+\gamma z^2+2a''x+\delta$,

$$H(u)=a\beta\gamma\delta-a''^2\beta\gamma;\ \text{hence},$$

i. for a central conicoid referred to the centre $H(u)=a\beta\gamma\delta=\Delta\delta$,

ii. for a paraboloid referred to the vertex $H(u)=-a''^2\beta\gamma$,

iii. for a cylinder referred to a point in the axis $H(u)=0$, a cylinder being a particular case of a cone.

398. *Equation of the polar plane of a point* (f, g, h).

By (2), the equation of a tangent plane at (x, y, z) is

$$(ax+c'y+b'z+a'')\,\xi+\ldots=0,$$

and since this plane passes through (f, g, h),

$$(ax+c'y+b'z+a'')f+\ldots=0;$$

hence, the locus of the points of contact of all tangent planes through (f, g, h), is the plane whose equation is (2) Art. 395, which is the polar plane of (f, g, h).

COR. The polar plane of the origin is given by

$$a''x+b''y+c''z+d=0.$$

The definition of a polar plane given in Art. 280 is proved to hold for any conicoid by taking the fixed point as origin, so that, if $x=lr$, &c., $r_1^{-1}+r_2^{-1}=-2\,(a''l+b''m+c''n)/d$.

399. The equation of the enveloping cone may be found precisely as in Art. 265 or 266.

400. *Condition that* $lx+my+nz=p$ *may be the equation of a tangent plane to a conicoid.*

Comparing with the equation (2) Art. 395,

$$af+c'g+b'h+a''=l\rho,\ \text{\&c., and}\ lf+mg+nh-p=0;$$

$$\therefore\quad \begin{vmatrix} a, & c', & b', & a'', & l \\ c', & b, & a', & b'', & m \\ b', & a', & c, & c'', & n \\ a'', & b'', & c'', & d, & -p \\ l, & m, & n, & -p, & \end{vmatrix}=0.$$

In the expansion of this determinant the coefficients of $-l^2$, $-2lm$, $2lp$, &c., are the minors of a, c', a'', &c. in the determinant $H(u)$.

401. *Equation of the cone reciprocal to the cone whose equation is* $u_2 = 0$.

The tangent plane at (f, g, h) is $(af + c'g + b'h)x + \ldots = 0$, and if (l, m, n) be the direction of a normal to this plane,

$$af + c'g + b'h = lp, \text{ \&c. and } lf + mg + nh = 0;$$

$$\therefore \begin{vmatrix} a, & c', & b', & l \\ c', & b, & a', & m \\ b', & a', & c, & n \\ l, & m, & n, & 0 \end{vmatrix} = 0,$$

In the expansion of this determinant the coefficients of $-l^2, -lm, \ldots$ are the minors of a, c', &c. in the determinant Δ.

The reciprocal cone is the locus of lines through the origin drawn perpendicular to the tangent planes to $u_2 = 0$, hence, with the notation of Art. 391, its equation is

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy = 0.$$

By Art. 391. the reciprocal of this cone is reducible to $u_2 = 0$, as ought to be the case.

402. *Cone reciprocal to the cone enveloping a conicoid* $x^2/a + y^2/b + z^2/c = 1$.

The equation of the enveloping cone referred to axes through the vertex (f, g, h) parallel to those of the conicoid, if $f^2/a + \ldots - 1 \equiv u_0$, is

$$u_0 (x^2/a + y^2/b + z^2/c) = (fx/a + gy/b + hz/c)^2.$$

In this case $A = (u_0/b - g^2/b^2)(u_0/c - h^2/c^2) - g^2h^2/b^2c^2$,

$$A' = f^2gh/a^2bc + (u_0/a - f^2/a^2)\, gh/bc,$$

and the equation of the reciprocal cone is

$$(f^2 - a)\, x^2 + \ldots + 2ghyz + \ldots = 0,$$

or $ax^2 + by^2 + cz^2 = (fx + gy + hz)^2$, as in Art. 373.

Analysis of the General Equation.

403. We now proceed to the classification of the different surfaces which can be represented by the general equation; the first step being to examine whether there is a single centre or a locus of centres; after this to shew how the equation may be reduced to the forms included in the equation

$$\alpha x^2 + \beta y^2 + \gamma z^2 + 2a''x + \delta = 0.$$

404. *To find the centre or the locus of centres of a surface of the second degree.*

Let (ξ, η, ζ) be a centre of the locus of $u = 0$, and let the origin be transformed to this point; the transformed equation is $f(x + \xi, y + \eta, z + \zeta) = 0$, and, since the new origin is the point of bisection of all chords drawn through it, each of the coefficients of x, y, z must vanish;

$$\begin{aligned} \therefore\ a\xi + c'\eta + b'\zeta + a'' &= 0, \\ c'\xi + b\eta + a'\zeta + b'' &= 0, \\ b'\xi + a'\eta + c\zeta + c'' &= 0. \end{aligned} \qquad (1)$$

Considering ξ, η, ζ as current coordinates, these three equations represent three planes in each of which the centre lies.

The three planes generally intersect in one point (I), but they may have one line common to them all (II), or they may all three coincide (III).

I. In the first case, there will be one centre which may be at a finite (i) or an infinite distance (ii).

i. If the centre be at a finite distance, its coordinates will be given by $\Delta\xi = A''$, $\Delta\eta = B''$, $\Delta\zeta = C''$, and Δ must be finite.

Cor. If the centre be upon the surface, the surface will be a cone. Multiplying the equations (1) by ξ, η, and ζ, since $u = 0$ we obtain a fourth equation $a''\xi + b''\eta + c''\zeta + d = 0$, and eliminating ξ, η, and ζ from these four equations, we obtain $H(u) = 0$ for the condition that the surface may be a cone, as in Art. 396.

ii. The centre will be at an infinite distance if any of its coordinates be infinite; thus, if ξ be infinite, $\Delta = 0$, and A'' or

$$-\sqrt{A}\,(a''\sqrt{A} + b''\sqrt{B} + c''\sqrt{C})$$

must be finite; and we may notice that η cannot at the same time be finite unless $B = 0$.

II. In the second case there will be a line of centres, which may be at a finite (i), or an infinite distance (ii).

i. The coordinates of the centre must be indeterminate, for which we have the conditions that $\Delta = 0$ and that the minors A'', B'', C'' all vanish, or $a''\sqrt{A} + b''\sqrt{B} + c''\sqrt{c} = 0$; this condition may be written, if A', B', C' be finite, $a''/A' + b''/B' + c''/C' = 0$.

ii. The line of centres will be at an infinite distance in four cases.

(*a*) If the three planes be parallel, and not more than two of them coincident; the conditions for this are $a/c' = c'/b = b'/a'$ and $c'/b' = b/a' = a'/c$, and that $a'a''$, $b'b''$, $c'c''$ shall not be all equal, hence in this case all the minors A, A', &c. vanish.

(*b*) If one plane be at an infinite distance, and the other two be parallel or coincident; in this case, if the first plane be that at an infinite distance, a, c', b' must all vanish and a'' be finite, also $b/a' = a'/c$ and these must not be equal to b''/c'' if the two planes be parallel, but will be equal to b''/c'' if they be coincident.

(*c*) If one be indeterminate and the others parallel but not coincident; suppose the first to be that which is indeterminate, a, c', b', and a'' must all vanish, and $a'c''$, cb'' must be unequal.

(*d*) If two be at an infinite distance, or if one be at an infinite distance and a second be indeterminate; in this case all the quantities a, b, c, a', b', c' vanish except one of the first three; if c be finite, a'' and b'' will be either one or both finite.

Hence, for every case of (ii), A', B', and C' all vanish.

III. In the third case there will be a plane of centres, which may be at an infinite distance.

In order that the three planes may coincide, we must have

$$a/c' = c'/b = b'/a' = a''/b'' \text{ and } c'/b' = b/a' = a'/c = b''/c'';$$

therefore all the minors vanish, and $a'a'' = b'b'' = c'c''$.

If the plane be at an infinite distance, all the coefficients of u_2 must vanish, while one at least of a'', b'', c'' is finite.

405. We have shewn that when Δ or $H(u_2)$ is finite, the terms included in u_1 may be removed by transformation, without altering the directions of the axes, but that for every departure from the general case, in which there is a single centre at a finite distance, one of the conditions is that Δ shall vanish, and this condition is independent of all coefficients of u except those of terms of the second degree; it is also the condition that the part u_2 containing the terms of the second degree shall be the product of two factors, real or imaginary, see Art. 91. So that, in every case except where there is a single centre at a finite distance, by choosing coordinate planes, two of which bisect the angles between the planes $u_2 = 0$, the general equation can be reduced to the form

$$\beta y^2 + \gamma z^2 + 2\alpha''x + 2\beta''y + 2\gamma''z + \delta = 0.$$

This is further reducible to $\beta y^2 + \gamma z^2 + 2\alpha''x + \delta = 0$, by moving the origin in the plane of yz; and if α'' be not zero, this finally reduces to $\beta y^2 + \gamma z^2 + 2\alpha''x = 0$, or to $\beta y^2 + \gamma z^2 + \delta = 0$ if $\alpha'' = 0$. If the two factors of u_2 be equal, the equation is reducible to $\gamma z^2 + 2\alpha''x = 0$ or $\gamma z^2 + \delta = 0$.

406. The loci of equations of the second degree may therefore be classified according to the nature of their centres.

I. Single Centre.

i. At a finite distance.

Ellipsoid.
Hyperboloids of one and two sheets.
Cone, real.
Cone, imaginary (or point-ellipsoid).

ii. At an infinite distance.

Paraboloids, elliptic and hyperbolic.

II. Line of Centres.

i. At a finite distance.

Cylinders, elliptic and hyperbolic.
Line cylinder (limit of ellip. cylinder).
Two planes, intersecting (limit of hyperb. cylinder).

ii. At an infinite distance.

Cylinder, parabolic.

III. Plane of Centres.

i. At a finite distance.

Two planes, parallel (limit of parab. cylinder).

ii. At an infinite distance.

Two planes, one at an infinite distance.

Two planes, both at an infinite distance.

407. We have now to shew that it is always possible to choose such directions of the axes, that the transformed equation shall contain no terms involving yz, zx, and xy, the axes being in both cases supposed to be rectangular.

408. Since our objects in this chapter are, either to determine what kinds of surfaces can be the loci of the general equation; or, given a particular equation, to identify the surface which is its locus, we may avoid complications by considering that if only one of the rectangles, say xy, appear in the equation, we can by rotation of the axes of x and y make this term disappear, so that the equation will be reduced to the form

$$\alpha x^2 + \beta y^2 + \gamma z^2 + 2\alpha'' x + 2\beta'' y + 2\gamma'' z + \delta = 0,$$

and the nature and position of the locus will be at once determined.

In dealing with the general case we shall not therefore always examine the particular modification of the formulæ which would be required if two of the three quantities a', b' and c' were to vanish.

409. *To shew that u_2 can always be reduced to the form $\alpha x^2 + \beta y^2 + \gamma z^2$ by transformation of coordinates, α, β and γ being real quantities.*

The quadric $s(x^2 + y^2 + z^2) - u_2$ will become the product of two linear factors, real or imaginary, if s satisfy the equation

$$f(s) \equiv (s-a)(s-b)(s-c)$$
$$- a'^2(s-a) - b'^2(s-b) - c'^2(s-c) - 2a'b'c' = 0, \text{ Art. 91.}$$

Since the equation is a cubic, one of the values of s must be real, and for this value $s(x^2+y^2+z^2) - u_2 = 0$ is the equation of two planes which, whether real or imaginary, have a real line of intersection; let this line be the axis of z in a new system of coordinates, so that $s(x^2+y^2+z^2) - u_2$ becomes $Ax^2 + 2Bxy + Cy^2$ on transformation, since $Ax^2 + 2Bxy + Cy^2 = 0$ represents two planes passing through the axis of z, and, from the formulæ for transformation, A, B, C are real; the term xy may then be made to disappear by simple rotation of the axes of x and y.*

* This method was adopted by Archibald Smith in his Notes on the "Undulatory Theory of Light."—*Camb. Math. Jour.*, vol. I., p. 5.

Hence, referred to these new axes, u_2 would be reduced to $\alpha x^2 + \beta y^2 + \gamma z^2$, in which α, β, γ are real, although any one or two may vanish, the corresponding cubic being

$$(s-\alpha)(s-\beta)(s-\gamma)=0.$$

410. The cubic given in the last article is called the *discriminating cubic*, the coefficients of which, as we have seen in Art. 156, are invariants.

Since the last term is $-\Delta$, it follows that whenever all the roots of the discriminating cubic are different from zero, the locus of the general equation is a central surface, Art. 404.

411. *To separate the roots of the discriminating cubic.*

The discriminating cubic is

$$f(s) \equiv (s-a)(s-b)(s-c) - a'^2(s-a) - \ldots - 2a'b'c' = 0;$$

suppose a', b' and c' all finite, then, if $a - b'c'/a' = \lambda$, $b - c'a'/b' = \mu$, $c - a'b'/c' = \nu$, this cubic equation becomes, by writing $\lambda + b'c'/a$ for a &c.,

$$f(s) \equiv (s-\lambda)(s-\mu)(s-\nu) - (s-\mu)(s-\nu)b'c'/a' - \ldots = 0,$$

$$\therefore f(\lambda) = -(\lambda-\mu)(\lambda-\nu)a'b'c'/a'^2, \text{ \&c.}; \quad (1)$$

hence, if λ, μ, ν be in descending order, these quantities separate the roots.

If one of the quantities, as a', vanish

$$f(s) \equiv (s-a)(s-b)(s-c) - b'^2(s-b) - c'^2(s-c),$$

suppose $b > c$, $f(\infty)$, $f(b)$, $f(c)$, and $f(-\infty)$ have signs $+$, $-$, $+$, $-$; $\therefore$ b and c separate the roots.

NOTE. If two roots of the cubic be equal, two of the three λ, μ, ν must be equal to each of the equal roots by (1). Suppose $\lambda = \mu$, $f(s)$ is divisible by $(s-\lambda)^2$; $\therefore$ $\lambda = \nu$, and $a - b'c'/a' = b - c'a'/b' = c - a'b'/c'$.

412. Cauchy's method of proof that the three roots of the discriminating cubic are real, is more simple although not so symmetrical; it may be stated as follows:

Write the cubic in the form

$$f(s) \equiv (s-a)\{(s-b)(s-c) - a'^2\} - b'^2(s-b) - c'^2(s-c) - 2a'b'c' = 0.$$

Let s' be a value of s which makes $(s-b)(s-c) = a'^2$;

$$\therefore (s'-b)f(s') = -b'^2(s'-b)^2 - 2a'b'c'(s'-b) - a'^2c'^2 = -\{b'(s'-b) + a'c'\}^2;$$

but since $(s'-b)^2 + (b-c)(s'-b) - a'^2 = 0$, the two values of $s'-b$ are real and of opposite signs, let s_1, s_2 be the values of s' greater and less than b respectively; hence $f(\infty)$, $f(s_1)$, $f(s_2)$, $f(-\infty)$ are alternately $+$ and $-$, and the equation has its three real roots separated by s_1 and s_2.

413. *Consequences of the vanishing of the first and second invariants derived from the discriminating cubic.**

Write the cubic in the form $s^3 - I_1 s^2 + I_2 s - \Delta = 0$, I_1, I_2, and Δ being the three invariants.

i. If $I_1 \equiv a + b + c = 0$, let a straight line which lies in the surface be taken for the axis of x, so that the coefficients of x^2 and x and the constant term disappear; $\therefore$ $a = 0$ and $b + c = 0$, and the equation of the surface is

$$b(y^2 - z^2) + 2a'yz + 2b'zx + 2c'xy + 2b''y + 2c''z = 0,$$

which, by turning the axes of y and z into positions parallel to the asymptotes of the rectangular hyperbola $b(y^2 - z^2) + 2a'yz = \text{constant}$, and moving the origin to a point $(-b''/c', 0, 0)$, assumes the form $ayz + \beta(x - h)z + \gamma xy = 0$, shewing that two sets of perpendicular lines lie on the surface,

$$\text{viz.}\quad y = 0,\ z = 0;\quad z = 0,\ x = 0;\quad x = h,\ y = 0;$$

$$\text{and}\quad ay = \beta h,\ az = -\gamma h;\quad az = -\gamma h,\ x = h;\quad x = 0,\ ay = \beta h.$$

Conversely, to find the condition that three perpendicular lines may lie on the surface, take the axis of x parallel to one of them, then, for some values of y and z, the equation is true for all values of x, $\therefore$ $a = 0$, and similarly b and $c = 0$; $\therefore$ $I_1 = 0$.

ii. If $I_2 = 0$, and a, β, γ be the roots of the discriminating cubic, $a^{-1} + \beta^{-1} + \gamma^{-1} = 0$, which shews that the cone reciprocal to the cone $ax^2 + \beta y^2 + \gamma z^2 = 0$ has three perpendicular generating lines, and therefore that the original cone has three perpendicular tangent planes. The vanishing of this invariant shews that the cone $u_2 = 0$ has three perpendicular tangent planes.

Conversely, if the cone $u_2 = 0$ have three perpendicular tangent planes, taking these for coordinate planes, $z = 0$ makes $ax^2 + by^2 + 2c'xy$ a complete square; $\therefore$ $ab = c'^2$, and similarly $bc = a'^2$ and $ca = b'^2$; $\therefore$ $I_2 = 0$.

414. *To find the conditions that the discriminating cubic may have two equal roots,† or that the corresponding surface may be one of revolution.*

Let the roots of the cubic be a, β, γ, then since $s(x^2+y^2+z^2)-u_2$ is transformed to $s(x^2 + y^2 + z^2) - \{ax^2 + \beta(y^2 + z^2)\}$, and each becomes the product of two linear factors when s is a root of the cubic, $\beta(x^2 + y^2 + z^2) - u_2$ becomes by transformation $(\beta - a)x^2$, and is therefore a complete square, viz. $\{(\beta - a)x - c'y - b'z\}^2/(\beta - a)$, and equating the coefficients of yz, $-(\beta - a)a' = b'c'$;

$$\therefore\ \beta = a - b'c'/a' = b - c'a'/b' = c - a'b'/c',$$

which are the conditions when a', b', and c' are finite.

If $a' = 0$, then $b'c'$ must $= 0$, suppose a' and $b' = 0$, then $\beta = c$, and $(c - a)x^2 + (c - b)y^2 - 2c'xy$ must be a complete square;

$$\therefore\ c'^2 = (c - a)(c - b).$$

415. *To find the equations of the coordinate axes which make the terms in u_2 involving yz, zx, xy disappear.*

When u_2 has been reduced by transformation to $ax^2 + \beta y^2 + \gamma z^2$, one of the new axes is the intersection of the two planes whose

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* These were first noticed by Routh.

† This method is given in Wolstenholme's *Problems*, p. 354.

equation is, referred to the original axes,

$$u_2 - s(x^2 + y^2 + z^2) = 0, \text{ where } s = \alpha, \beta \text{ or } \gamma;$$

therefore, by Art. 92, the equations of the axes are found by writing α, β, γ successively for s in

$$x\{b'c' - (a-s)a'\} = y\{c'a' - (b-s)b'\} = z\{a'b' - (c-s)c'\}.$$

These equations do not give the position of the axes directly, if two of the three quantities a', b', c' vanish; but, if a', b' be the two which vanish, it is obvious, from the original equation, that the axis of z will be in the direction of one of the axes.

416. The method just described is the most direct for obtaining the necessary transformation of coordinates, but the following investigation of the position of the principal planes is very easy of application.

417. *To find the equation of the locus of middle points of a system of parallel chords of a central conicoid determined by the general equation,* $u_2 + d = 0$.

Let the equation of the conicoid be $f(x, y, z) = 0$, and let (λ, μ, ν) be the direction of the chords to be bisected, (ξ, η, ζ) the middle point of any chord.

Then the equation $f(\xi + \lambda r, \eta + \mu r, \zeta + \nu r) = 0$ must have its roots equal and of opposite signs.

This gives the condition $\lambda \dfrac{df}{d\xi} + \mu \dfrac{df}{d\eta} + \nu \dfrac{df}{d\zeta} = 0,$

$$\text{or } (a\xi + c'\eta + b'\zeta)\lambda + (c'\xi + b\eta + a'\zeta)\mu + (b'\xi + a'\eta + c\zeta)\nu = 0,$$

which is the equation of the diametral plane.

418. *To determine the principal planes of a central conicoid.*

A principal plane being perpendicular to the chords which it bisects, we shall have the direction-cosines given by the three equations

$$\begin{aligned} a\lambda + c'\mu + b'\nu &= s\lambda, \\ c'\lambda + b\mu + a'\nu &= s\mu, \\ b'\lambda + a'\mu + c\nu &= s\nu, \end{aligned} \qquad (1)$$

where s is a constant given by the cubic

$$(s-a)(s-b)(s-c) - a'^2(s-a) - b'^2(s-b) - c'^2(s-c) - 2a'b'c' = 0,$$

the discriminating cubic which has been already discussed. There are, in general, three and only three principal planes, since to each of the three values of s there corresponds one system of values of $\lambda : \mu : \nu$; namely, those given by

$$\{b'c' + a'(s-a)\}\lambda = \{c'a' + b'(s-b)\}\mu = \{a'b' + c'(s-c)\}\nu,$$

as in Art. 415.

If, as in the case of a surface of revolution, there are an infinite number of values of $\lambda : \mu : \nu$, we obtain from equations (1)

$$(a-s)/c' = c'/(b-s) = b'/a', \text{ and } c'/b' = (b-s)/a' = a'/(c-s);$$

$$\therefore\ s = a - b'c'/a' = b - c'a'/b' = c - a'b'/c', \text{ as in Art. 414.}$$

419. *To shew that the three principal planes of any conicoid are mutually at right angles.*

Let s_1, s_2, s_3 be the three roots of the discriminating cubic, and let the corresponding values of λ, μ, ν be denoted by the same suffixes; we shall then have

$$\begin{aligned} a\lambda_1 + c'\mu_1 + b'\nu_1 &= s_1\lambda_1, \\ c'\lambda_1 + b\mu_1 + a'\nu_1 &= s_1\mu_1, \\ b'\lambda_1 + a'\mu_1 + c\nu_1 &= s_1\nu_1. \end{aligned} \qquad (1)$$

Multiplying by λ_2, μ_2, ν_2 and adding, we obtain

$$(a\lambda_2 + c'\mu_2 + b'\nu_2)\lambda_1 + (c'\lambda_2 + b\mu_2 + a'\nu_2)\mu_1 + (b'\lambda_2 + a'\mu_2 + c\nu_2)\nu_1$$
$$= s_1(\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2);$$

$$\therefore\ s_2\lambda_2.\lambda_1 + s_2\mu_2.\mu_1 + s_2\nu_2.\nu_1 = s_1(\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2),$$

$$\text{whence } (s_2 - s_1)(\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2) = 0.$$

Hence, if two roots of the cubic be unequal the corresponding principal planes will be at right angles.

We may make use of equations (1) to shew that the equation of the surface referred to the principal planes as coordinate planes will be of the form $\alpha x^2 + \beta y^2 + \gamma z^2 + \delta = 0$, in which α, β, γ are the roots of the discriminating cubic, for, on transformation, the coefficient of x^2 will be

$$a\lambda_1^2 + b\mu_1^2 + c\nu_1^2 + 2a'\mu_1\nu_1 + 2b'\nu_1\lambda_1 + 2c'\lambda_1\mu_1 = s_1, \text{ by (1)},$$

and similarly, $\beta = s_2$, $\gamma = s_3$.

420. *To distinguish the surfaces represented by an equation for which the roots of the discriminating cubic are finite.*

In this case there is a centre at a finite distance, to which if the origin be transferred, the direction of a new system of axes can be chosen, Art. 409, such that the equation

$$u \equiv ax^2 + by^2 + cz^2 + dw^2$$
$$+ 2a'yz + 2b'zx + 2c'xy + 2a''xw + 2b''yw + 2c''zw = 0$$

will become by transformation $\alpha x^2 + \beta y^2 + \gamma z^2 + \delta w^2 = 0$, w being written for the unit.

The transformation will be effected by substituting $lx + my + nz + \xi w$ for x, and similar expressions for y and z, w being unchanged; the discriminants, being invariants, Art. 397, are therefore equal, since the modulus of transformation $= 1$; $\therefore H(u) = \alpha\beta\gamma\delta$, and the transformed equation is $\alpha x^2 + \beta y^2 + \gamma z^2 = -H(u)/\alpha\beta\gamma$.

Hence, we have the following table for the case in which $\alpha\beta\gamma$ or Δ is finite, and $\alpha > \beta > \gamma$, by which it may be seen how the

loci are distinguished:

α	β	γ	$H(u)$	
+	+	+	−	Ellipsoid
+	+	−	+	Hyperboloid, one sheet
+	−	−	−	Hyperboloid, two sheets
+ +	+ −	− −	0	Cone, real
+	+	+	0	Cone, imaginary, or point
+	+	+	+	Imaginary locus

In order that α, β, and γ may be all positive, $a+b+c$ and Δ must be positive. If the locus be a point, or rather an indefinitely small ellipsoid, the section of $u_2 = 0$ by each coordinate plane must be a point-ellipse; therefore each of the quantities $bc - a'^2$, $ca - b'^2$, and $ab - c'^2$ must be positive.

421. *To distinguish the surfaces represented by an equation, for which one of the roots of the cubic vanishes, and the centre is single and at an infinite distance.*

The conditions that the centre may be at an infinite distance are that $\Delta = 0$, and that one or more of the three quantities below shall be finite,

$$a''A + b''C' + c''B',$$
$$a''C' + b''B + c''A',$$
$$a''B' + b''A' + c''C.$$

The surfaces will be the elliptic or hyperbolic paraboloid, according as the roots of $h^2 - (a+b+c)h + A + B + C = 0$ have the same or opposite signs, *i.e.* as $A+B+C$ is + or −; but, by Art. 391, A, B, C have the same sign, hence

A, B, and C + gives an elliptic paraboloid,

A, B, and C − „ hyperbolic paraboloid.

422. *To distinguish the surfaces represented by the general equation when there is a line of centres at a finite distance.*

The conditions that there may be a line of centres are $\Delta = 0$ and $H(u) = -\{a''\sqrt{A} + b''\sqrt{B} + c''\sqrt{C}\}^2 = 0$; or, if A', B', C' be finite, $a''/A' + b''/B' + c''/C' = 0$. The equations of the line of centres are $A'\xi - a'a'' = B'\eta - b'b'' = C'\zeta - c'c'' = \rho$ suppose; therefore, if we transfer the origin to any point in the line of centres defined by some value of ρ, the equation of the surface will become

$$u_2 + a''\xi + b''\eta + c''\zeta + d = 0,$$

$$\text{or } u_2 + a'a''^2/A' + b'b''^2/B' + c'c''^2/C' + d = 0,$$

since the coefficient of ρ vanishes.

If a', b', c' be all finite, A', B', C' will be so also, but if a', for instance, vanish, the other two being finite, A' will be finite, and, by Art. 391, if $B'=0$, then $A=0$, and $C'=0$, and b and c vanish; hence, recurring to the original equations for determining the centre, we easily obtain the equation $u_2 - 2a''b''/c' + ab'''^2/c'^2 + d = 0$, and the condition $b'c''=c'c''$.

If two roots of the discriminating cubic be finite, since u_2 is reducible to the form $\beta y^2 + \gamma z^2$, the surfaces represented by the equation will be in the general case in which

i. $a'a'''^2/A' + b'b'''^2/B' + c'c'''^2/C' + d$ is finite,

A, B, and $C+$, an elliptic cylinder,

A, B, and $C-$, a hyperbolic cylinder;

ii. when $a'a'''^2/A' + \ldots + d = 0$,

A, B, and $C+$, a line cylinder,

A, B, and $C-$, two intersecting planes.

If only one root be finite, u_2 is reducible to γz^2, but in this case, since $A+B+C=0$, A, B, C being all of the same sign, must vanish separately, from which it follows that A', B', C' also vanish, and there cannot be a line of centres at a finite distance.

423. *To distinguish the surfaces when there is a line of centres at an infinite distance.*

In this case $A'=B'=C'=0$; therefore $A=B=C=0$; two of the roots of the discriminating cubic must therefore vanish; also $a'a''$, $b'b''$, $c'c''$ must not be all equal.

Since $aa'=b'c'$, &c., u_2 can be put into the form

$$a'b'c'(x/a' + y/b' + z/c')^2,$$

and the only surface represented is a parabolic cylinder.

424. *To distinguish the surfaces for which there is a plane of centres.*

In this case, as in the last, the minors all vanish, and we have in addition $a'a''=b'b''=c'c''$; the equation may therefore be written

$$a'b'c'(x/a' + y/b' + z/c')^2 + 2a'a''(x/a' + y/b' + z/c') + d = 0.$$

The surface represented consists of two parallel planes unless $a'b'c'd = a'^2a''^2$, or $d = a''^2/a = b''^2/b = c''^2/c$, in which case they are coincident.

One of the planes will be at an infinite distance if a, b, c, a', b', c' all vanish, while one at least of the other quantities remains finite.

425. The results in the general case may be tabulated as follows, if v be written for $a'a'''^2/A' + b'b'''^2/B' + c'c'''^2/C' + d$, and v'

for $(a'a''-b'b'')^2+(b'b''-c'c'')^2$, where f denotes 'finite' and α, β, γ are the roots of the discriminating cubic, $\alpha>\beta>\gamma$.

Δ	α	β	γ	$H(u)$	A, B, C	v	v'	
+	+	+	+	−				Ellipsoid
−	+	+	−	+				Hyperboloid, one sheet
+	+	−	−	−				Hyperboloid, two sheets
− +	+ +	+ −	− −	0				Cone, real
+	+	+	+	0	+			Cone, imaginary, or point-ellipsoid
0	0	+	+	f	+			Paraboloid, elliptic
0	0	+	−	f	−			Paraboloid, hyperbolic
0	0	+	+	0	+	f		Cylinder, elliptic
0	0	+	+	0	+	0		Cylinder, line
0	0	+	−	0	−	f		Cylinder, hyperbolic
0	0	+	−	0	−	0		Planes, intersecting
0	0	0	+	0	0		f	Cylinder, parabolic
0	0	0	+	0	0		0	Planes, parallel

For coincident planes $d=a''^2/a=b''^2/b=c''^2/c$.

For two planes, one at an infinite distance, $a, b, c, a', b', c'=0$, one at least of a'', b'', c'' finite.

For two planes at an infinite distance, d alone finite.

426. *Processes for finding the locus of any given equation.*

When a particular equation of the second degree is presented to us, in order to discover what species of surface it represents, we would recommend the student first to form the discriminating cubic, and it will then be seen whether the last term Δ vanishes or not.

I. If Δ be different from zero, we must find the centre, transfer the origin to it, and by changing the directions of the axes reduce the equation to the form $\alpha x^2+\beta y^2+\gamma z^2+\delta=0$, where α, β, γ are the roots of the discriminating cubic, which can always be found approximately, at all events their signs can be determined by Des Cartes' rule; and δ has been shewn to be $H(u)/\Delta$, or in particular cases may be found more easily without the use of the determinants.

II. If $\Delta=0$, and $A+B+C$ be not zero, in which case the two roots β, γ will be finite, it will be best to determine the directions of the axes which correspond to the three roots 0, β, γ, and to suppose the origin so chosen that the equation becomes either $\beta y^2+\gamma z^2+2a''x=0$, (1), or $\beta y^2+\gamma z^2+\delta=0$, (2). If we do not require the position of the vertex, the absolute value of a'' in (1) can be found by equating the discriminants, by which we obtain

$$\beta\gamma a''^2=-(a''A+b''C'+c''B')^2/A,$$

and if we find that a'' vanishes, we take case (2).

If more particulars be required it is perhaps the simplest to transfer to some point (ξ, η, ζ), viz. the vertex in (1) or a point on the axis in (2), in which cases $F(x+\xi, y+\eta, z+\zeta)\equiv u_2+2a''(lx+my+nz)$ or $u_2+\delta$, (l, m, n)

being the direction of the new axis of x. We thus have equations

$$a\xi + c'\eta + b'\zeta + a'' = l\iota'', \text{ or } 0, \text{ for (1) or (2)},$$
$$c'\xi + b\eta + a'\zeta + b'' = m\alpha'', \text{ or } 0,$$
$$b'\xi + a'\eta + c\zeta + c'' = n\alpha'', \text{ or } 0.$$

In case (1) $a''\xi + b''\eta + c''\zeta + d + (l\xi + m\eta + n\zeta)\alpha'' = 0$, since (ξ, η, ζ) is the vertex, and therefore on the surface;

$$\therefore\ a''A + b''C' + c''B' = (lA + mC' + nB')\alpha'',$$

whence the latera recta $-2\alpha''/\beta$ and $-2\alpha''/\gamma$ are known, and the fourth equation with two of the former give the vertex.

In case (2) $f(\xi, \eta, \zeta) \equiv a''\xi + b''\eta + c''\zeta + d = \delta$, and, since $a''A + b''C' + c''B' = 0$, the first three equations are equivalent to two, giving the axis of the cylinder; and the fourth equation combined with two of the first three gives, for determining δ,

$$\delta(a'c' - bb') = b''(b'b'' - a'a'') + c''(b'a'' - c'b'') + d(a'c' - bb').$$

III. If $\Delta = 0$, and $A + B + C = 0$, by Art. 422, $aa' = b'c'$, &c., and the original equation will appear in the form

$$a'b'c'(x/a' + y/b' + z/c')^2 + 2(a''x + b''y + c''z) + d = 0.$$

i. If $a'a''$, $b'b''$, $c'c''$ are equal the surface is obviously two parallel planes.

ii. If $a'a''$, $b'b''$, $c'c''$ are not all equal the surface is a parabolic cylinder.

When the line of vertices and latus rectum of the principal parabolic section are required, let (l, m, n), (l', m', n') be the directions of the new axes of x and z, in order that the equation may be reduced to the form $\gamma z^2 + 2\alpha''x = 0$, and, as in case II, transferring the origin to one of the vertices (ξ, η, ζ), u becomes

$$\gamma(l'x + m'y + n'z)^2 + 2\alpha''(lx + my + nz),$$

and it can be shewn, as before, that

$$\alpha''^2 = a''^2 + b''^2 + c''^2 - (a''l' + b''m' + c''n')^2,$$

and that the locus of the vertices is given by

$$\gamma(l'\xi + m'\eta + n'\zeta) + a''l' + b''m' + c''n' = 0,$$
$$2(a''\xi + b''\eta + c''\zeta) + (a''l' + b''m' + c''n')^2/\gamma + d = 0,$$
$$\text{where } l'a' = m'b' = n'c' = \sqrt{\{a'b'c'/(a + b + c)\}}.$$

IV. The condition for a surface of revolution being

$$a - b'c'/a' = b - c'a'/b' = c - a'b'/c' = \beta,$$

the equation assumes the form

$$\beta(x^2 + y^2 + z^2) + a'b'c'(x/a' + y/b' + z/c')^2 + 2a''x + 2b''y + 2c''z + d = 0,$$

which may be written

$$\beta\{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2\} + a'b'c'(x/a' + y/b' + z/c' + \sigma)^2 = 0, \quad (1)$$
$$\text{if } -\beta\xi + b'c'\sigma = a'', \ -\beta\eta + c'a'\sigma = b'', \ -\beta\zeta + a'b'\sigma = c'',$$
$$\text{and } \beta(\xi^2 + \eta^2 + \zeta^2) + a'b'c'\sigma^2 = d,$$

eliminating ξ, η, and ζ, we have a quadratic in σ,

$$\text{and } a'(\beta\xi + a'') = b'(\beta\eta + b'') = c'(\beta\zeta + c'') = a'b'c'\sigma,$$

which determine the coordinates of two points which are, as the form (1) shews, the foci of the generating curve.

427. It should be noticed that if the general form only of the surface be required, this can always be found by the simple process of expressing u_2 in the form of three, two, or one square, in which cases the equation appears in one of the forms $u^2 \pm v^2 \pm w^2 = \delta$, $u^2 \pm v^2 = w$, or $u^2 = w$, when u, v, w are linear expressions.

For example, $au_2 \equiv (ax + c'y + b'z)^2 + Cy^2 - 2A'yz + Bz^2$, where $Cy^2 - 2A'yz + Bz^2$ may vanish, or be a square, or the sum of two squares.

428. *To find the surface whose equation is*

$$32x^2 + y^2 + z^2 + 6yz - 16zx - 16xy - 6x - 12y - 12z + 18 = 0.$$

The discriminating cubic is

$$(s - 32)(s - 1)^2 - 9(s - 32) - 128(s - 1) - 6.64 = 0,$$

the last term of which is 0 and the roots 0, 36, -2.

The directions of the axes corresponding to these roots are given by the equations of Art. 415, viz.

$$\{64 + 3(s - 32)\}\, x = \{-24 - 8(s - 1)\}\, y = \{-24 - 8(s - 1)\}\, z,$$

which give $2x = y = z$; $x = -4y = -4z$; $x = 0$, $y = z$; the direction-cosines being $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$; $\frac{2}{3}\sqrt{2}, -\frac{1}{6}\sqrt{2}, -\frac{1}{6}\sqrt{2}$; $0, \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}$.

The equation is reducible to $36y^2 - 2z^2 + 2a''x = 0$, since the centre is at an infinite distance, and $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ being the direction of the axis of x, the equation is, when transformed to axes parallel to the original directions,

$$u_2 + \tfrac{2}{3}a''(x + 2y + 2z) = 0 \quad \text{or} \quad F(x + \xi, y + \eta, z + \zeta) = 0,$$

where (ξ, η, ζ) is the vertex of the paraboloid,

$$\begin{aligned} \therefore\ \ 32\xi - 8\eta - 8\zeta - 3 &= \tfrac{1}{3}a'' \\ -8\xi + \eta + 3\zeta - 6 &= \tfrac{2}{3}a'' \\ -8\xi + 3\eta + \zeta - 6 &= \tfrac{2}{3}a'' \\ -3\xi - 6\eta - 6\zeta + 18 &= 0; \end{aligned}$$

$$\therefore\ \ a'' = -9 \ \text{ and } \ 2\xi = \eta = \zeta = \tfrac{2}{3}.$$

429. *To find the surface whose equation is*

$$x^2 - 2y^2 + 2z^2 + 3zx - xy - 2x + 7y - 5z - 3 = 0.$$

As an illustration of Art. 427, the form of the surface may be found by arranging the terms of the second degree in the form of squares

$$x^2 + (3z - y)\, x + \tfrac{1}{4}(3z - y)^2 - \tfrac{1}{4}(z^2 - 6yz + 9y^2),$$

and those of the first degree being

$$-2(x - \tfrac{1}{2}y + \tfrac{3}{2}z) - 2(z - 3y),$$

$$(x + \tfrac{1}{2}y + \tfrac{3}{2}z - 1)^2 - (\tfrac{1}{2}z - \tfrac{3}{2}y + 2)^2 = 0.$$

The surface represented is the two planes

$$(x - 2y + 2z + 1)(x + y + z - 3) = 0.$$

By the general method, the discriminating cubic is $s(s^2 - s - 13/2) = 0$, and the centre has a locus $\xi = -4\eta + 7 = \frac{1}{3}(-4\zeta + 5)$, the equation may therefore be reduced to $\beta y^2 + \gamma z^2 + \delta = 0$, and referring to axes through (ξ, η, ζ) parallel to the original axes this is identical with $F(x + \xi, y + \eta, z + \zeta) = 0$; hence, equating coefficients, we find that $\delta = -\xi + \frac{7}{2}\eta - \frac{5}{2}\zeta - 3 = 0$. The surface is therefore two planes, since β, γ have opposite signs.

430. *To find the equation of any confocal of a central conicoid given by the general equation.*

Let the general equation be $u_2 = 1$, and when the coordinate axes are transformed to the principal axes, let u_2 become $x^2/\alpha + y^2/\beta + z^2/\gamma$, the equation of any confocal must become $x^2/(\alpha + k) + y^2/(\beta + k) + z^2/(\gamma + k) = 1$,

$$\text{or } x^2(\beta + k)(\gamma + k) + y^2(\gamma + k)(\alpha + k) + z^2(\alpha + k)(\beta + k) = (\alpha + k)(\beta + k)(\gamma + k).$$

The invariants I_1, I_2, Δ of the quadric u_2 are the coefficients of k, k^2, and k^3 in $(1 + k/\alpha)(1 + k/\beta)(1 + k/\gamma)$ and $(\beta + k)(\gamma + k) = \alpha\beta\gamma/\alpha + (\alpha + \beta + \gamma)k + k^2 - k\alpha$, hence, the equation of the confocal, referred to the principal axes, is

$$x^2/\alpha + y^2/\beta + z^2/\gamma + (I_2 k + \Delta k^2)(x^2 + y^2 + z^2) - \Delta k(\alpha x^2 + \beta y^2 + \gamma z^2) = 1 + I_1 k + I_2 k^2 + \Delta k^3,$$

and referring back to the original axes, by Art. 401,

$$u_2 + (I_2 k + \Delta k^2)(x^2 + y^2 + z^2) - k(Ax^2 + \ldots + 2A'yz + \ldots) = 1 + I_1 k + I_2 k^2 + \Delta k^3.$$

431. *To find the focal conics of the conicoid given by the general equation.*

The focal conic corresponding to $k=-\gamma$ may be obtained by writing $-\gamma+\sigma$ for k in the equation

$$x^2(\beta+k)(\gamma+k)+y^2(\gamma+k)(\alpha+k)+z^2(\alpha+k)(\beta+k)=(\alpha+k)(\beta+k)(\gamma+k),$$

and equating to zero the coefficients of σ and the term independent of σ, hence the equations of the focal conic are, since $1-I_1\gamma+I_2\gamma^2-\Delta\gamma^3=0$,

$$u_2-(I_1-\gamma^{-1})(x^2+y^2+z^2)+\gamma(Ax^2+\ldots+2A'yz+\ldots)=0,$$

$$\text{and } (I_2-2\Delta\gamma)(x^2+y^2+z^2)-Ax^2-\ldots-2A'yz-\ldots=I_1-2\gamma I_2+\Delta\gamma^2.$$

XXIX.

(1) Find the nature of the surfaces represented by the following equations:

1. $3z^2-x^2-y^2+4xy=a^2$.
2. $yz+zx+xy=a^2$.
3. $x^2+y^2+z^2+2xy+2yz+4zx=1$.
4. $x^2+y^2+2(xy+yz+zx)=a^2$.
5. $2z^2-5x^2-2y^2+10xy+4yz+4y+16z+16=0$.
6. $x^2+2(yz+zx+xy)+2(z-y-1)=0$.
7. $x^2+y^2+3z^2+3yz+zx+xy-7x-14y-25z+d=0$.
8. $5y^2-2x^2-z^2-4xy-6yz+8zx=1$.

Prove the following results:

1. Hyperboloid of one sheet. 2. Hyperboloid of revolution, eccentricity of generating hyperbola $=\sqrt{\frac{3}{2}}$. 3. Hyperboloid of one sheet, axes $\sqrt{6}\pm\sqrt{2}$ and 2; direction-cosines of axes in the ratios $1, \pm\sqrt{3}-1, 1$ and $1, 0, 1$. 4. Hyperbolic cylinder. 5. Hyperboloid of one sheet, centre $(6, 6, -10)$. 6. Cone, direction-cosines of axes $(0, \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$, $(\pm\frac{1}{2}\sqrt{2}, \frac{1}{2}, \frac{1}{2})$. 7. Ellipsoid, point or impossible, according as $d <=> 55$. 8. Hyperbolic cylinder.

(2) The equation $(cy-bz)^2+(az-cx)^2+(bx-ay)^2=1$ represents a right circular cylinder, the equations of whose axis are $x/a=y/b=z/c$.

(3) The surface represented by the equation

$$a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy=1$$

is a hyperboloid of one sheet, and the sum of the squares on its real axes is equal to the square on its conjugate axis.

(4) Prove that, when $bb'=c'a'$, and $cc'=a'b'$, the equation

$$ax^2+\ldots+2a'yz+\ldots+2a''x+\ldots+d=0,$$

represents in general a paraboloid, whose axis is parallel to the straight line $x=0$, $c'y+b'z=0$.

(5) The equation

$$ax^2+4y^2+9z^2+12yz+6zx+4xy+2a''x+2b''y+2c''z+d=0$$

will in general represent an elliptic paraboloid, a parabolic cylinder, or a hyperbolic paraboloid, according as $a >=< 1$. What surfaces will it represent in the following cases:

i. $3b''=2c''$, $a >=$ or <1. ii. $6a''=3b''=2c''$, $a=1$.

(6) Find the equation of the hyperboloid, three of whose generating lines are $x=0$, $y=a$; $y=0$, $z=a$; and $z=0$, $x=a$; shew that it is a surface of revolution, and find the eccentricity of a meridian section.

(7) Find the position and magnitude of the axes of $yz+\sqrt{3}(z-y)x=1$, and shew that the eccentricities of the focal conics are reciprocals of each other.

(8) Find the locus of the centres of the surfaces represented by the equation $x^2+y^2-z^2+2pzx+2qyz-2ax-2by+2cz=0$, a, b, c being given positive numbers, and p and q variable parameters. i. When p and q vary in every possible manner. ii. When they vary in such a manner that the equation may always represent a cone.

(9) If the equation $ax^2+by^2+cz^2+2b'zx+2c'xy+2a''x=0$ represent a paraboloid of revolution, prove that $c=b\pm a$. Taking the upper sign, prove that the axis is $z=0$, $x\sqrt{a}+y\sqrt{b}=0$.

XXX.

(1) If $ax^2+by^2+cz^2+2a'yz+2b'zx+2c'xy=0$ represent two planes, shew that the two planes which bisect the angles contained by the former planes will be given by

$$\begin{vmatrix} x, & ax+c'y+b'z, & (aa'-b'c')^{-1} \\ y, & c'x+by+a'z, & (bb'-c'a')^{-1} \\ z, & b'x+a'y+cz, & (cc'-a'b')^{-1} \end{vmatrix}=0.$$

(2) Prove that the direction-cosines of the axis corresponding to the root α of the discriminating cubic can be put in the symmetrical form

$$(\alpha-b)(\alpha-c)-a'^2=(\alpha-\beta)(\alpha-\gamma)l^2, \text{ \&c.}$$

(3) If the surface represented by the general equation be a hyperbolic paraboloid, prove that the tangent of the angle between the planes to which the generators are parallel will be $2\sqrt{(a'^2+b'^2+c'^2-ab-ac-bc)}/(a+b+c)$.

(4) $u=0$ is the equation of a conicoid in rectangular coordinates, shew that the equation of the locus of points, from which three tangent lines mutually at right angles can be drawn, is

$$\frac{d^2u^{\frac{1}{2}}}{dx^2}+\frac{d^2u^{\frac{1}{2}}}{dy^2}+\frac{d^2u^{\frac{1}{2}}}{dz^2}=0.$$

(5) The equation $a(y-z)^2+b(z-x)^2+c(x-y)^2=d$ represents a cylinder, which is hyperbolic when $bc+ca+ab$ is negative; and which, when $bc+ca+ab$ is positive and less than $\frac{1}{3}(a+b+c)^2$, is elliptic or impossible according as $a+b+c$ is positive or negative; if $a+b+c=0$, the principal section will be a rectangular hyperbola.

(6) The equation $ax^2+by^2+cz^2+2a'yz+2b'zx+2c'xy=0$ will represent a right cone, whose vertical angle is θ, if

$$a-b'c'/a'=b-c'a'/b'=c-a'b'/c'=(a+b+c)(1+\cos\theta)/(1+3\cos\theta).$$

(7) If a cone whose vertex is the origin and base a plane section of the surface $ax^2+by^2+cz^2=1$ be a cone of revolution, the plane must touch one of the cylinders

$$(b-a)y^2+(c-a)z^2=1,\quad (c-b)z^2+(a-b)x^2=1,\quad (a-c)x^2+(b-c)y^2=1.$$

(8) The area of the section of the conicoid

$$ax^2+by^2+cz^2+2a'yz+2b'zx+2c'xy=1$$

by a plane through the extremities of its principal axes is

$$\frac{2\pi}{3\sqrt{3}}\sqrt{\left(\frac{a+b+c}{abc+2a'b'c'-aa'^2-bb'^2-cc'^2}\right)}.$$

(9) The surface whose equation, referred to axes inclined at angles α, β, γ, is $ayz+bzx+cxy=1$, will be one of revolution if $(\pm1+\cos\alpha)/a=(\pm1+\cos\beta)/b=(\pm1+\cos\gamma)/c$, one or three of the ambiguities being taken negative.

(10) Apply the methods of Art. 409 and 415 to find the directions of the axes of a cone enveloping an ellipsoid; and, independently, of the reciprocal cone, Art. 402.

XXXI.

(1) Discuss the different surfaces represented by the equation

$$x^2 + (2m^2 + 1)(y^2 + z^2) - 2(yz + zx + xy) = 2m^2 - 3m + 1,$$

as m varies from $-\infty$ to $+\infty$; considering particularly the critical values -1, $\frac{1}{2}$, and 1.

(2) The radius r of the central circular sections of the surface $ayz+bzx+cxy=1$ is given by the equation $abcr^6 + (a^2+b^2+c^2)r^4 = 4$; and the direction-cosines of the sections by the equations $(m^2+n^2)\,l/a = (n^2+l^2)\,m/b = (l^2+m^2)\,n/c = -lmnr^2$.

(3) The equation of every surface of revolution of the second degree which passes through the two lines $x = 0$, $y = 0$, and $y = a$, $z = 0$ is included in

$$\cos\theta\,(y^2 - ay \mp xz) + \sin\theta\,\{yz \pm x(y-a)\} = xz.$$

(4) The surface whose equation, referred to axes inclined at angles λ, μ, ν, is $ax^2 + by^2 + cz^2 = 1$, will be one of revolution if

$$a\cos\lambda/(\cos\lambda - \cos\mu\cos\nu) = b\cos\mu/(\cos\mu - \cos\nu\cos\lambda) = c\cos\nu/(\cos\nu - \cos\lambda\cos\mu).$$

(5) The equation $ax^2 + by^2 + cz^2 = 1$, may, when referred to oblique axes, be transformed into the equation $2m(yz + zx + xy) = 1$ in an infinite number of ways. If a', b', c' be the cosines of the angles between the axes, shew that

$$m/a + m/b + m/c = a' + b' + c' - \tfrac{3}{2},$$

$$m^2(a+b+c)/abc = b'c' + c'a' + a'b' - a' - b' - c',$$

$$\text{and } 2m^3/abc = 1 - a'^2 - b'^2 - c'^2 + 2a'b'c'.$$

If the oblique axes be mutually inclined at angles of 60°, shew that either $-2a = b = c$, $a = -2b = c$, or $a = b = -2c$.

(6) Shew that the hyperboloid, whose equation, referred to oblique axes inclined at angles whose cosines are a', b', c', is

$$(1-a')\,yz + (1-b')\,zx + (1-c')\,xy = d,$$

is a hyperboloid of revolution, whose equation, referred to its principal axes, is $ax^2 - y^2 - z^2 = 2d$, where

$$a = 2(1-a')(1-b')(1-c')/(1 - a'^2 - b'^2 - c'^2 + 2a'b'c').$$

(7) If two conicoids $2ayz + 2bzx + 2cxy = 1$ and $2a'yz + 2b'zx + 2c'xy = 1$ can be placed so as to be confocal,

$$abc/(a^2+b^2+c^2) + a'b'c'/(a'^2+b'^2+c'^2) = 0,$$

$$\text{and } 27a^2b^2c^2/(a^2+b^2+c^2)^3 + a'^2b'^2c'^2/(a'^2+b'^2+c'^2)^3 = 1.$$

(8) When the general equation represents a paraboloid of revolution, its focus and directrix plane are given by the equations

$$a'(sx + a'') = b'(sy + b'') = c'(sz + c'') = \tfrac{1}{2}(a''^2 + b''^2 + c''^2 - sd)/(a''/a' + b''/b' + c''/c'),$$

and

$$2(a''/a' + b''/b' + c''/c')(x/a' + y/b' + z/c') + (a''^2 + b''^2 + c''^2 - ds)/a'b'c' = 0.$$

where $2s = a + b + c$, and a', b', c' are finite.

(9) Three fixed rectangular axes are taken, and a fixed line through the origin whose direction-cosines are λ, μ, ν. If any rigid surface be turned about this line through an angle θ, the equation of such a surface in its new position may be derived from its equation in the old one by changing x into $x\cos\theta + \lambda(\lambda x + \mu y + \nu z)(1 - \cos\theta) + (\mu z - \nu y)\sin\theta$, with similar changes for y and z.

Hence, defining an axis of a conicoid as a diameter, such that by revolution about it through two right angles every point of the surface returns to the surface again, deduce the ordinary cubic equation for the determination of the axes.

CHAPTER XVII.

DEGREES AND CLASSES OF SURFACES. DEGREES OF CURVES AND TORSES. COMPLETE AND PARTIAL INTERSECTIONS OF SURFACES.

432. Having already fully investigated the nature of the surfaces represented by the general equation of the second degree, we will proceed to the loci of equations of higher degrees, which we may consider as equations either in three-plane or four-plane coordinates: in the latter case we may suppose the equations homogeneous, without loss of generality.

433. Surfaces which are represented by rational and integral algebraical equations are arranged according to the degrees of these equations when plane coordinates are used, and according to classes when tangential or point coordinates are used.

A surface is of the n^{th} *degree* when the equation of which it is the locus is of the n^{th} degree in the coordinates of any point of the locus; the geometrical equivalent being that a surface is of the n^{th} degree when an arbitrary straight line intersects it in n points, real or imaginary.

A surface is of the n^{th} *class* when n tangent planes, real or imaginary, can be drawn to it through an arbitrary straight line.

If p', q', r', s' and p'', q'', r'', s'' be the point coordinates of two planes, the coordinates of any plane passing through their line of intersection will be $lp' + mp''$, $lq' + mq''$..., Art. 128, $l : m$ being an arbitrary ratio, and the particular planes which touch a surface whose tangential equation is $F(p, q, r, s) = 0$, supposed a homogeneous algebraical equation of the n^{th} degree, will be determined by the values of $l : m$ which satisfy the equation $F(lp' + mp'', \ldots) = 0$; the number of values of the ratio will be n, and this will therefore be the class of the surface.

434. Curves and Torses are arranged according to their degrees.

A curve of the n^{th} degree is one which intersects an arbitrary plane in n points, real or imaginary.

A torse of the n^{th} degree is one to which n tangent planes, real or imaginary, can be drawn through an arbitrary point.

435. Among the various methods of treating of curves which have been proposed, one is to consider them as the intersection of surfaces whose equations are given. In this method the difficulty arises, to which allusion has been made in Art. 14, viz. that

extraneous curves may be introduced which are not the subjects of investigation.

If any curve be supposed to be given in space, it is impossible generally to determine two surfaces which shall contain no other common points but points which lie on the proposed curve; but among all the surfaces which may be drawn through a curve, it is desirable to obtain the simplest forms of surfaces of which the curve shall be the partial intersection.

436. The number of points in which three surfaces intersect, which are of the m^{th}, n^{th}, and p^{th} degrees respectively, is mnp, unless they intersect in a common curve, in which case it is infinite.

For the proof of this proposition, the student is referred to Salmon's *Treatise on Higher Algebra*, Lesson VIII., on the number of solutions of three equations in three unknown quantities.

The student may be able to satisfy himself of the truth of the proposition, by considering that the number of points in which the surfaces intersect will, by the law of continuity, be unaltered, if we substitute particular instead of the general forms of the surfaces. If the surfaces respectively consist of m, n, p arbitrary planes, it is obvious that the number of their common points of intersection will be mnp, each point being the intersection of three planes, taken one from each system.

437. *The complete intersection of two surfaces of the m^{th} and n^{th} degrees respectively, is a curve of the mn^{th} degree.*

Let a plane intersect the surfaces, the number of points of intersection of the plane with the surfaces is mn, this is therefore the number of points in which the plane cuts the curve, and the curve is of the mn^{th} degree.

438. *To find the number of conditions which a surface of the n^{th} degree may be made to satisfy.*

The number of constants in the general equation of the n^{th} degree is evidently the number of homogeneous products of four things of n dimensions, and is therefore $(n+1)(n+2)(n+3)/6$; but in estimating the number of constants with reference to the number of conditions which the locus can be made to satisfy, we must diminish this number by one, since the generality of the equation is unaltered if we divide by any one of the constants.

The number of disposable constants, so obtained, is

$$\tfrac{1}{6}(n+1)(n+2)(n+3)-1 \equiv \tfrac{1}{6}n(n^2+6n+11) \equiv \phi(n),$$

thus $\phi(2)=9$, $\phi(3)=19$, $\phi(4)=34$, $\phi(5)=55$, $\phi(6)=83$.

Since, when a point is given, we may substitute its coordinates in the general equation of a given degree, and thus obtain a linear equation of condition between the constants, a surface of the third degree may be made to pass through 19 arbitrarily chosen points, and one of the fourth through 34, &c., and $\phi(n)$ arbitrarily chosen

points will completely determine the position and dimensions of a surface of the n^{th} degree.

A surface of the n^{th} degree is also determined by $\phi(n)$ independent linear equations of any kind between its coefficients.

439. *All surfaces of the n^{th} degree which pass through $\phi(n)-1$ given points have a common curve of intersection.*

If $u=0$, $v=0$ be the equation of two surfaces passing through the given points, $\lambda u+\mu v=0$ will be the equation of another surface of the n^{th} degree which passes through the $\phi(n)-1$ given points; and since, by giving proper values to the ratio $\lambda:\mu$, this surface may be made to pass through any additional point which is not common to the two surfaces $u=0$, $v=0$, this equation will be the general equation of all surfaces which contain the $\phi(n)-1$ given points. But this equation is also satisfied by the coordinates of all points which lie on the curve of intersection of $u=0$ and $v=0$; this curve, which is of the degree n^2, is therefore the common curve of intersection of all surfaces containing the $\phi(n)-1$ points.

440. By reasoning similar to the above, it can be seen that, if a surface be of such a nature that m points or m linear equations of condition completely determine it, we may assert, that if $m-1$ such conditions be given, all surfaces of this kind which satisfy these conditions will have a common curve of intersection.

441. We shall give the name of *cluster* to the series of surfaces of the n^{th} degree determined by the equation $\lambda u+\mu v=0$, and call the curve of the degree n^2, which is the common intersection of the surfaces, the *base* of the cluster.

We have adopted *cluster* as the equivalent of the term *faisceau* used by French writers.

442. We may remark that, if $\phi(n)-1$ points be given it will be possible to eliminate from the general equation of the surface of the n^{th} degree all the constants but one, which will enter into the resulting equation in the first power only. This equation will then be of the form $u+\lambda v=0$, where u, v are of the n^{th} degree, and λ an undetermined constant. All surfaces represented by this equation will pass through the curve given by the equations $u=0$, $v=0$, which curve is therefore completely determined. For example, eight points determine a curve which is the complete intersection of two conicoids.

In the case of complete intersections of surfaces the nature of the curve is not given when the degree is given, except in the case of prime numbers, when it must be a plane curve.

For example, a curve of the twelfth degree might be the complete intersection of pairs of surfaces of the degrees (1, 12), (2, 6), (3, 4), and these different species, belonging to the same degree,

would require a different number of given points to determine completely the surfaces.

The following proposition serves to obtain the number of given points sufficient to determine a surface of the n^{th} degree which, by its complete intersection with a surface of a lower degree, gives a curve of the nq^{th} degree: this is given by Plücker, but may also be proved directly by a theorem given by Cayley.*

443. *All surfaces of the n^{th} degree containing $\phi(n) - \phi(n-q) - 1$ given points of a surface of the q^{th} degree cut this last surface in one and the same curve.*

Of $\phi(n) - 1$ points, $\phi(n-q)$ lie on a surface of the p^{th} degree, where $n = p + q$, whose equation is $u_p = 0$, and if $\phi(n) - 1 - \phi(p)$ given points lie on a surface of the q^{th} degree, $u_q = 0$, $u_p u_q = 0$ is one of the surfaces of the n^{th} degree which contain $\phi(n) - 1$ points. Therefore, if $u = 0$, $v = 0$ be any two surfaces containing these $\phi(n) - 1$ points, since $u_p u_q = 0$ is one of the cluster of surfaces represented by $\lambda u + \mu v = 0$, the base of the cluster will be a curve which lies on the two surfaces $u_p = 0$ and $u_q = 0$.

Hence if $\phi(n) - \phi(n-q) - 1$ points be taken on any fixed surface $u_q = 0$, all surfaces of the n^{th} degree, which pass through these points, will intersect the surface of the q^{th} degree in the same curve.

Thus, if $q = 2$, the proposition is reduced to the following:

All surfaces of the n^{th} degree which pass through $n(n+2)$ points on a conicoid intersect the conicoid in the same curve.

444. When it is said that a curve or surface is determined by a certain number of points, these points must be supposed arbitrarily taken, for it is possible so to select the points that this number would not be sufficient. Thus, a plane cubic is generally determined by 9 points, but, if those be the nine points of intersection of two of such curves, an infinite number may be drawn through them. A curve of the fourth degree of one species, namely the intersection of two conicoids, can be determined completely by 8 arbitrary points, but if these given points be the intersections of three conicoids which have not a common curve of intersection, taking these surfaces two and two, we may obtain three curves of that species passing through the same eight points.

445. If two surfaces of the n^{th} degree pass through a curve of the nr^{th} degree situated on a surface of the r^{th} degree, they will also intersect in a curve of the $n(n-r)^{th}$ degree, situated on a surface of the $(n-r)^{th}$ degree, because one of the surfaces which passes through the intersection of the two n^{tic} surfaces will be the complex surface formed of two of the degrees r and $n-r$ respectively.

Thus, if a conicoid intersect a cubic surface in three conics, the planes of each of these will intersect the cubic in a straight line, the conic and straight line forming the plane cubic curve which is the complete intersection; and since the three planes form a cubic surface, part of whose curve of intersection with the original cubic surface lies on a conicoid, the three straight lines will lie in one plane.

* *Nouvelles Annales*, XII., p. 396.

446. The theory of *partial* intersections of surfaces was first discussed by Salmon.* Without an examination of such partial intersections it is not possible to analyse different species of curves of the same degree. If we considered only *complete* intersections of surfaces, curves of the third degree could only be considered as plane curves, whereas it will be seen that they may also be *partial* intersections of conicoids.

447. *To find the surfaces of which a given curve is the partial intersection.*

In order to find the surfaces which may contain a curve of the m^{th} degree, it is observed that a surface of the k^{th} degree can be made to pass through $\phi(k)$ points. Now, the total number of points which are common to a proper curve of the m^{th} degree and such a surface, supposing the curve not to lie entirely on the surface, is mk, since this is the number of points in which k planes intersect the curve; and the law of continuity makes the statement general. If more than mk points lie on the surface, the curve will lie entirely on the surface.

If $\phi(k) = mk + 1$, one such surface can be drawn containing the curve; if $\phi(k) > mk + 1$, two surfaces of the k^{th} degree can be drawn, and therefore an infinite number. Thus, for a curve of the third degree, if $k = 2$, $\phi(k) = 9 > 3.2 + 1$, hence an infinite number of conicoids may be drawn containing any curve of the third degree.

When $\phi(k) = mk + 1$, one surface of the k^{th} degree contains the curve, and another of the $(k+1)^{\text{th}}$ must also contain it, for $\phi(k+1) - \phi(k) = \frac{1}{2}(k+2)(k+3)$, therefore

$$\phi(k+1) - \{m(k+1) + 1\} = \tfrac{1}{2}(k+2)(k+3) - m,$$

which is always positive, since $m = \frac{1}{6}(k^2 + 6k + 11) - k^{-1}$.

Modifications are required when the surfaces are not proper surfaces. Salmon gives as examples of this modification a plane curve of the third degree through which it is possible to describe an infinite number of conicoids, but since each conicoid must necessarily consist of the plane of the curve and an arbitrary plane, the intersection of the plane and conicoid will not determine the curve. Again, if a curve of the fifth degree, which, according to the above laws, ought necessarily to be determined by surfaces of the third degree, lie entirely on a conicoid, every surface of the third degree which contains the curve may be a compound of the conicoid and a plane, and we must advance to surfaces of the fourth degree to determine the curve.

If a curve be given of the m^{th} degree, and k, l be the lowest degrees of surfaces upon which it can lie, any surface of the k^{th}

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* *Quarterly Journal*, vol. v.

degree constructed to pass through $mk+1$ points will contain the curve, and similarly for the other surface.

If $ml+1$ points known to lie on the curve be given, and $l>k$, all the rest can be found.

448. The number of *arbitrary* points through which a curve of the m^{th} degree can be drawn cannot exceed a certain superior limit which is easily determined, for suppose k arbitrary points to be given, and a cone to be constructed containing the curve, and having its vertex in one of the assumed points, the degree of this cone will be $m-1$, since any plane through the vertex must contain $m-1$ points of the curve besides the vertex, and therefore $m-1$ generating lines of the cone, and the number of its generating lines sufficient for its complete determination is the same as that of the number of points necessary to determine a plane curve of the $(m-1)^{\text{th}}$ degree, viz. $\frac{1}{2}m(m+1)-1$.

The greatest value of k for which such a cone can be constructed is $\frac{1}{2}m(m+1)$; this is therefore a superior limit, although lower limits to the number k may be obtained in general from other considerations.

Thus, a curve of the third degree cannot be made to pass through more than six arbitrarily chosen points.

449. *If $\phi(n)-2$ points be given, all surfaces of the n^{th} degree, which can be drawn through these points, will pass through $n^3-\phi(n)+2$ additional fixed points.*

Let $u=0$, $v=0$, $w=0$ be the equations of three surfaces of the n^{th} degree which pass through $\phi(n)-2$ points, and which have not a common curve of intersection, they will pass through n^3 common points, and $\lambda u+\mu v+\nu w=0$ is the equation of another surface of the n^{th} degree, which passes through the same points, and by giving different values to $\lambda:\mu:\nu$ we can obtain *all* surfaces which pass through these points. Any surface will be particularized when two points are given which do not lie on all three of the surfaces, or both on the same two; and all such surfaces will contain $n^3-\phi(n)+2$ common points besides the given points.

Thus, all conicoids which pass through seven points will pass through a fixed eighth, as is easily seen when each conicoid consists of two parallel planes, the seven points being angular points of a parallelepiped.

A surface of the third degree, drawn through 17 points, passes through 10 others.

450. The following propositions, connected with this part of the subject, are of importance in some investigations in which it is required to determine the number of points of intersection of three surfaces: the surfaces under consideration in particular cases may have common lines in any degree of multiplicity, and it becomes necessary to determine to how many points of intersection these lines are equivalent.

451. *Three surfaces of the m^{th}, n^{th}, and p^{th} degrees, contain a multiple straight line in the degrees of multiplicity μ, ν, and π respectively; to find the number of points of intersection to which this multiple line corresponds.*

The number of points of intersection of three surfaces will be unaltered, if we suppose each surface to degenerate into a set of proper surfaces of inferior degrees, so long as the sum of the degrees of the set is the degree of the surface so broken up.

We will, therefore, suppose the surface of the m^{th} degree to consist of μ planes, and of a proper surface of the $(m-\mu)^{th}$ degree; and similarly for the others.

The whole number mnp of points of intersection will then be made up of intersections, (i) of the three proper surfaces, (ii) of proper surfaces from two systems with planes from the remaining system, (iii) of a proper surface of one system with planes from the two remaining systems, and (iv) of planes from the three systems; the numbers of these intersections are

i. $(m-\mu)(n-\nu)(p-\pi)$, ii. $(n-\nu)(p-\pi)\mu+\ldots$,
iii. $(m-\mu)\nu\pi+\ldots$, iv. $\mu\nu\pi$.

If now we suppose all the planes to pass through the same straight line, we shall have the case of surfaces with multiple lines; and those of the mnp points, which will lie on the multiple line, will be clearly taken from the groups (iii) and (iv).

The multiple line therefore corresponds to the number of points $(m-\mu)\nu\pi + (n-\nu)\pi\mu + (p-\pi)\mu\nu + \mu\nu\pi \equiv m\nu\pi + n\pi\mu + p\mu\nu - 2\mu\nu\pi$, which coincides with the particular case given by Salmon.*

452. *Three surfaces of the m^{th}, n^{th}, and p^{th} degrees have a common curve line of the μ^{th}, ν^{th}, π^{th} degrees of multiplicity respectively, the curve being the intersection of two surfaces of the degrees k and l; to find the number of points to which this multiple line corresponds.*

Let the surfaces be broken up into proper surfaces, and the multiple lines be thrown out of gear.

The first shall be composed of μ surfaces of the degree k and one of the degree $m-\mu k$, the second of ν surfaces of the degree k, ν of degree l and one of the degree $n-\nu k-\nu l$, the third of π of the degree l, and one of the degree $p-\pi l$.

The number of points which lie on the intersection of surfaces of the degrees k and l will be $(m-\mu k)\nu k.\pi l + (n-\nu l-\nu k)\mu k.\pi l + (p-\pi l)\mu k.\nu l + \pi l.\mu k.(\nu k+\nu l)$ $\equiv lk\{m\nu\pi + n\pi\mu + p\mu\nu - \mu\nu\pi(l+k)\}$, which is the number of points required, coinciding with the result of the preceding proposition when $l=k=1$.

Application to the Four-point System.

453. It is easy to express in the language of four-point coordinates the results of this chapter.

Thus, a surface of the n^{th} class is determined if $\phi(n)$ tangent planes be given.

If surfaces of the n^{th} class be drawn touching $\phi(n)-1$ tangent planes, they will be touched by one common developable surface.

If three surfaces of the n^{th} class touch $\phi(n)-2$ given planes, they will touch $n^3-\phi(n)+2$ additional fixed planes.

Similarly for other theorems.

In illustration of the points which have been considered in this chapter, relating to the intersection of surfaces, we give here some elementary properties of cubic and quartic curves.

* *Camb. and Dub. Math. Jour.* II. p. 71.

Cubic Curves.

454. If two conicoids have a common generating line, any plane which does not contain this generating line will intersect the two conicoids in two conics which have four points in common, one of which will be in the generating line; hence the curve which with the generating line forms the complete intersection of the conicoids, being met by an arbitrary plane in three points, is a curve of the third degree; such a curve is called a *cubic curve.*

Conversely, if we take any seven points upon a given cubic curve and an eighth on any chord of the curve, we can make an infinite number of conicoids pass through these eight points, which will have for their common curve of intersection the cubic curve and the chord, for each conicoid meets the curve in seven points and the chord in three, and therefore contains both entirely.

455. *A cubic curve, which is the intersection of two conicoids having a common generating line, intersects all the generating lines of the same system as the common line in two points, and those of the opposite system in one point only.*

Call the two conicoids A and B, and the common line L. Any generating line of A intersects B in two points, neither of which will lie on L, if it be of the same system as L, but one will lie on L, if it be of the system opposite to that of L; but the points which do not lie on L must lie on the cubic curve, which proves the proposition.

456. *The common generating line of two conicoids which determine a cubic curve is twice crossed by the curve.*

A plane which contains the common generating line intersects each of the conicoids in a generating line of the opposite system, and these two lines intersect in one point only; but the plane contains three points of the curve; hence two of the three points must lie on the common generating line.

457. *When two cubic curves lie on a given conicoid, to find the number of points in which they intersect.*

Each of the cubic curves is the partial intersection of the given conicoid with another which has a common generating line with it.

Call the three conicoids A, B, and B', and the curves C, C', and let the complete intersection of A and B be the curve C and the line L, and let that of A and B' be the curve C' and the line L'.

The eight points which are common to A, B, and B' must be the intersections of the complex curves CL and $C'L'$; and two distinct cases arise according as L. L' are of the same or of opposite systems.

If they be of the same system, L will meet B' in two points both of which will be on C'; L' and C will intersect in two points, therefore C and C' will intersect in four points.

If they be of opposite systems the two points in which L intersects B' will lie one on L' and the other on C'; hence L, L'; L, C'; and L', C will intersect in three points, and therefore C and C' in the five remaining points.

If L and L' coincide, by Art. 451, the coincidence is equivalent to four points of intersection, and C, C' will intersect in four points.

Quartic Curves.

458. The intersection of two conicoids is a quartic curve, since a plane must meet the two conicoids in two conics which intersect in four points; but this is a particular kind of quartic curve. An arbitrary quartic curve will intersect an arbitrary conicoid in 2×4 points, and only one conicoid can be constructed which will contain nine points of the curve, and therefore the entire curve.

The general quartic curve may therefore be considered as the partial intersection of a conicoid and a cubic surface drawn through $3 \times 4 + 1$ points of the curve, and the remaining portion of the complete intersection must be either (i) two straight lines which do not intersect, or (ii) a conic which may be two intersecting straight lines.

i. In the first case a generating line of the conicoid which is of the same system as the two straight lines common to the two surfaces, meets the cubic surface in three points which must be on the quartic curve, while one of the opposite system meets the cubic surface in one point only, besides the points in which it cuts the two common lines, and therefore intersects the quartic curve once.

ii. In the second case every generator of the conicoid meets the common conic in one point, therefore two of the three points in which it intersects the cubic surface lie on the quartic curve.

If $u_2 = 0$ be the equation of a conicoid containing the quartic curve, $u_1 = 0$ that of the plane of the common conic, the equation of the cubic surface must be of the form $v_1 u_2 + u_1 v_2 = 0$, and the quartic curve, in this case, must be of the particular kind which is the base of a cluster of conicoids, viz. that determined by the equation $\lambda u_2 + \mu v_2 = 0$ for all arbitrary values of λ and μ.

459. *To find the number of points of intersection of two quartic curves which both lie on the same cubic surface.*

Let the surface be denoted by S_3 and the conicoids which contain the two curves by S_2 and S_2', the remaining parts of the complete intersections may be two non-intersecting lines, or a conic, denote these by L. M, and C_2.

i. Let the complete intersections of S_3 and S_2 be C_4, L, M, and that of S_3 and S_2' be C_4', L', M'; the three surfaces S_3, S_2, S_2' intersect in 12 points, of which two lie on L', L, viz the points in which L and S_2' intersect, and similarly two lie on L, M, and M', and there remain only 4 which must be the intersections of C_4 and C_4'.

This supposes that the lines L, L' do not intersect, but if they intersect, the modification is easily made; for example, if the four lines form a skew quadrilateral, the number of points belonging to these lines will be reduced to four, and C_4, C_4' will intersect in eight points.

ii. Let the complete intersection be C_4, C_2, and C_4', C_2'; the section of S_2' made by the plane of C_2 is a conic which intersects C_2 in 4 points, therefore 8 of the 12 points lie on the two curves C_2, C_2', leaving 4 for the number of intersections of C_4 and C_4'.

If the three surfaces all contain the same conic, this will, by Art. 452, count as 8 points of intersection, and C_4, C_4' will intersect, as before, in four.

XXXII.

(1) Every cone containing a curve of the third degree, in which the vertex lies, is of the second degree.

(2) No straight line can cut a curve of the n^{th} degree, not plane, in more than $n-1$ points.

(3) If the base of a cluster of conicoids cross itself, all the conicoids will touch at the point of crossing, and if it cross itself twice it will consist of two conics.

(4) Through a curve of the third degree, and a straight line meeting the curve in one point only, a conicoid can be drawn, of which the generating lines, which intersect the given line, meet the curve each in two points.

(5) If P, Q be two points on a cubic curve, all the conicoids which contain the curve and the chord PQ have common tangent planes at P and Q.

(6) Prove that an infinite number of curves of the third degree can be drawn through five points arbitrarily chosen in space, but that six determine the curve; what limitations are necessary that such a curve shall pass through the points?

(7) Through any point in space a straight line can be drawn which meets a curve of the third degree, not a plane curve, in two points.

(8) The projection upon any plane of a curve of the third degree, not plane, by straight lines drawn from a given point, is a curve of the third degree having a double point.

XXXIII.

(1) A conicoid can be drawn through a given chord of a cubic curve containing the curve and touching a given plane through the chord at a given point of the chord.

(2) Three conicoids which have a common generating line meet only in four points besides the generating line.

(3) Through five points of a conicoid, we can draw two curves of the third degree lying entirely in the conicoid.

(4) The locus of the centres of a cluster of conicoids is a cubic curve.

(5) A quartic curve is the intersection of two conicoids, prove that a cubic surface can be constructed which contains the curve and two given conics, one on each conicoid, if these conics do not lie in the same plane.

(6) Shew that, if normals be drawn to a conicoid from every point of a straight line, their feet will lie on a quartic curve.

(7) Among the conicoids forming a cluster there are four cones, real or imaginary; each of these cones has four of its sides tangents to the curve which is the base of the cluster, and the four points of contact are in one plane.

(8) The projection of the base of a cluster of conicoids on a plane is a curve of the fourth degree having two double points, real or imaginary.

XXXIV.

(1) The eight points given by the equations $lx^2 = my^2 = nz^2 = rw^2$, are so related that any conicoid passing through seven of them will pass through the eighth.

(2) Through a given straight line planes are drawn touching the sections of a conicoid made by a plane passing through a second straight line; shew that the locus of the points of contact is a quartic curve passing through the four points in which the two straight lines intersect the conicoid, and four of whose tangents intersect both of these straight lines.

(3) If three straight lines be the complete intersection of a cubic surface with a plane, three planes through these lines will intersect the surface in three conics; prove that one conicoid can be drawn containing the three conics.

(4) Each of two quartic curves which lie on the same hyperboloid intersects the generating lines of one system in three points, prove that the curves intersect in six points if the generating lines be of the same system for both curves, and in ten points if the systems be opposite.

(5) Find the number of points in which two curves of the fifth degree on the same surface of the third degree, and on two conicoids, intersect.

(6) Prove that the projection upon a given plane of a quartic curve, through which only one conicoid can be drawn, by straight lines drawn through an arbitrary point on the conicoid, is a quartic curve having a triple point.

(7) Through a chord AB of a quartic curve, which is the base of a cluster of conicoids, a plane is drawn determining a second chord ab; shew that, as the plane turns round AB, the chord ab generates a conicoid; shew also that a plane which passes through ab and a fixed point E of the curve also passes through a fixed chord EF.

(8) Through every sextic curve in space one, and in general only one, cubic surface can be drawn. One of two sextic curves which lie on the same cubic surface, lies also on a conicoid, and the other lies also on another cubic surface, find the number of points in which they intersect.

(9) Three cones of the same degree have their vertices on a straight line, and parts of two of their three curves of intersection are plane curves; shew that part of the third curve is also plane, and that the planes of the three curves intersect in a straight line.

CHAPTER XVIII.

TANGENT LINES AND PLANES. NORMALS. SINGULAR POINTS. SINGULAR TANGENT PLANES. POLAR EQUATION OF TANGENT PLANES. ASYMPTOTES.

460. In this chapter we shall confine ourselves to the consideration of surfaces whose equations are given in Cartesian coordinates, and in discussing singularities of contact we shall only consider those of a simpler kind.

461. It will be convenient to state here, that we shall often employ the following notation: when the function $F(x, y, z)$, for which we shall write F, is used, U, V, W will be written for $\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz}$, and u, v, w, u', v', w' for $\frac{d^2F}{dx^2}, \frac{d^2F}{dy^2}, \frac{d^2F}{dz^2}, \frac{d^2F}{dy\,dz}, \frac{d^2F}{dz\,dx}, \frac{d^2F}{dx\,dy}$, and when $z = f(x, y)$, p, q, r, s, t will be written for $\frac{dz}{dx}, \frac{dz}{dy}, \frac{d^2z}{dx^2}, \frac{d^2z}{dx\,dy}, \frac{d^2z}{dy^2}$.

462. *To find the relation between the direction-cosines of a tangent to a surface at a given ordinary point of a surface.*

Let the equation of the surface be $F \equiv F(\xi, \eta, \zeta) = 0$, ξ, η, ζ being the current coordinates of a point, and let (x, y, z) be the point P at which the line is a tangent.

The equations of a line through P, whose direction-cosines are λ, μ, ν are $(\xi - x)/\lambda = (\eta - y)/\mu = (\zeta - z)/\nu = r$. (1).

At the points where this line meets the surface, the values of r are given by the equation $F(x + \lambda r, y + \mu r, z + \nu r) = 0$, and, since $F(x, y, z) = 0$, this equation is $rDF + \frac{1}{2}r^2D^2F + \ldots = 0$, (2), where D denotes the operation $\lambda\frac{d}{dx} + \mu\frac{d}{dy} + \nu\frac{d}{dz}$.

One value of r is zero, whatever be the direction of the line (1), since P is on the surface, but if we so choose the direction of the line that $DF \equiv \lambda U + \mu V + \nu W = 0$ (3), a second value of r will vanish; therefore for this direction another point Q will become coincident with P, and the line will be a tangent line.

At an ordinary point U, V, W do not all vanish, but there may exist points on a surface for which this does happen; such points are called *singular points:* we shall presently consider this peculiarity.

463. *To find the equations of the tangent plane and the normal to a surface at an ordinary point.*

The equation of the locus of all the tangent lines which can be drawn through an ordinary point is found by eliminating λ, μ, ν between the equations (1) and (3), which gives

$$(\xi - x) U + (\eta - y) V + (\zeta - z) W = 0,$$

shewing that the tangent lines all lie in a plane, which is called the *tangent plane*.

The normal is perpendicular to this plane, and its equations are $(\xi - x)/U = (\eta - y)/V = (\zeta - z)/W = r/\sqrt{(U^2 + V^2 + W^2)}$; the equation (3) represents that the normal is perpendicular to every tangent line.

464. *To find the number of normals which can be drawn from a given point to a surface of the n^{th} degree.*

Let $F(\xi, \eta, \zeta) = 0$ be the surface. The number of normals will be the same from whatever point they be drawn, the number may therefore be found by investigating the number of normals which can be drawn from a point at an infinite distance, which we may assume in Ox produced.

The number will therefore be equal to the number of normals parallel to Ox, together with the number of normals to the section by a plane at an infinite distance.

If (x, y, z) be a point at which the normal is parallel to Ox, $V = 0$, and $W = 0$, which combined with the equation $F(x, y, z) = 0$ give $n(n-1)^2$ solutions.

Again, any plane section of the surface will be of the n^{th} degree, and the number of normals drawn to any curve $f(x, y) = 0$ of the n^{th} degree is, in like manner, the number of normals parallel to Ox, together with the normals which can be drawn to points at an infinite distance, the number of the latter is n, and the number of normals parallel to Ox is given by the number of solutions of $f'(y) = 0$, and $f(x, y) = 0$, which is $n(n-1)$; hence, the number of normals to the plane section at an infinite distance is n^2.

Therefore the number of normals which can be drawn to the surface from any point is $n(n-1)^2 + n^2 \equiv n^3 - n^2 + n$.

465. *To obtain the form of the equation of the tangent plane when $F(\xi, \eta, \zeta)$ is represented as the sum of a series of homogeneous functions.*

$$\text{Let } F(x, y, z) \equiv F_n + F_{n-1} + \ldots + F_1 + c,$$

where F_s denotes a homogeneous function of the s^{th} degree in x, y, z; then, by a known property of homogeneous functions,

$$xU_s + yV_s + zW_s = sF_s;$$

therefore the equation of the tangent plane may be written

$$\xi U + \eta V + \zeta W = nF_n + (n-1) F_{n-1} + \ldots + F_1,$$

$$\text{or, since } F_n + F_{n-1} + \ldots + c = 0,$$

$$\xi U + \eta V + \zeta W + F_{n-1} + 2F_{n-2} + \ldots + (n-1) F_1 + nc = 0.$$

466. *To find the equation of a tangent plane to a surface, when the direction of the plane is given.*

Let (l, m, n) be the given direction, and $l\xi + m\eta + n\zeta = p$ the equation of a tangent plane to the surface $F(\xi, \eta, \zeta) = 0$; then if (x, y, z) be the point of contact, since this equation must be identical with $\xi U + \eta V + \zeta W = xU + yV + zW$, we have

$$U/l = V/m = W/n = (xU + yV + zW)/p,$$

and these equations, with that of the surface, give the coordinates of the points of contact of any tangent plane in the given direction, and also determine a relation between l, m, n and p, such as was found in Art. 256 in the case of a conicoid; this relation is the tangential equation with the Boothian coordinates l/p, m/p, n/p.

467. *To find the locus of the points of contact of tangent planes drawn to a given surface from a given point.*

Let $F \equiv F(\xi, \eta, \zeta) = 0$ be the equation of the given surface of the n^{th} degree, and let (f, g, h) be the given point. If (x, y, z) be one of the points of contact, the tangent plane to the surface at (x, y, z) must pass through (f, g, h).

This gives the condition $(f - x)\,U + (g - y)\,V + (h - z)\,W$ (4), which, combined with the equation of the surface, determines the required locus, or the *curve of contact*.

It has been shewn, Art. 465, that $xU + yV + zW$ may by means of the equation of the surface be reduced to an expression of the $(n-1)^{\text{th}}$ degree in x, y, z; the equation (4) so reduced gives a surface of the $(n-1)^{\text{th}}$ degree, called the *first polar*, whose intersection with the given surface is the curve of contact. The curve of contact for any conicoid is therefore a conic, the first polar being in this case a plane, and it is obvious that this plane is always real, whether the points of contact be real or imaginary.

Singular Points.

468. *To find the relation between the direction-cosines of a tangent line at a singular point.*

Since at a singular point P, U, V and W separately vanish, the coefficient of r in equation (2), Art. 462, vanishes for all values of λ, μ, ν, which shews that the line (1) meets the surface in two coincident points, in whatever direction it be drawn through P; in this case we find the direction of any tangent line by taking a point Q near the double point P, and moving it up to P until a third value of r vanishes, the direction of PQ will then be that of a tangent line, and the relation between its direction-cosines will be $D^2F = 0$, or, written in full,

$$u\lambda^2 + v\mu^2 + w\nu^2 + 2u'\mu\nu + 2v'\nu\lambda + 2w'\lambda\mu = 0. \quad (5)$$

If all the partial differential coefficients as far as those of the $(s-1)^{\text{th}}$ order vanish, the equation (2) will have s roots equal to zero, and the point will be a multiple point of the s^{th} degree; it is

easily seen that the direction-cosines of any tangent line will satisfy the equation $D^sF=0$, (6), which shews that the tangent lines all lie in a cone whose vertex is the point P. This cone is called a *tangent cone.*

469. *Equation of the tangent cone at a multiple point.*

If the multiple point be of the s^{th} degree, the equation of the tangent cone is found by eliminating λ, μ, ν between the equations (6) and (1), we thus obtain

$$\left\{(\xi - x)\frac{d}{dx} + (\eta - y)\frac{d}{dy} + (\zeta - z)\frac{d}{dz}\right\}^s F = 0,$$

where it must be remembered that in the performance of the operation indicated $\xi - x$, $\eta - y$ and $\zeta - z$ must be treated as constant; in other words, the symbol of operation must be expanded before the differentiations are performed.

470. *Equation of the normal cone at a double point.*

The equation of the tangent cone at a double point is, by (5), Art. 468,

$$u(\xi - x)^2 + v(\eta - y)^2 + w(\zeta - z)^2$$
$$+ 2u'(\eta - y)(\zeta - z) + 2v'(\zeta - z)(\xi - x) + 2w'(\xi - x)(\eta - y) = 0,$$

and that of the tangent plane at any point of a generating line of this cone whose coordinates are $x + \lambda r$, $y + \mu r$, $z + \nu r$ is

$$(u\lambda + w'\mu + v'\nu)(\xi - x) + (w'\lambda + v\mu + u'\nu)(\eta - y)$$
$$+ (v'\lambda + u'\mu + w\nu)(\zeta - z) = 0;$$

hence, if $(x - \xi)/\lambda' = (y - \eta)/\mu' = (z - \zeta)/\nu'$ be the equation of the normal to this plane at (x, y, z), we obtain

$$\begin{vmatrix} u, & w', & v', & \lambda' \\ w', & v, & u', & \mu' \\ v', & u', & w, & \nu' \\ \lambda', & \mu', & \nu', & 0 \end{vmatrix} = 0, \qquad (7)$$

and the equation of the locus of the normals to all the tangent planes to the tangent cone is, by eliminating λ', μ', ν',

$$p(\xi - x)^2 + q(\eta - y)^2 + r(\zeta - z)^2$$
$$+ 2p'(\eta - y)(\zeta - z) + 2q'(\zeta - z)(\xi - x) + 2r'(\xi - x)(\eta - y) = 0,$$

where $p = vw - u'^2$, $p' = v'w' - uu'$, &c.

471. The condition that the tangent cone shall degenerate into two tangent planes is

$$\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} = 0, \text{ Art. 91,}$$

and the equation (7) becomes $(p\lambda' + r'\mu' + q'\nu')^2/p = 0$, Art. 392 so that the normal cone degenerates into two coincident planes; this may be accounted for geometrically in the following manner: the generating lines of the normal

cone are each perpendicular to the plane containing two of the generating lines of the tangent cone taken indefinitely near to one another; if then the tangent cone become two planes, we can take the two generating lines on one plane, which gives a normal to that plane; or we may take one on each close to the line of intersection of the planes, which will give a normal in any direction we please in the plane perpendicular to the line of intersection, and a double plane will be formed, because these two generators may be on either side of the double point.

The equations of the line of intersection of the two tangent planes will, by Art. 92, be

$$(v'w'-uu')(\xi-x)=(w'u'-vv')(\eta-y)=(u'v'-ww')(\zeta-z),$$
$$\text{or } p'(\xi-x)=q'(\eta-y)=r'(\zeta-z),$$

and that of one of the coincident planes in which the normals lie

$$p(\xi-x)+r'(\eta-y)+q'(\zeta-z)=0,$$

and this plane is perpendicular to the line of intersection, since $pp'=q'r'$, Art. 391.

If H, H' be the determinants, the vanishing of which give the conditions that the tangent and normal cone respectively may become two planes, and P, Q, R, P', Q', R' be the minors of H', $H'=Pp+R'r'+Q'q'$, but

$$P=qr-p'^2=uH,\quad P'=q'r'-pp'=u'H, \text{ \&c., Art. 391};$$
$$\therefore\ H'=H(up+w'r'+v'q')=H^2.$$

472. *To find the equation of the tangent plane and normal at any point of the surface given by the equation $\zeta=f(\xi,\eta)$.*

Let a line be drawn through x, y, z, whose equations are

$$\xi-x=m(\zeta-z),\quad \eta-y=n(\zeta-z),$$

the points in which this line meets the surface are those for which ζ is given by the equation $\zeta-z=f\{x+m(\zeta-z), y+n(\zeta-z)\}$ $-f(x,y)=(mp+nq)(\zeta-z)+\frac{1}{2}(rm^2+2smn+tn^2)(\zeta-z)^2+\ldots$, and if the line be a tangent line two values of $\zeta-z$ are zero; therefore $1=mp+nq$, and eliminating m and n by means of the equations of the line, we obtain the locus of the tangent lines $\zeta-z=p(\xi-x)+q(\eta-y)$, which is the equation of the tangent plane, unless p and q assume the indeterminate form 0/0. The equation is deducible immediately from that of Art. 463 by means of the equations $U+pW=0$ and $V+qW=0$. The equations of the normal are

$$\xi-x+p(\zeta-z)=0 \text{ and } \eta-y+q(\zeta-z)=0.$$

473. Before we consider the properties of the curve of intersection of a surface with its tangent plane, we should notice that among all the tangent lines drawn at an ordinary point, whose locus is the tangent plane, there are two whose direction-cosines satisfy the equation $D^2F=0$ as well as $DF=0$, and that for these lines three points become coincident, so that they have a closer contact with the surface than any of the other tangent lines; these tangent lines are called *inflexional tangents*. In the case of a conicoid three points on one line cannot coincide, unless the line lie entirely in the surface, and the two inflexional tangents are the two generating lines which pass through the point of contact.

Among the tangent lines at a double point it will be seen similarly that there are generally six which have a closer contact than the rest.

474. *Geometrical explanation of the nature of the intersection of a surface with its tangent plane at any point.*

Every plane intersects a surface of the n^{th} degree in a curve which is of the same degree; hence a tangent plane at any point intersects the surface in a curve of the n^{th} degree, passing through the point of contact.

Now when a tangent plane exists, since it is the locus of the tangent lines at the point of contact, and each of these tangent lines contains two points which coincide in the point of contact, it follows that any line, drawn in the tangent plane through the point of contact meets the curve of intersection in two points at the point of contact.

The point of contact is, therefore, a singular point in the curve of intersection.

The singular point may be either a conjugate point, as in the case of contact with an ellipsoid; or a multiple point, as in the case of a hyperboloid of one sheet; or a point through which two coincident lines pass, as in the case of a cylinder.

If the surface be of the second degree the curve of intersection will be of the second degree, and, since it must contain a singular point, the only admissible lines of intersection will be either an indefinitely small circle or ellipse, or else two straight lines which cross one another, or are coincident.

475. *If a plane intersect a surface in a curve which contains a singular point, the plane will generally be a tangent plane to the surface at that singular point.*

For a straight line drawn in any direction in the plane through a singular point, meets the surface in two coincident points, and therefore generally satisfies the condition of being a tangent line to the surface.

If the point which is a singular point in the curve of intersection be also a singular point in the surface, the condition of passing through two coincident points will not be sufficient to define a tangent line.

Thus, if at any point of a surface there be a conical tangent, there may be a singular point in the curve of intersection of a plane through the vertex of the conical tangent, which will not make the cutting plane a tangent plane at the multiple point.

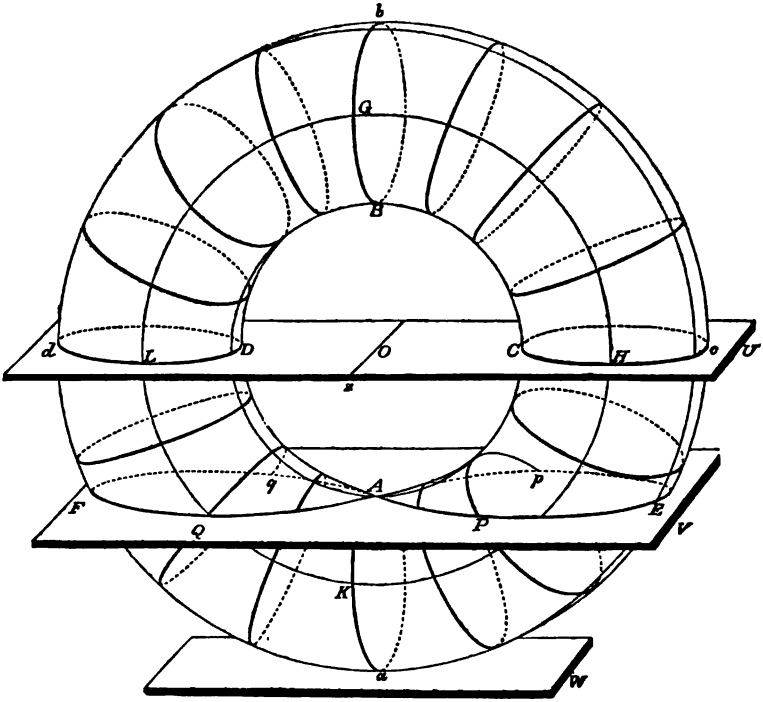
476. The form of the curve of intersection of a surface with the tangent plane at any point may be illustrated by taking the case of an anchor ring or *tore*, supposed to be generated by the revolution of a circle about an axis in its plane not intersecting the circle.

The figure represents the ring, with the generating circle in different positions as it revolves about the axis Oz.

The plane U is drawn through the axis Oz, intersecting the surface in the circles CHc, DLd.

Suppose this plane to move, parallel to itself, towards the position V, the closed curves in which it intersects the surface become elongated until they meet one another in the point A, forming for the position V of the plane a figure of eight, viz. $EPAqFQAp$ which has a double point at A. Here we observe that the concavities of the circles AKa and $ACBD$, which are sections by planes perpendicular to V and to each other, lie in opposite directions with regard to the plane V, and that the tangent lines at A lie in that plane, which is therefore the tangent plane at A; and it is a tangent plane at no other point of the curve of intersection.

The sections by planes through A perpendicular to V change the directions



of their concavities as they pass from the position AKa to $ACBD$, when they pass through the tangents to the branches pAQ, PAq at the multiple point.

If the plane move past V to the position W, the curve of intersection will gradually assume an oval form, which will degenerate into a conjugate point at a.

It is clear also that a plane may meet the ring in the circle $GHKL$, in which case it is a tangent plane at every point of the curve in which it meets the surface; this curve is composed of two coincident circles, as may be seen by moving the plane inwards parallel to itself.

It will be shewn also, that a tangent plane, drawn through a line COD perpendicular to Oz, intersects the ring in two circles.

477. *To find the equations of the tangent line at any point of the curve of intersection of a surface with its tangent plane.*

Let the equation of the surface be $F(\xi, \eta, \zeta)=0$; that of the tangent plane at (x, y, z) will be

$$(\xi-x)U+(\eta-y)V+(\zeta-z)W=0, \qquad (1)$$

Let the equation of the tangent line at any point (x', y', z') of the curve of intersection be $(\xi-x')/\lambda=(\eta-y')/\mu=(\zeta-z')/\nu$, since this line lies in (1), $\lambda U+\mu V+\nu W=0$, (2), and since it meets the surface in two coincident points at (x', y', z'), $\lambda U'+\mu V'+\nu W'=0$, (3); these two equations determine $\lambda:\mu:\nu$ when (x', y', z') is an ordinary point on the curve and the surface.

478. *To find the singular points of the curve of intersection with the tangent plane at any point.*

If the point be a singular point on the curve of intersection, any line drawn through this point will have two points coincident at the point considered; hence, the two equations obtained in the preceding article will be satisfied by an infinite number of values of $\lambda:\mu:\nu$; this will happen in any of the following three cases:

i. When U, V, and W vanish simultaneously, which occurs when there is a singular point at (x, y, z), in which case every plane through that point has one of the characteristics of a tangent plane.

ii. When U', V', W' vanish simultaneously in which case (x', y', z') is a singular point on the surface.

iii. When $U/U'=V/V'=W/W'$, in which case the tangent plane at (x, y, z) is a tangent plane at (x', y', z') also.

In case i. one of the tangent planes to the tangent cone touching it along a generating line (λ', μ', ν') must be the plane considered, and the equation (2) must be replaced by

$$(u\lambda'+w'\mu'+v'\nu')\lambda+(w'\lambda+v\mu'+u'\nu)\mu+(v'\lambda'+u'\mu'+w\nu')\nu=0,$$

thus the ratio $\lambda:\mu:\nu$ will be determined, except in cases where (x', y', z') is a singular point on the surface, or where the tangent plane considered is also a tangent plane to the surface at (x', y', z').

In case ii. a third point at least must be coincident with (x', y', z'), and the equation (3) must be replaced by

$$\left(\lambda\frac{d}{dx'}+\mu\frac{d}{dy'}+\nu\frac{d}{dz'}\right)^s F(x', y', z')=0,$$

where s is 2, 3, ... according to the degree of multiplicity of the singular point (x', y', z').

In case iii., if neither (x, y, z) nor (x', y', z') be singular points of the surface, the equations which determine $\lambda:\mu:\nu$ will be $\lambda U+\mu V+\nu W=0$, $\left(\lambda\frac{d}{dx'}+\mu\frac{d}{dy'}+\nu\frac{d}{dz'}\right)^2 F(x', y', z')=0$, whether (x', y', z') be coincident with (x, y, z) or not.

This case includes the singular tangent plane, a portion of whose curve of intersection consists of two coincident curve lines, which will be considered immediately.

Ruled Surfaces.

479. The student is already familiar with certain surfaces which are capable of being generated by straight lines, or through every point of which some straight line may be drawn which will lie, throughout its length, on the surface.

For example,—a plane, a cone, a cylinder, a hyperboloid of one sheet, a hyperbolic paraboloid.

He is aware that any portion of two of these, the cone and the cylinder, may, if supposed perfectly flexible, be developed into a plane without tearing or rumpling.

We shall now give some account of the general character of surfaces which have this property, distinguishing them from those which, although capable of being generated by the motion of a straight line, are incapable of development into a plane.

480. Def. A *Ruled Surface* is a surface which can be generated by the motion of a straight line, or a surface through every point of which a straight line can be drawn, which will lie entirely in the surface.

A ruled surface, on which each generating line intersects that which is next consecutive, is called a *Developable Surface*, or *Torse.*

A ruled surface, on which consecutive generating lines do not intersect, is called a *Skew Surface*, or *Scroll.*

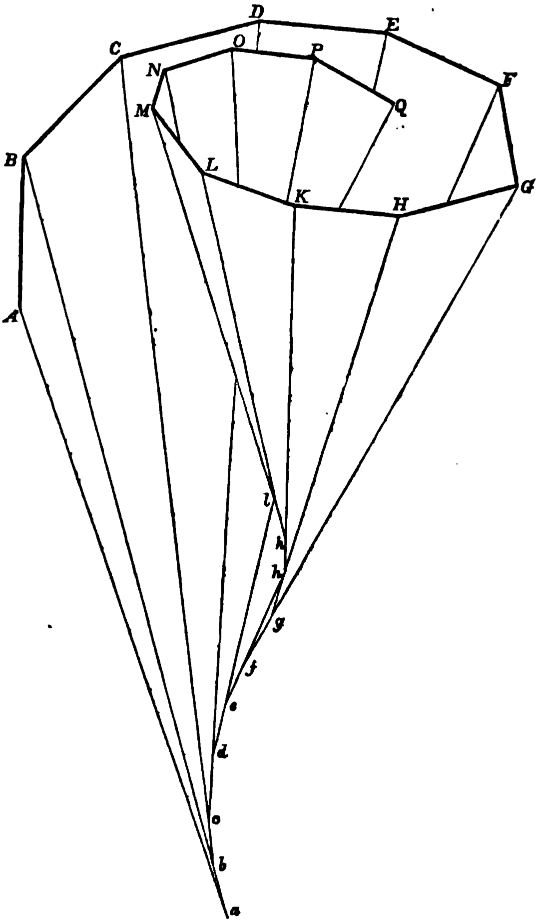
Developable Surfaces or Torses.

481. *Development of a torse into a plane.*

Let Aa, Bb, Cc, ... be a series of straight lines taken in order, according to any proposed law, so as to satisfy the condition that each intersects the preceding, namely, in the points a, b, c, ...

Since Aa, Bb intersect in a, they lie in the same plane, similarly the successive pairs of lines Bb and Cc, Cc and Dd, &c., lie in one plane; thus, a polyhedral surface is formed by the successive plane elements AaB, BbC, &c.

This surface may be developed into one plane by turning the face AaB about Bb, until it forms a continuation of the plane BbC, and again turning the two, now forming one face, about Cc until the three AaB, BbC, CcD are in one plane, and so on; the whole surface may, therefore, be developed into one plane without tearing or rumpling; the same being true, however near the lines Aa, Bb, ... are taken, will be true in the limit, when the surface will become what we have defined as a developable surface, this name being derived from the property just proved.



Edge of Regression.

482. The polygon *abcd* ... whose sides are in the direction of the lines *Bb*, *Cc*, ... becomes in the limit a curve, always tortuous, which is called the *Edge of Regression*, from the fact that the surface bends back at this curve so as to be of a cuspidal form. Every generating line of the system is a tangent to the edge of regression, which is therefore the envelope of all the generating lines.

In the case of a cylinder, the edge of regression is at an infinite distance.

For a practical construction of a developable surface having a given edge of regression, see Thomson and Tait, *Nat. Phil.*, vol. I, Art. 149.

483. *To find the general nature of the intersection of a tangent plane to a developable surface with the surface.*

The plane containing the element DdE of the surface represented by the figure evidently becomes in the limit a tangent plane to the developable surface at any point D in the generating line Dd, since it contains two tangent lines, viz. Dd and the limiting position of a line joining such points as D and E, which ultimately coincide; and again, supposing DdE in the plane of the paper, Ff meets this plane in e, Ggf meets it in some point f', Hhg in g', &c., and similarly for Cc, Bb, ... on the other side.

The complete intersection of the surface and tangent plane is therefore the double line formed by the coincidence of Dd, Ee, and the limit of the polygon $a'b'cdef'g'$ ... which is a curve touching the double line Dd at the edge of regression.

COR. *To find the nature of the contact of the edge of regression and the tangent plane.*

The plane containing the generating lines Dd, Ee contains the three angular points c, d, e of the polygon in the limit, therefore the tangent plane contains two consecutive elements of the edge of regression, and is, as will be seen later on, what is called the osculating plane at that point.

484. *To shew that the equation of the tangent plane to a developable surface contains only one parameter.*

Since the general equations of a straight line involve four arbitrary constants, we must, in order to generate any ruled surface, have three relations connecting the constants, so that it may be possible, between these equations and the two equations of the generating line, to eliminate the four constants, and thus obtain the equation of the surface which is the locus of all the straight lines. In developable surfaces the generating straight lines are such that any two consecutive ones intersect, and the plane containing them is ultimately a tangent plane to the surface. The equation of this plane will then involve the four parameters, and by means of the three relations we may eliminate three, so that the general equation of the tangent plane to a developable surface will involve only one parameter, and we may write it in the form

$$z = ax + \phi(a)\, y + \psi(a),$$

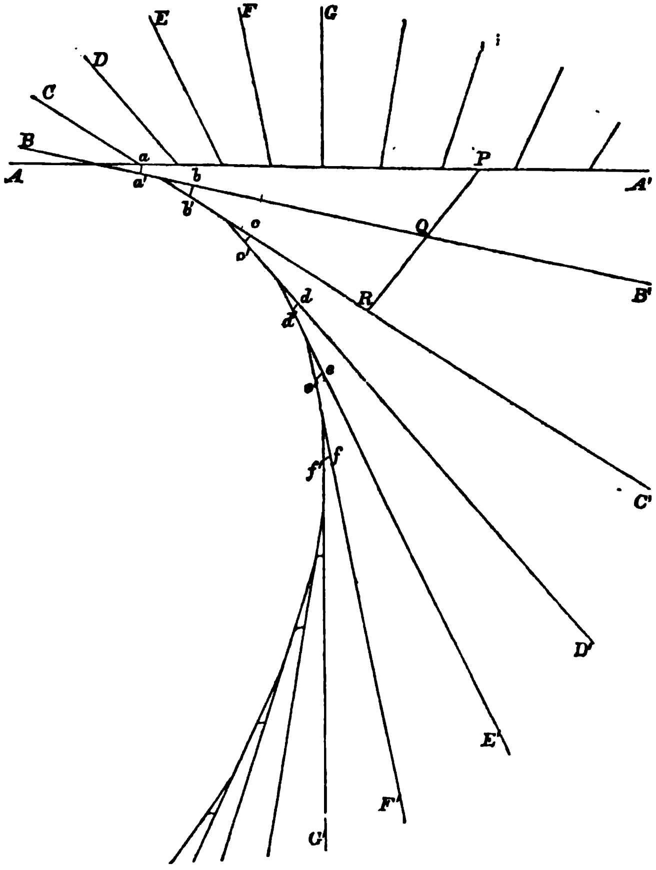
a being the parameter, and $\phi(a)$, $\psi(a)$ functions of that parameter, given in any particular case.

Skew Surfaces or Scrolls, and their Curves of Greatest Density, or Lines of Striction.

485. Let AA', BB', CC', DD', &c., be straight lines drawn according to some fixed law, such that none intersects the next consecutive; let aa', bb', cc', dd', ... be the shortest distances.

Suppose now that we take two of the generating lines as CC', DD', and imagine DD' twisted about c' so as to be parallel to CC', and united with it by means of an uniform elastic membrane: if now DD' be returned to its original position, the portion of the membrane near cc' being unstretched will be denser than any other portion. If the same process be adopted for every line, the series of membranes will generate a surface which will ultimately, as the lines approach nearer to one another, become a *skew* or twisted surface.

The curve which is the limit of the polygon formed by joining a, b, c, d,... the points at which the imagined membranes would have the greatest density, is called the *curve of greatest density;* the *line of striction* is a better name.



It may be observed, that the shortest distances between the consecutive generating lines of a scroll are not generally elements of the line of striction.

486. *To explain the nature of a tangent plane to a scroll at any point.*

Let P be any point of a scroll, AA' the generating line passing through P, suppose a plane to be drawn through P containing BB' the next consecutive position of the generating line, this plane will intersect the third line CC' in some point R, and, if PR be joined, it will meet BB' in Q; PR will therefore be an inflexional tangent line at P, since it passes through three consecutive points, and, if the surface were of the second order, it would lie entirely in the surface. The tangent plane at P is the plane containing AA' and PR; R will change its position for any change of position of P, thus the tangent plane at any point in AA' will always contain AA', but it will move about AA' through all positions, as the point of contact moves along AA'.

Hence, the section of a scroll by a tangent plane at any point is made up of the generating line through that point, and of a curve of the $(n-1)^{\text{th}}$ degree, which must thus be a straight line in the case of a hyperboloid or hyperbolic paraboloid.

487. *A plane passing through a generator of a scroll touches the scroll at one and only one point in the generator.*

Since the position of a tangent plane at a point P in the generator AA' depends only on the consecutive generators BB', CC', the determinate conicoid, of which AA', BB', CC' are generators, will have the same tangent planes as the scroll at every point of AA', and for every plane through AA' there is only one point of contact with the conicoid.

488. ***The points in a generator of a scroll, at which the tangent planes to the scroll are at right angles, form a system of points in involution, the foci of which are imaginary, and the centre the point in which the line of striction crosses the generator.***

The first part has been proved by A. J. C. Allen* as follows. Take the generator as the axis of z, any point O in it being the origin, and let Ox be in the tangent plane at O, Oy the normal; the equation of the conicoid of the last article will then be

$$ax^2 + by^2 + 2a'yz + 2b'zx + 2c'xy + + 2b''y = 0.$$

Let P be any point in the generator, $OP = \gamma$, the tangent plane at P will be $\gamma(a'y + b'x) + b''y = 0$, (1), and if θ be the angle between the tangent planes at O and P, then $\tan\theta = y/x = -b'\gamma/(b'' + a'\gamma)$, (2); and if $OP' = \gamma'$ give a point at which the tangent plane is perpendicular to that at P,

$$b'^2\gamma\gamma' + (b'' + a'\gamma)(b'' + a'\gamma') = 0,$$

$$\text{whence } \{\gamma + a'b''/(a'^2 + b'^2)\}\{\gamma' + a'b''/(a'^2 + b'^2)\} = -b'^2b''^2/(a'^2 + b'^2)^2,$$

which proves the first part. Also the centre C, for which $z = -a'b''/(a'^2 + b'^2)$, is the point of striction; for the consecutive generator has the equations

$$x = \lambda(by + 2a'z + 2b'') \text{ and } y = -\lambda(ax + 2b'z + 2c'y),$$

where λ is indefinitely small, and thus, x and y being small, the projection on the plane of xy is $b'x + a'y = 2b'b''\lambda$; and therefore the projection of the element

* *Mess. of Math.*, vol. XII., p. 26.

of the line of striction is $a'x = b'y$, hence at the point where it meets the consecutive generator $x/b' = y/a' = 2b'b''\lambda/(a'^2 + b'^2)$ and $2b'z = -y/\lambda$;

$$\therefore \; z = -a'b''/(a'^2 + b'^2).$$

It is also clear that, since $CP.CP'$ is constant, the centre of the involution is the point where the tangent plane is perpendicular to the tangent plane at infinity along the generating line.

489. *If two scrolls have a common generator and touch at three points along it, they will touch at all points along the generator.*

This follows from equation (2) of the last Article.

490. *If tangent planes to two scrolls at three points along a common generator be inclined at the same angle, they will make the same angle at all points along the generator.*

This follows from the last Article by turning one of the scrolls round the common generator.

491. *The normals along a generator of a scroll generate a hyperbolic paraboloid.*

For, by (1), the locus of all the normals is $(b'y - a'x)z = b''x$.

492. *On every scroll represented by an equation of the n^{th} degree there is generally a double curve which is intersected by every generator of the surface in $n-2$ points.**

Any plane drawn through a generator meets the surface in that generator and a curve of the $(n-1)^{th}$ degree, which is cut by the generator in $n-1$ points; of these $n-1$ points one is a unique point being the one point, Art. 487, of the generator at which the plane is a tangent plane; the remaining $n-2$ points are the intersections of the generator with $n-2$ non-consecutive generators of the scroll, and are fixed points for all positions of the plane. They are generally double points on the surface, and their locus for all the generators is a double curve.

493. Taking the axes as in Art. 488, the plane zx being the tangent plane at the origin, the equation of the scroll may be written in ascending powers of x and y, $2b'zx + 2(b'' + a'z)y + ax^2 + 2c'xy + by^2 + \&c. = 0$, where b' is a function of z of the $(n-1)^{th}$ degree, and the remaining coefficients of the terms written are of the $(n-2)^{th}$ degree.

When $b'z = 0$ and $b'' + a'z$ is finite, $y = 0$ is a tangent plane, but the origin is the only point of contact of the plane $y = 0$, Art. 487, $\therefore$ for the $n-2$ values of z given by $b' = 0$, $b'' + a'z$ must vanish, it must therefore be of the form $b'(p + qz) \equiv b'u$, and the equation must be

$$2b'(zx + uy) + ax^2 + 2c'xy + by^2 + \&c. = 0.$$

Let z_1 be a root of $b' = 0$, P_1 the corresponding point on Oz, then at all points near P_1, for which $z = z_1 + \zeta$,

$$2\frac{db'_1}{dz_1}\zeta(z_1x + u_1y) + a_1x^2 + 2c'_1xy + b_1y^2 = 0,$$

which is the equation of the conical tangent at the double point P_1.

494. *To find the line of striction of a scroll.*

Let the equations of a generating line be

$$\eta = m\xi + \alpha, \quad \zeta = n\xi + \beta, \tag{1}$$

where the constants are functions of one parameter θ; the equations of a consecutive generator corresponding to a value $\theta + d\theta$ of

* Cayley, *Camb. and Dub. Math. Journ.*, vol. VII., p. 171.

the parameter are

$$\eta = (m + dm)\,\xi + \alpha + d\alpha, \quad \zeta = (n + dn)\,\xi + \beta + d\beta. \tag{2}$$

Let P be a point in the line of striction, PQ being the shortest distance between (1) and (2), x, y, z, and $x + \delta x$, $y + \delta y$, $z + \delta z$ the coordinates of P and Q; PQ is perpendicular to both generators,

$$\therefore\ \delta x + m\delta y + n\delta z = 0 \quad\text{and}\quad \delta x + (m + dm)\,\delta y + (n + dn)\,\delta z = 0;$$

$$\therefore\ \delta y / dn = \delta z / (-dm) = \delta x / (ndm - mdn), \tag{3}$$

and, by the equations (1) and (2),

$$\delta y - m\delta x = xdm + d\alpha, \quad \delta z - n\delta x = xdn + d\beta;$$

hence

$$(xdm + d\alpha)\{(1 + n^2)dm - mndn\} + (xdn + \beta)\{(1 + m^2)dn - mndm) = 0; \tag{4}$$

if we eliminate θ from this equation, and the equations $y = mx + \alpha$, $z = nx + \beta$, we shall have two equations, which will be those of the line of striction.

NOTE. If the equation of the scroll be given, an equation

$$U\delta x + V\delta y + W\delta z = 0$$

will be obtained, and (4) will be replaced by

$$U(ndm - mdn) + Vdn - Wdm = 0.$$

495. *Line of striction of a hyperboloid of one sheet.*

i. If we use the equations (1) of Art. 213 with the upper sign, the condition of perpendicularity of the line joining (x, y, z) and $(x + \delta x, y + \delta y, z + \delta z)$ gives

$$a \sin \alpha\delta x - b \cos \alpha\delta y + c\delta z = 0, \quad\text{and}\quad a \cos \alpha\delta x + b \sin \alpha\delta y = 0;$$

$$\therefore\ \delta x / -b \sin \alpha = \delta y / a \cos \alpha = c\delta z / ab,$$

$$\text{and}\ \ x\delta x / a^2 + y\delta y / b^2 - z\delta z / c^2 = 0,$$

$$\therefore\ -x \sin \alpha / a^3 + y \cos \alpha / b^3 - z / c^3 = 0,$$

$$\text{also}\ \ (z^2/c^2 + 1) \sin \alpha = y / b + xz / ac,$$

$$(z^2/c^2 + 1) \cos \alpha = x / a - yz / bc,$$

$$\text{whence}\ \ (b^{-2} - a^{-2})\, xy / ab = z / c\,(x^2/a^4 + y^2/b^4 + z^2/c^4 + c^{-2});$$

therefore the lines of striction of one set of generators lie in the intersection of this cubic surface and the hyperboloid.

ii. If we employ the method of Art. 210, we obtain a quartic surface on which the lines of striction of both sets of generators lie.

$$\text{For}\ \ l\delta x + \ldots = 0,\ dl\,\delta x + \ldots = 0,\ \text{and}\ \ ax\delta x + \ldots = 0,$$

$$\text{whence}\ \ ax\,(mdn - ndm) + \ldots = 0,$$

$$\text{also}\ \ aldl + \ldots = 0,\ ldl + \ldots = 0;$$

$$\therefore\ ldl / (b - c) = mdm / (c - a) = ndn / (a - b) = mn\,(mdn - ndm) / a;$$

$$\therefore\ la^2x + \ldots = 0,\ lax + \ldots = 0,\ l^2a + \ldots = 0,$$

$$\text{and}\ \ (b - c)^2 / ax^2 + (c - a)^2 / by^2 + (a - b)^2 / cz^2 = 0. \tag{1}$$

The equation of the lines of striction for both sets of generators found by (i) is, with the notation of (ii), $abx^2y^2 + cz^2\,(a^2x^2 + b^2y^2 + c^2z^2 - 1)^2 = 0$, apparently a sextic surface, but reducible to (1) by rejecting the factor $1 - cz^2$ not zero.

Singular Tangent Planes.

496. DEF. A *singular tangent plane* is a plane which, instead of touching a surface in any finite number of points, touches along the whole of a curve line.

If the curve of intersection of any plane with the surface be composed, in part at least, of two or more coincident lines, the other part being made up of simple curves, either the plane will be a tangent plane to the surface at every point of such a multiple curve, or it will contain a multiple line of the surface, such as would be generated by the rotation of a cross round any fixed line not passing through the angle of the cross.

Conversely, if a tangent plane touch along a curve line on the surface, this curve line will be a multiple line on the tangent plane.

Thus, in the case of the anchor ring, Art. 476, the plane which touches the ring along a curve has for its curve of intersection the two circles coincident in LKH; also the tangent plane to a cone contains two generating lines which ultimately coincide, and is therefore a tangent plane at every point of the generating line which it contains; any more general developable surface is an example of the case of a tangent plane which contains a double line, at every point of which it is a tangent, combined, as shewn in Art. 483, with another simple curve.

A surface of the fourth degree admits of the case of a double conic, as in the example of the anchor ring, or of a quadruple straight line, as when it is made up of two cones touching along a generating line.

A surface of the fifth degree might be composed of one of the third degree and one of the second, in which case a tangent plane might meet the former in a triple and the latter in a double straight line.

497. *To find the condition that a tangent plane may be singular.*

The conditions of the existence of a singular tangent plane may be found by considering that the point of contact, as determined by the equations of Art. 466, may be any point in a curve line, and its coordinates are therefore indeterminate.

498. *The analytical conditions of the singularity may be investigated as follows.*

Since a line, drawn in any direction in the tangent plane through any point of the double curve in which the tangent plane touches the surface, will contain two coincident points, but if it be drawn in the direction of the two coincident tangents to the curve of contact it will contain four coincident points, we have to express that at every point of the double curve there are two coincident tangents, and that a line in their direction contains four coincident points. Hence $DF=0$, $D^2F=0$, and $D^3F=0$, D denoting the operation mentioned in Art. 462; moreover, the two inflexional tangents coincide; therefore, if λ, μ, ν be looked upon as current coordinates, $DF=0$ will be a tangent plane to the cone $D^2F=0$;

$$\therefore\ (u\lambda+w'\mu+v'\nu)/U=(w'\lambda+v\mu+u'\nu)/V=(v'\lambda+u'\mu+w\nu)/W,$$

and $\lambda U+\mu V+\nu W=0$ hold simultaneously;

$$\therefore\ \Delta\equiv\begin{vmatrix} u, & w', & v', & U \\ w', & v, & u', & V \\ v', & u', & w, & W \\ U, & V, & W, & 0 \end{vmatrix}=0;$$

the condition that the fourth point may be coincident is found by substituting in $D^3F=0$ for λ, μ, ν the coefficients of U^2, V^2, W^2 in the expansion of $\Delta=0$.

499. For a surface given by the unsymmetrical equation $\zeta = f(\xi, \eta)$, the equation of a tangent plane at any point (x, y, z) is $\zeta - z = p(\zeta - x) + q(\eta - y)$; and a tangent line whose equations are $(\xi - x)/\lambda = (\eta - y)/\mu = (\zeta - z)/\nu = \rho$ meets the surface in points for which ρ is given by

$$\nu\rho = (p\lambda + q\mu)\rho + \tfrac{1}{2}\rho^2(r\lambda^2 + 2s\lambda\mu + t\mu^2) + \tfrac{1}{6}\rho^3\left(\lambda\frac{d}{dx} + \mu\frac{d}{dy}\right)^3 z + \ldots.$$

If the tangent plane be singular, for all the points in which it meets the surface, $\nu = p\lambda + q\mu$, and for all the points of the double curve four values of ρ are zero, and the two inflexional tangents coincide; $\therefore\ r\lambda^2 + 2s\lambda\mu + t\mu^2 = 0$ and $\left(\lambda\frac{d}{dx} + \mu\frac{d}{dy}\right)^3 z = 0$, and the former has equal roots, therefore $rt = s^2$, and either $r\lambda + s\mu = 0$ or $s\lambda + t\mu = 0$.

500. Every tangent plane to a developable surface is a singular tangent plane, since it contains two consecutive generating lines, hence its curve of intersection with the surface consists of two coincident straight lines, and, as shewn in Art. 483, a single curve line. The analytical conditions of singularity are satisfied, since, if λ, μ, ν be the direction-cosines of the double line, which lies entirely in the surface, the coefficients of *all* the powers of ρ will vanish, and $rt = s^2$ in consequence of the coincidence of the two lines.

At any point of the single curve, the values of p, q being p', q', the direction cosines λ, μ, ν of the tangent are given by $\nu = \lambda p + \mu q$ and $\nu = \lambda p' + \mu q'$, which are independent equations, since p', p and q', q are generally unequal.

Polar Equation of the Tangent Plane.

501. *To find the polar equation of the tangent plane to a surface at a given point.*

Let the equation of the surface be $r'^{-1} = u' = f(\theta', \phi')$, and let u, θ, ϕ be the coordinates of the point of contact of the tangent plane.

The equation of the tangent plane is of the form

$$pu' = \cos\alpha\cos\theta' + \sin\alpha\sin\theta'\cos(\phi' - \beta),\ \text{Art. 77,}$$

and the constants p, α, and β are to be determined from the consideration that the tangent plane contains not only the point of contact but adjacent points which have moved up to and ultimately coincided with that point.

Hence the values of $\dfrac{du}{d\theta}$ and $\dfrac{du}{d\phi}$ at the point of contact are the same for both tangent plane and surface, let v, w be those values;

$$\begin{aligned} \therefore\ pu &= \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos(\phi - \beta), & (1)\\ pv &= -\cos\alpha\sin\theta + \sin\alpha\cos\theta\cos(\phi - \beta), & (2)\\ pw &= -\sin\alpha\sin\theta\sin(\phi - \beta); & (3) \end{aligned}$$

$$\therefore\ p(u\sin\theta + v\cos\theta) = \sin\alpha\cos(\phi - \beta),$$

$$p(u\cos\theta - v\sin\theta) = \cos\alpha;$$

the last three equations give readily the values of the constants; and the equation of the tangent plane becomes

$$\begin{aligned} u' &= (u\cos\theta - v\sin\theta)\cos\theta'\\ &\quad + (u\sin\theta + v\cos\theta)\cos(\phi' - \phi)\sin\theta'\\ &\quad + w\operatorname{cosec}\theta\sin(\phi' - \phi)\sin\theta'. \end{aligned}$$

This equation can also be written in the form

$$r^2/r' = \frac{d}{d\theta}\left[r\left\{\sin\theta\cos\theta' - \cos\theta\sin\theta'\cos(\phi'-\phi)\right\}\right]$$
$$-\operatorname{cosec}\theta\sin\theta'\sin(\phi'-\phi)\frac{dr}{d\phi}.$$

502. *To find the perpendicular distance from the pole upon the tangent plane.*

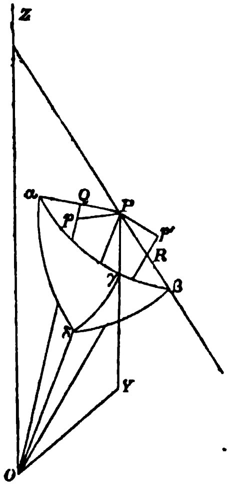
By equations (1), (2), and (3) of the last Article

$$p^2(u^2+v^2+w^2\operatorname{cosec}^2\theta)=1,$$

$$\text{or}\quad p^{-2}=u^2+\left(\frac{du}{d\theta}\right)^2+\left(\frac{du}{d\phi}\right)^2\operatorname{cosec}^2\theta.$$

503. We may arrive at the above result by the following process, which serves to shew the geometrical signification of the partial differential coefficients, and will be useful as an exercise.

Let P be the point of contact, PR a tangent line passing through OZ, and PQ a tangent line in the plane through OP perpendicular to the plane POZ; take R and Q points very near to P, and in OQ, OR take Op, Op' each equal



to OP; then $Pp=r\sin\theta d\phi$ and $Pp'=rd\theta$ ultimately, and Qp, $-Rp'$ are respectively the values of dr due to changes of θ and ϕ, considering the other constant, $\therefore \frac{dr}{rd\theta}=-\frac{Rp'}{Pp'}=-\cot OPR$, and $\frac{dr}{r\sin\theta d\phi}=\frac{Qp}{Pp}=-\cot OPQ$.

Draw OY perpendicular to the tangent plane QPR, and on a sphere, whose centre is P, let $\alpha\delta\beta$ be a spherical triangle with its angular points in PQ, PO, PR, join $\delta\gamma$, γ being the intersection of PY and $\alpha\beta$, then $\delta\gamma$ is perpendicular to $\alpha\beta$, and $\alpha\delta\beta$ is a right angle. Hence $\cot\alpha\delta=\cot\delta\gamma\cos\alpha\delta\gamma$, and $\cot\beta\delta=\cot\delta\gamma\sin\alpha\delta\gamma$; $\therefore$ $\cot^2\alpha\delta+\cot^2\beta\delta=\cot^2\delta\gamma=(r^2-p^2)/p^2$;

$$\therefore\ p^{-2}=r^{-2}+r^{-4}\left(\frac{dr}{d\theta}\right)^2+r^{-4}\operatorname{cosec}^2\theta\left(\frac{dr}{d\phi}\right)^2.$$

Four-plane Coordinates.

504. *To find the equation of the tangent plane at any point of a surface referred to a four-plane or tetrahedral coordinate system.*

Let the equation of the surface be $F(\xi, \eta, \zeta, \omega) = 0$, (1) in a homogeneous form, and let (x, y, z, w) be any point P on the surface, then, by Art. 103, the equations of any straight line through P is $(\xi - x)/\lambda = (\eta - y)/\mu = \ldots = r$, where r is the distance of any current point in the line from P. When this straight line is a tangent at P two values of r are zero;

$$\therefore\ \lambda F'(x) + \mu F'(y) + \nu F'(z) + \rho F'(w) = 0;$$

hence, writing $\xi - x$ for λ, &c., and observing that

$$xF'(x) + yF'(y) + \ldots = nF(x, y, z, w) = 0,$$

we have, for any point in any tangent line at P,

$$\xi F'(x) + \eta F'(y) + \zeta F'(z) + \omega F'(w) = 0; \qquad (2)$$

this is therefore the equation of the tangent plane at P.

505. *Polar of a point with respect to a surface.*

From a point Q let a line QP touch the surface (1) in P, then, since Q lies in the tangent plane at P, the coordinates $\alpha, \beta, \gamma, \delta$ of Q, and x, y, z, w of P, satisfy the equation (2);

$$\therefore\ \alpha F'(x) + \beta F'(y) + \gamma F'(z) + \delta F'(w) = 0$$

is the equation of a surface which contains the points of contact of all tangents drawn from Q to the surface. It is called the *First Polar* of Q with respect to the surface, and is of the $(n-1)^{\text{th}}$ degree.

COR. 1. If the surface be a conicoid the first polar is a plane, and its equation may be written

$$xF'(\alpha) + yF'(\beta) + zF'(\gamma) + wF'(\delta) = 0.$$

COR. 2. The centre of the conicoid, being the pole of the *plane* at infinity $x + y + z + w = 0$, is given by the equation

$$F'(\alpha) = F'(\beta) = F'(\gamma) = F'(\delta).$$

506. *Conical envelope of a conicoid referred to tetrahedral coordinates.*

The equation of the conical envelope can be found, as in Art. 266, viz. $4F(\alpha, \beta, \gamma, \delta)\, F(x, y, z, w) = \{xF'(\alpha) + yF'(\beta) + \ldots\}^2$.

507. *Equation of the tangent cone at a multiple point of a surface referred to tetrahedral coordinates.*

At a multiple point for all values of λ, μ, ν, ρ the coefficient of r must vanish, or $\lambda F'(x) + \ldots = 0$, but with tetrahedral coordinates $\lambda + \mu + \nu + \rho = 0$, Art. 103;

$$\therefore\ F'(x) = F'(y) = F'(z) = F'(w) = \{xF'(x) + \ldots\}/(x+y+z+w) = 0,$$

and a tangent line is obtained by making the coefficient of r^2 vanish,

and writing in the resulting equation $\xi - x$ for λ, &c., whence we obtain the equation of the tangent cone

$$\left\{(\xi-x)\frac{d}{dx}+(\eta-y)\frac{d}{dy}+(\zeta-z)\frac{d}{dz}+(\omega-w)\frac{d}{dw}\right\}^2 F(x, y, z, w)=0.$$

Four-point Coordinates.

508. *To find the equation of the point of contact of any tangent plane to a surface, represented by an equation in a four-point system.*

Let $F(p, q, r, s)=0$, (1) be the tangential equation of the surface of the n^{th} degree, p', q', r', s' the coordinates of a tangent plane U', p'', q'', r'', s'' those of a plane U'' through any line L in U'; by Art. 130, the coordinates of any plane V through L are $p=lp'+mp''$, $q=lq'+mq''$, &c.

V is a tangent plane if $F(lp'+mp'', \ldots)=0$,

or $l^n F(p', q', r', s')+l^{n-1}m\{p''F'(p')+q''F'(q')+\ldots\}=0.$

For all positions of L, determined by U'', one position of V is given by $m=0$, viz. where it coincides with U', but, if L pass through the point of contact, two positions of V will coincide with U', hence the coefficient of m will vanish, and the coordinates of U'' must satisfy the equation

$$p''F'(p')+q''F'(q')+r''F'(r')+s''F'(s')=0,$$

hence, the coordinates of all planes passing through the point of contact must satisfy the equation

$$pF'(p')+qF'(q')+rF'(r')+sF'(s')=0, \qquad (2)$$

which is therefore the equation required.

509. *Pole of a given plane with respect to a conicoid given by a tangential equation.*

Let (1) be the equation of the conicoid, and let $(\alpha, \beta, \gamma, \delta)$ be a plane cutting the conicoid in a curve on which the point lies whose equation is (2), the tangent plane at which is (p', q', r', s');

$$\therefore\ \alpha F'(p')+\beta F'(q')+\gamma F'(r')+\delta F'(s')=0,$$

$$\text{or } p'F'(\alpha)+q'F'(\beta)+r'F'(\gamma)+s'F'(\delta)=0; \qquad (3)$$

hence, the tangent planes at every point of the plane section made by $(\alpha, \beta, \gamma, \delta)$ pass through a point whose equation is (3), p', q', r', s' being current coordinates, and this point is the pole of $(\alpha, \beta, \gamma, \delta)$.

510. *Centre of the conicoid given by a tangential equation.*

The centre is the pole of the plane at infinity, for which $\alpha=\beta=\gamma=\delta$, therefore the equation of the centre is

$$F'(p')+F'(q')+F'(r')+F'(s')=0.$$

511. *Tangential equations of an enveloping cone touching a conicoid along a given plane section.*

Let (p', q', r', s') be the plane of the given section, and $u \equiv F(p, q, r, s)=0$ the tangential equation of the conicoid, and let

$u' \equiv F(p', q', r', s')$. The equation of the vertex of the enveloping cone is $v \equiv pF'(p') + qF'(q') + rF'(r') + sF'(s') = 0$.

Since every tangent plane of the cone must touch the conicoid, and also pass through the vertex, their coordinates must satisfy both $u = 0$ and $v = 0$, which are therefore the tangential equations of the cone corresponding to the equations of the curve of contact in tetrahedral coordinates.

512. *Tangential equation of the curve of contact of the enveloping cone.*

The equation $4u'u = v^2$ represents a conicoid which touches the surface $u = 0$, where the polar plane of $v = 0$ meets it, and since it is satisfied by p', q', r', s', the polar is a tangent plane; therefore the conicoid is a flat surface bounded by the curve of contact, and the equation in this sense is the tangential equation of the curve of contact.

513. *To shew that if two tetrahedrons be so related that each angular point of one is the pole of a face of the other with respect to any conicoid, the lines of intersection of corresponding faces will lie on one conicoid, which touches the faces of both tetrahedrons.*

Let $ABCD$, $abcd$ be the two tetrahedrons, and let the equation of the conicoid referred to $ABCD$ be

$$px^2 + qy^2 + rz^2 + sw^2 + 2lyz + 2mzx + 2nxy + 2l'xw + 2m'yw + 2n'zw = 0.$$

The equation of bcd, being the polar of A, is

$$px + ny + mz + l'w = 0,$$

and the equations of the line of intersection of planes BCD, bcd are

$$ny + mz + l'w = 0, \quad x = 0, \qquad (1),$$

of CDA, cda, $$nx + lz + m'w = 0, \quad y = 0, \qquad (2),$$

of DAB, dab, $$mx + ly + n'w = 0, \quad z = 0, \qquad (3),$$

of ABC, abc, $$l'x + m'y + n'z = 0, \quad w = 0, \qquad (4).$$

Equations of any line intersecting (1) and (2) are

$$kx + ny + mz + l'w = 0, \quad nx + k'y + lz + m'w = 0, \qquad (5).$$

The conditions that this line may intersect (3) and (4) are

$$\begin{vmatrix} k, & n, & l' \\ n, & k', & m' \\ m, & l, & n' \end{vmatrix} = 0, \quad \text{and} \quad \begin{vmatrix} k, & n, & m \\ n, & k', & l \\ l', & m', & n' \end{vmatrix} = 0,$$

the same for both; hence the four lines (1)...(4) are generators of the same system of the conicoid generated by (5).

Eliminating k and k' from the equations of the line (5) and either of these determinants, we find the equation of the conicoid generated by it

$$mnl'x^2 + nlm'y^2 + lmn'z^2 + l'm'n'w^2 + (mm' + nn')(lyz + l'xw) + \ldots = 0;$$

since the faces of $ABCD$ meet this surface in straight lines they will be tangent planes to it; and since it involves five constants, it may be considered as the general equation of a conicoid inscribed in $ABCD$.

Cor. If $ll' = mm' = nn'$, the conicoid in which the intersections of the corresponding faces lie becomes two coincident planes, viz.

$$\{ll'(l'x + m'y + n'z) + l'm'n'w\}^2 = 0,$$

and the intersections lie in one plane.

Asymptotes.

514. Def. An *asymptote* to a surface is a straight line which meets the surface in two points at least, at an infinite distance, while the line itself remains at a finite distance.

An *asymptotic plane* is a tangent plane whose point of contact is at an infinite distance, the plane itself being at a finite distance.

An *asymptotic surface* is a surface which is enveloped by all the asymptotic planes to the surface.

515. *General considerations on asymptotes.*

If we imagine any tangent plane to a surface, and consider the result of supposing its point of contact to be at an infinite distance, we shall be led to the following conclusions:

Since the plane at infinity intersects the surface in a curve, real or imaginary, there are generally an infinite number of directions in which a point of contact may be supposed to move off to infinity; to each of these directions will correspond an asymptotic plane.

Each asymptotic plane is the locus of all the corresponding asymptotes, and these asymptotes will all be parallel, since they pass through the same point at infinity at which they are tangents.

Since there are two tangents in every tangent plane at an ordinary point which pass through three consecutive points, viz. the tangents to the curve of intersection at the point of contact, there are in each asymptotic plane two corresponding inflexional asymptotes which pass through three points at an infinite distance.

Since any plane which passes through an inflexional tangent intersects the surface in a curve which has a point of inflexion at the point of contact of such a tangent, the curve of intersection of the surface and any plane drawn through an inflexional asymptote has a point of inflexion at an infinite distance.

516. The peculiarities which arise in the case of singular points at an infinite distance can be examined without much difficulty by a comparison with what takes place at a finite distance.

If, for example, there be a double point at infinity, in the place of the conical tangent at a finite distance, there will be a cylinder of the second degree formed by the asymptotes which correspond to the direction in which the double point lies.

Of the generating lines of this asymptotic cylinder there are six which meet the surface in four points at infinity.

The curve of intersection with any plane parallel to these generating lines has a double point at infinity.

The curve of intersection with any tangent plane to the cylindrical asymptote has a cusp at infinity.

517. *To find the asymptotes to a given surface.*

Let $F \equiv F(\xi, \eta, \zeta) = 0$ be the equation of the given surface, (x, y, z) any point in an asymptote, $(\xi - x)/\lambda = (\eta - y)/\mu = (\zeta - z)/\nu = r$ its equations; and let $F(\lambda, \mu, \nu)$ be arranged in a series of homogeneous functions of the degrees $n, n-1, \ldots$, so that $F(\lambda, \mu, \nu) \equiv \phi_n + \phi_{n-1} + \ldots + \phi_1 + c$.

The points in which the asymptote meets the surface are given by the equation $F(x+\lambda r, y+\mu r, z+\nu r)=0$, or, if D denote the operation

$$x\frac{d}{d\lambda}+y\frac{d}{d\mu}+z\frac{d}{d\nu},$$

$$r^n\phi_n+r^{n-1}(D\phi_n+\phi_{n-1})+r^{n-2}(\tfrac{1}{2}D^2\phi_n+D\phi_{n-1}+\phi_{n-2})+\ldots=0.$$

Now for a simple asymptote two roots are infinite;

$$\therefore\ \phi_n=0,\quad (1)\qquad\text{and}\qquad D\phi_n+\phi_{n-1}=0.\quad (2)$$

The equation (1) shews that all asymptotes are parallel to generating lines of the cone $F_n(\xi,\eta,\zeta)=0$, (3) where F_n consists of the terms of the n^{th} degree in F.

The equation (2) shews that all the asymptotes parallel to any generating line of the cone (3) lie in one plane, which is the asymptotic plane parallel to the tangent plane touching the cone along the generating line.

Again, corresponding to inflexional tangents in tangent planes at points at a finite distance, there are generally two asymptotes in each asymptotic plane which meet the surface in three points at an infinite distance, the condition of this is $\frac{1}{2}D^2\phi_n+D\phi_{n-1}+\phi_{n-2}=0$, (4) and the two inflexional asymptotes are the lines of intersection of the conicoid (4) with the plane (2). It can be shewn that the conicoid and plane insersect in two parallel or coincident lines by proving that, if (x, y, z) be any point in which they intersect, a line drawn through this point in the direction (λ, μ, ν) lies entirely in both surfaces.

Write $x+\lambda r$ for x, &c., and Δ for $\lambda\dfrac{d}{d\lambda}+\mu\dfrac{d}{d\mu}+\nu\dfrac{d}{d\nu}$, λ, μ, ν being considered constant in the differentiations; then (2) becomes $(D+r\Delta)\phi_n+\phi_{n-1}=r\Delta\phi_n=rn\phi_n$, and (4) becomes $\frac{1}{2}(D+r\Delta)^2\phi_n+(D+r\Delta)\phi_{n-1}+\phi_{n-2}=r\Delta D\phi_n+\frac{1}{2}r^2\Delta^2\phi_n+r\Delta\phi_{n-1}=r(n-1)(D\phi_n+\phi_{n-1})+\frac{1}{2}r^2n(n-1)\phi_n=\frac{1}{2}r^2n(n-1)\phi_n$; therefore, since $\phi_n=0$, (2) and (4) are satisfied for all values of r.

518. Should the student be interested in the discrimination of the various singularities which may occur, he will find a guide in two articles by Painvin,* who has nearly adopted our method of treatment, and has carefully followed out the consequences of supposing the conicoid (4) to have the various forms of which it is capable.

519. A singular asymptotic plane is one which touches the surface along a line at infinity, if considered as the limit of a tangent plane; and if considered as the locus of asymptotic lines, it is a plane such that lines drawn in *any* direction in it meet the surface in two points at an infinite distance.

The analytical conditions are obtained by considering that the equation $D\phi_n+\phi_{n-1}=0$ must be independent of the values of λ, μ, ν.

Asymptotic Surfaces.

520. *To find the asymptotic surface of a given surface.*

The asymptotic surface being the surface enveloped by the asymptotic planes, which are tangent planes whose points of contact are at an infinite distance, is a torse circumscribing the surface along the curve of intersection with the plane at infinity.

Write U_n, V_n, W_n and u, v, w, u', v', w' for the first and second differential coefficients of ϕ_n with respect to λ, μ, and ν; the equation of an asymptotic plane is $P=xU_n+yV_n+zW_n+\phi_{n-1}=0$, by (2), where $\phi_n=0$ and $\lambda^2+\mu^2+\nu^2=1$.

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* Crelle's *Journal*, vol. 65.

Considering a consecutive position of the asymptotic plane we have the equations

$$\frac{dP}{d\lambda}d\lambda + \frac{dP}{d\mu}d\mu + \frac{dP}{d\nu}d\nu = 0,$$
$$U_n\,d\lambda + V_n\,d\mu + W_n\,d\nu = 0,$$
$$\lambda d\lambda + \mu d\mu + \nu d\nu = 0;$$

from which it follows that $\dfrac{dP}{d\lambda}/U_n = \dfrac{dP}{d\mu}/V_n = \dfrac{dP}{d\nu}/W_n$. (5)

These equations and (2) are equivalent to two distinct equations which are those of a generating line of the asymptotic surface; that of the surface itself is found by eliminating λ, μ, ν from $\phi_n = 0$ and these two equations, all being homogeneous in λ, μ, ν.

If $\phi_{n-1} \equiv 0$, since $(n-1)\,U_n = \lambda u + \mu w' + \nu v'$, and $\dfrac{dP}{d\lambda} = xu + yw' + zv'$, the equations (5) are reduced to $x/\lambda = y/\mu = z/\nu$, and the asymptotic surface becomes the cone $\phi_n = 0$, as in the case of $ax^3 + by^3 + cz^3 = 1$.

In the general case it is easily seen that the generating line passes through the centre of the conicoid which determines the position of the inflexional asymptotes, for which $\dfrac{dP}{d\lambda} = \dfrac{dP}{d\mu} = \dfrac{dP}{d\nu} = 0$.

521. *To find the degree of the asymptotic surface.*

We shall find how many generating lines intersect an arbitrary straight line $(x-\alpha)/l' = (y-\beta)/m' = (z-\gamma)/n' = r$. If we equate to $(n-1)\,\rho$ each member of equations (5) of a generating line, the equations may be written

$$(x-\lambda\rho)\,u + (y-\mu\rho)\,w' + (z-\nu\rho)\,v' + U_{n-1} = 0,$$
$$(x-\lambda\rho)\,w' + (y-\mu\rho)\,v + (z-\nu\rho)\,u' + V_{n-1} = 0,$$
$$(x-\lambda\rho)\,v' + (y-\mu\rho)\,u' + (z-\nu\rho)\,w + W_{n-1} = 0,$$

or, if $H \equiv \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}$,

$$(x-\lambda\rho)\,H + \left(U_{n-1}\frac{d}{du} + V_{n-1}\frac{d}{dw'} + W_{n-1}\frac{d}{dv'}\right)H = 0;$$

therefore at the point of intersection

$$(\alpha + l'r)\,H - \lambda\rho H + \left(U_{n-1}\frac{d}{du} + \ldots\right)H = 0,$$

and similar equations; or, eliminating r and ρ,

$$\begin{vmatrix} \alpha H + \left(U_{n-1}\dfrac{d}{du} + \ldots\right)H, & \lambda, & l' \\ \beta H + \left(U_{n-1}\dfrac{d}{dw'} + \ldots\right)H, & \mu, & m' \\ \gamma H + \left(U_{n-1}\dfrac{d}{dv'} + \ldots\right)H, & \nu, & n' \end{vmatrix} = 0,$$

now the degrees of $\dfrac{dH}{du} \equiv vw - u'^2$ and U_{n-1} are $2(n-2)$ and $n-2$; the degree of the equation is therefore $3n-5$, and the number of values of λ, μ, ν which satisfy this equation and $\phi_n = 0$ is $n(3n-5)$, which is the degree of the asymptotic surface.

522. Or we may proceed thus:

The asymptotic surface contains $3n(n-2)$ lines in the plane at infinity which are the intersections of the planes of inflexion of the cone $\phi_n = 0$,

and contains, moreover, the curve of the n^{th} degree, in which the plane at infinity intersects the cone; hence, the number of points in which the asymptotic surface is met by an arbitrary line in the plane at infinity

$$= 3n(n-2) + n \equiv n(3n-5).$$

For limitations of the number arising from the existence of singular points, see Painvin's second article.*

Method of Approximation.

523. Although it is necessary to know general methods of handling the equations of surfaces, yet in order to find the shape at particular points or at an infinite distance, it is most instructive for the student to employ peculiar methods to suit peculiar cases.

The method of approximation by transferring the origin to the particular point in question, and rejecting all terms which can be shewn to be small compared with those retained, gives immediately conical tangents or any other form which nearly coincides with a surface in the neighbourhood of a singular point.

The form of a surface at an infinite distance may be found by a careful consideration of the relative magnitude of the coordinates in the same manner as the author has treated the subject in his treatise on Curve Tracing. The kind of consideration required may be seen by the following example.

524. *To find the plane and parabolic asymptotes of the surface whose equation is*

$$x^3 + y^3 + z^3 - 3xyz - 3a(yz + zx + xy) = 0.$$

The equation may be written $(u+a)v = au^2$,

$$\text{where } u \equiv x + y + z, \quad v \equiv x^2 + y^2 + z^2 - yz - zx - xy.$$

For points at an infinite distance, two cases occur; i. u may be finite while v becomes infinite, which gives $u + a = 0$ a plane asymptote. ii. u and v may both be infinite, in which case $v = au$ for a first approximation, and for a second approximation $v = a(u-a)$, a paraboloid of revolution.

The same results may be obtained by making the line $x = y = z$ a new axis of x, so that the equation becomes $(\sqrt{3}x + a)(y^2 + z^2) = 2ax^2$, the asymptotes being $\sqrt{3}x + a = 0$, and $3(y^2 + z^2) = 2a(\sqrt{3}x - a)$.

We have selected the following illustrations of the points which have been considered in this chapter, and we call attention especially to those relating to cubic surfaces and the wave surface.

525. *Tangent plane to an anchor ring or tore.*

Let the plane containing the centres of the generating circles be taken for the plane of xy, and the axis of rotation for the axis of z; and let r be the distance of any point (x, y, z) from the axis, c that of the centre of the generating circle, a its radius; then $r^2 = x^2 + y^2$ and $z^2 + (r-c)^2 = a^2$; the equation of the anchor ring is $(\xi^2 + \eta^2 + \zeta^2 + c^2 - a^2)^2 - 4c^2(\xi^2 + \eta^2) = 0$; that of the tangent plane at a point (x, y, z) is

$$x(r-c)(\xi - x) + y(r-c)(\eta - y) + zr(\zeta - z) = 0,$$

$$\text{or } (r-c)(x\xi + y\eta) + rz\zeta = r^2(r-c) + rz^2 = r\{a^2 + c(r-c)\}.$$

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* Crelle's *Journal*, vol. 65.

526. *To find the curve of intersection of the surface with a tangent plane which passes through the centre.*

Suppose that it passes through the axis of y, and is inclined at an angle α to that of x, so that $a = c \sin \alpha$; and at any point of the curve of intersection $r = c - a \cos\theta$, $z = a \sin\theta$, $x = z \cot \alpha$;

$$\therefore\ y^2 = r^2 - x^2 = c^2 - 2ac\cos\theta + a^2\cos^2\theta - a^2\cot^2\alpha\sin^2\theta$$
$$= c^2 - 2ac\cos\theta + a^2\cos^2\theta - (c^2 - a^2)\sin^2\theta = (c\cos\theta - a)^2;$$
$$\therefore\ (y \pm a)^2 = c^2\cos^2\theta, \text{ and } x^2 + z^2 = c^2\sin^2\theta:$$
$$\therefore\ x^2 + (y \pm a)^2 + z^2 = c^2;$$

hence the curve is two circles which intersect in the points of contact, forming two double points.

527. *To find the form of the curve EAF in the figure of the ring,* p. 199.

The equation of the tangent plane is $\xi = c - a$, and the form of the curve of intersection is given by the equation

$$\{\eta^2 + \zeta^2 + 2c(c-a)\}^2 = 4c^2\{\eta^2 + (c-a)^2\},$$
$$\text{or } (\eta^2 + \zeta^2)^2 - 4ac\eta^2 + 4c(c-a)\zeta^2 = 0.$$

When $c = 2a$, the curve is the lemniscate of Bernoulli.

528. *If a ring be formed by the revolution of any ellipse about any line in its plane which does not intersect it, the double tangent planes intersect the ring in two ellipses, the projections of which on a plane perpendicular to the axis of revolution have each one focus in that axis.*

For, Oz being the axis of revolution, Oy the intersection of two double tangents to the ring, their traces on the plane zx must be tangents to the generating ellipse in its two positions on zx, making equal angles with Ox, so that if $u_2 + 2u_1 + u_0 = 0$ be the equation of one trace on zx, $u_1 + u_0 = 0$ will be the polar of O, and $u_0u_2 - u_1^2 = 0$ the equation of the two tangents, which must be of the form $z^2 = x^2\tan^2\alpha$, the coefficient of xz must therefore vanish.

A convenient form for the equation of the ellipse is

$$(x-a)^2 - b^2 + 2mxz + nz^2 - 2cz = 0,$$

and for the two tangents through O

$$(a^2 - b^2)(x^2 + 2mxz + nz^2) - (ax + cz)^2 = 0,$$
$$\text{hence } m(a^2 - b^2) = ac, \text{ and } n(a^2 - b^2) - c^2 = b^2\cot^2\alpha. \tag{1}$$

If $x = r\cos\theta$, $y = r\sin\theta$, the equation of the ring will be

$$(r-a)^2 - b^2 + 2mrz + nz^2 - 2cz = 0.$$

The ring is intersected by the tangent plane $z = x\tan\alpha = r\cos\theta\tan\alpha$ in a curve, the equation of the projection of which on the plane of xy is, by (1),

$$\{a^2 - b^2 - (a + c\cos\theta\tan\alpha)r\}^2 = r^2b^2\sin^2\theta,$$
$$\text{or } a^2(x^2 + y^2) = (a^2 - b^2 - cx\tan\alpha \mp by)^2,$$

i.e. two ellipses, each having a focus in the origin, the squares of their eccentricities being $(c^2\tan^2\alpha + b^2)/a^2$.

529. *Tangent plane and normal to a Helicoid.*

Def. The Helicoid is a scroll generated by the motion of a straight line which intersects at right angles a fixed axis, about which it twists with an angular velocity which varies as the velocity of the point of intersection with the axis.

If the axis be taken for the axis of z, and that of x be one position of the generating line, the equation of the surface generated will be $\zeta = c\tan^{-1}\eta/\xi$, and the tangent plane at a point (x, y, z) will be $(x^2 + y^2)(\zeta - z) = c(x\eta - y\xi)$; at the point $(x, 0, 0)$, the equation becomes $x\zeta = c\eta$, hence the tangent of the angle which the tangent plane at any point makes with the axis varies as the distance

of the point from the axis. The equations of the normal at (x, y, z) are

$$(\xi - x)/y = -(\eta - y)/x = c(\zeta - z)/(x^2 + y^2);$$

and for the normal at $(x, 0, 0)$, $\xi = x$, $x\eta + c\zeta = 0$, hence the locus of the normals at points taken along a generating line is a hyperbolic paraboloid; which has been shewn to be true for any scroll.

530. *To find the singularities of the surface whose equation is*

$$(z^2 + 2x^2 + 2y^2)^2 - (x^2 + y^2)(x^2 + y^2 + 1)^2 = 0.$$

We consider this surface as represented by the given equation, in order to illustrate the general methods given in Arts. 468 and 496, for discussing singular points and planes; but the student will see clearly the results to which we shall be led, if he first trace the plane curve whose equation is $z^2 = x(1 - x)^2$,* and then imagine the form of the surface which would be generated by its revolution round the axis of z, which it is easily seen is the surface proposed.

To find a singular point we have, writing r^2 for $x^2 + y^2$,

$$U = 2x(4z^2 - 1 + 4r^2 - 3r^4) = 0,$$
$$V = 2y(4z^2 - 1 + 4r^2 - 3r^4) = 0,$$
$$W = 4z(z^2 + 2r^2) = 0.$$

The systems of values of x, y, z which simultaneously satisfy these equations and that of the surface are $z = 0$, and either i. $x = 0$, $y = 0$, or ii. $r^2 = 1$; i. shews that the origin is a singular point—it will be found that the tangent cone of Art. 469 becomes an infinitely slender cylinder or cone, given by $\lambda^2 + \mu^2 = 0$; ii. gives a circle of singular points—the conical tangent at any point $(x, y, 0)$ of this circle becomes the two tangent planes $(x\xi + y\eta - 1)^2 = \zeta^2$.

To find a singular tangent plane we have, by Art. 498, the equations

$$2(\lambda x + \mu y)(4z^2 - 1 + 4r^2 - 3r^4) + 4z(z^2 + 2r^2)\nu = 0,$$

and $$2(\lambda^2 + \mu^2)(4z^2 - 1 + 4r^2 - 3r^4) + 4\nu^2(3z^2 + 2r^2)$$
$$+ 32z(\lambda x + \mu y)\nu + 8(\lambda x + \mu y)^2(2 - 3r^2) = 0;$$

there will be two coincident tangents if

$$\nu = 0 \text{ and } 4z^2 - 1 + 4r^2 - 3r^4 = 0,$$

also by the equation of the surface $(z^2 + 2r^2)^2 - r^2(1 + r^2)^2 = 0$, the only solutions of these equations are $z^2 = 0$, $r^2 = 1$, and $z^2 = \frac{4}{27}$, $r^2 = \frac{1}{9}$, the first solution gives no tangent plane, but two cones intersecting in a circle, any generating line of either of which is a tangent line; the second solution gives two tangent planes $z = \pm\frac{2}{9}\sqrt{3}$, each of which is a singular tangent plane touching along a circle $x^2 + y^2 = \frac{1}{9}$, the direction of the tangent to which is given by $\lambda x + \mu y = 0$ and $\nu = 0$, the remaining part of the curve of intersection is a single circle of radius $\frac{4}{3}$.

In this case the condition of four points being coincident is

$$\frac{1}{\underline{|3}}\left(\lambda\frac{d}{dx} + \mu\frac{d}{dy}\right)^3 F = 0,$$

which becomes $\frac{20}{3}(\lambda x + \mu y)(\lambda^2 + \mu^2) - 8(\lambda x + \mu y)^3 = 0$,

it is therefore satisfied by $\lambda x + \mu y = 0$.

Wave Surface.

531. The equation of the Wave Surface may be written in either of the forms $x^2/(r^2 - a^2) + y^2/(r^2 - b^2) + z^2/(r^2 - c^2) = 1$, or $a^2x^2/(r^2 - a^2) + b^2y^2/(r^2 - b^2) + c^2z^2/(r^2 - c^2) = 0$, Art. 261, where $r^2 = x^2 + y^2 + z^2$; and we shall suppose $a > b > c$.

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* Frost's *Curve Tracing*, Plate II., Fig. 8.

The existence of singular tangent planes to this surface is of great importance in explaining a peculiarity in the transmission of light through a biaxial crystal.

In order to shew that such planes exist, we shall employ the method of Art. 497.

532. *To find the point of contact of a tangent plane whose equation is $l\xi + m\eta + n\zeta = p$, and the relation between l, m, n and p.*

Taking the first form of the equation, by Art. 466,

$$U/l = V/m = W/n = (Ux + Vy + Wz)/p = 2\sigma, \text{ suppose,}$$

$$\text{where } U = 2x/(r^2 - a^2) - 2xP, \text{ \&c.,} \qquad (1)$$

$$\text{and } P = x^2/(r^2 - a^2)^2 + y^2/(r^2 - b^2)^2 + z^2/(r^2 - c^2)^2;$$

$$\therefore \tfrac{1}{2}(Ux + Vy + Wz) = p\sigma = 1 - r^2P,$$

$$\tfrac{1}{4}(U^2 + V^2 + W^2) = \sigma^2 = P - 2P + r^2P^2 = -Pp\sigma;$$

$$\therefore 0 = 1 - (r^2 - p^2)P, \quad \sigma(r^2 - p^2) = -p,$$

$$\text{by (1), } \sigma l = x/(r^2 - a^2) - x/(r^2 - p^2);$$

$$\therefore lp(r^2 - a^2) = x(p^2 - a^2), \text{ \&c.,} \qquad (2)$$

and, multiplying by l, m, n, since $lx + my + nz = p$,

$$l^2(r^2 - a^2)/(p^2 - a^2) + \ldots = 1 = l^2 + m^2 + n^2,$$

$$\therefore l^2/(p^2 - a^2) + m^2/(p^2 - b^2) + n^2/(p^2 - c^2) = 0, \qquad (3)$$

$$x = lp\{(r^2 - p^2)/(p^2 - a^2) + 1\}, \text{ \&c.;} \qquad (4)$$

$$\therefore r^2 = p^2(r^2 - p^2)^2\{l^2/(p^2 - a^2)^2 + m^2/(p^2 - b^2)^2 + n^2/(p^2 - c^2)^2\} + p^2;$$

hence $r^2 - p^2$ is known, and by (4) x, y, z are determined in terms of the constants, (3) being the required relation between the constants.

533. If α, β, γ be the Boothian coordinates of the tangent plane, viz. l/p, m/p, n/p, and a', b', c' be the reciprocals of a, b, c, the tangential equation of the wave surface will be, by (2),

$$a'^2\alpha^2/(\rho^2 - a'^2) + b'^2\beta^2/(\rho^2 - b'^2) + c'^2\gamma^2/(\rho^2 - c'^2) = 0,$$

where $\rho^2 = \alpha^2 + \beta^2 + \gamma^2$, an equation of the same form as the Cartesian.

534. *To find the singular tangent planes of the wave surface.*

The point of contact, in the case of a singular tangent plane, being indeterminate, suppose y of the form $0/0$, or $m = 0$ and $p = b$; hence, by (2), $lb(r^2 - a^2) = x(b^2 - a^2)$ and $nb(r^2 - c^2) = z(b^2 - c^2)$, (5);

$\therefore$, eliminating r^2, $a^2 - c^2 = (a^2 - b^2)x/lb + (b^2 - c^2)z/nb$ and $b = lx + nz$,

$\therefore l^2/(a^2 - b^2) = n^2/(b^2 - c^2) = 1/(a^2 - c^2)$, (6).

The curve in which the plane touches the surface is a circle which is the intersection of the plane $lx + nz = b$ with either of the spheres (5).

If a, b, c be in order of magnitude, $l = 0$, $p = a$, and $n = 0$, $p = c$ give imaginary tangent planes. The four real singular tangent planes are given by (6) and $m = 0$.

535. *To find the singular points and the corresponding normal cones.*

The singular points may be found by investigating for what definite points of contact with the plane $l\xi + m\eta + n\zeta = p$, l, m, n can be indeterminate; by making m indeterminate, $y = 0$, $r = b$, and

$$a^2x^2/(a^2 - b^2) = c^2z^2/(b^2 - c^2) = a^2c^2/(a^2 - c^2). \qquad (7)$$

But, by (2), $-(a^2 - b^2)\,l/x = p - a^2/p$ and $(b^2 - c^2)\,n/z = p - c^2/p$;

$$\therefore\ (a^2 - b^2)\,l/x + (b^2 - c^2)\,n/z = (a^2 - c^2)/p,$$

and, by (7), $(a^2lx + c^2nz)(lx + nz) = a^2c^2(l^2 + m^2 + n^2)$;

by substituting the values of x and z, this reduces to

$$(a^2 - b^2)\,l^2 + (a^2 - c^2)\,m^2 + (a^2 - b^2)\,n^2 = (a/c + c/a)\,\sqrt{(a^2 - b^2)}\,\sqrt{(b^2 - c^2)}\,ln,$$

which, being a homogeneous equation, gives the equation of the normal cone at the singular point.

The four real singular points are given by $y = 0$ and (7).

Cubic Surfaces.

536. *On every surface of the third degree there are* 27 *straight lines and* 45 *triple tangent planes, real or imaginary.*

This theorem was first discovered by Cayley.*

An arbitrary straight line intersects a cubic surface in three points, given by an equation of the form

$$u + Du.r + \tfrac{1}{2}D^2u.r^2 + \tfrac{1}{6}D^3u.r^3 = 0.$$

Now the four constants in the equation of a line may be chosen so as to satisfy the equations $u = 0$, $Du = 0$, $D^2u = 0$, $D^3u = 0$, and, since the above equation will then be satisfied by all values of r, all straight lines having such constants will lie entirely in the surface; and the number of such straight lines will clearly be limited, speaking generally, although in particular cases, as in that of a cylindrical surface, it may be infinite.

If a plane be drawn in any direction through such a straight line, its curve of intersection with the surface will be composed of that straight line and a conic forming a group of the third degree; and the two double points in which the straight line intersects the the conic are two points of the surface at which the plane is a tangent plane to the cubic, Art. 475.

Now, there will be five positions of the plane for which the conic will become two straight lines.

For, if the axis of x be a line which lies entirely in the surface, the equation of the surface will be of the form $yu_2 + zv_2 = 0$, where u_2, v_2 are quadrics; and if the surface be cut by a plane whose equation is $y/\mu = z/\nu = r$, the conic, which is part of the line of intersection, will have for its equation $\mu u'_2 + \nu v'_2 = 0$, where u'_2, v'_2 have each the form $a_2r^2 + b_0x^2 + c_0 + 2d_0x + 2e_1r + 2f_1rx = 0$, in

* *Cambridge and Dublin Mathematical Journal*, vol. IV.

which a_2, e_1, f_1 are homogeneous functions of μ and ν of the degrees denoted by the suffixes.

Hence the equation of the conic will be of the form

$$\alpha_2 r^2 + \beta_1 x^2 + \gamma_1 + 2\delta_1 x + 2\varepsilon_2 r + 2\zeta_2 rx = 0;$$

it will therefore become two straight lines if

$$\alpha_2\beta_1\gamma_1 + 2\delta_1\varepsilon_2\zeta_2 - \alpha_2\delta_1^2 - \beta_1\varepsilon_2^2 - \gamma_1\zeta_2^2 = 0,$$

which gives five values of the ratio $\mu : \nu$.

In each of the five particular positions of the plane the complete intersection is three straight lines, which give three double points, and the plane is a triple tangent plane touching the surface at each of these double points.

Through each of the three straight lines in a triple tangent plane four other triple tangent planes besides the one considered can be drawn, giving rise to 12 new triple tangent planes and 24 new straight lines, making in all 27; and the surface cannot contain any but these 27 lines, for the point in which any line on the surface meets a triple tangent plane ABC must lie on one of the three lines AB, BC, CA, which form the complete intersection of ABC with the surface, and the plane which passes through the new line and AB supposing this to be the line which it cuts, must contain a third line, and, therefore, must be one of the five triple tangent planes drawn through AB; the line considered must therefore be one of the 27 lines.

Five triple tangent planes can be drawn through each of the 27 lines, which would make 5×27 planes in all; but since each plane contains three of the lines, we have in obtaining this number reckoned each three times, hence the number of triple tangent planes is 45.

537. *To find the equations of the 27 lines on the general cubic surface.*

The general equation of the cubic surface may be put into the form $uvw + u'v'w' = 0$, where u, v, &c. are, when equated to zero, equations of planes, since it contains 19 constants, the requisite number.

Now, whatever u, v, &c. may be, a, b, c, α, β, γ, a', &c. can always be chosen such that the two following equations are true identically, viz.

$$au + bv + cw + a'u' + b'v' + c'w' = 0,$$

$$\text{and } \alpha u + \beta v + \gamma w + \alpha' u' + \beta' v' + \gamma' w' = 0;$$

and therefore, for all values of ρ, $(a\rho - \alpha)u + (b\rho - \beta)v + \ldots \equiv 0$, which may be written $\lambda u + \mu v + \nu w + \lambda' u' + \mu' v' + \nu' w' \equiv 0$, hence the line of intersection of $\lambda u + \lambda' u' = 0$, $\mu v + \mu' v' = 0$ lies in the plane $\nu w + \nu' w' = 0$.

Now this particular line will lie altogether on the surface, if $\lambda\mu\nu = \lambda'\mu'\nu'$, or if ρ be chosen as one of the three roots of

$$(a\rho - \alpha)(b\rho - \beta)(c\rho - \gamma) = (a'\rho - \alpha')(b'\rho - \beta')(c'\rho - \gamma').$$

And, this being true of any other similar combinations, such as $\lambda u + \lambda' u' = 0$, $\mu w + \nu' w' = 0$, $\nu w + \mu' v' = 0$, since there are six such combinations, we have 18 lines out of the 27, the remaining 9 being the obvious ones $u = 0$, $u' = 0$; $v = 0$, $v' = 0$; &c.

If $\lambda\lambda_1\lambda_2$, $\mu\mu_1\mu_2$, &c. be successively written for λ, μ, &c. corresponding to the roots $\rho\rho_1\rho_2$ of the cubic, in the following table of equations:

$$\left.\begin{array}{l}\lambda u+\lambda' u'\\ \mu v+\mu' v'\\ \nu w+\nu' w'\end{array}\right\}=0,\quad(1)\qquad \left.\begin{array}{l}\lambda u+\lambda' u'\\ \mu v+\nu' w'\\ \nu w+\mu' v'\end{array}\right\}=0,\quad(2)\qquad \left.\begin{array}{l}\lambda u+\mu' v'\\ \mu v+\nu' w'\\ \nu w+\lambda' u'\end{array}\right\}=0,\quad(3)$$

$$\left.\begin{array}{l}\lambda u+\mu' v'\\ \mu v+\lambda' u'\\ \nu w+\nu' w'\end{array}\right\}=0,\quad(4)\qquad \left.\begin{array}{l}\lambda u+\nu' w'\\ \mu v+\lambda' u'\\ \nu w+\mu' v'\end{array}\right\}=0,\quad(5)\qquad \left.\begin{array}{l}\lambda u+\nu' w'\\ \mu v+\mu' v'\\ \nu w+\lambda' u'\end{array}\right\}=0,\quad(6)$$

the equations of the 18 lines are found.

538. Let these lines be named $1, 1_1, 1_2$; $2, 2_1, 2_2$; &c., each of the lines $1, 3, 5$ lies in the same plane as, and therefore intersects, each of $2, 4, 6$; but $1, 3, 5$ do not intersect, nor do $2, 4, 6$. Again, 1 intersects 3_1, 3_2, and 5_1, 5_2, 2 intersects 4_1, 4_2, and 6_1, 6_2, to prove which we shew that 1 and 3_1 intersect;

$$(\lambda u+\lambda' u')\,A/\lambda+(\mu v+\mu' v')\,B/\mu+(\nu w+\nu' w')\,C/\nu=0, \qquad \text{(i)}$$

$$\text{and}\quad(\lambda_1 u+\mu'_1 v')\,A/\lambda_1+(\mu_1 v+\nu'_1 w')\,B/\mu_1+(\nu_1 w+\lambda'_1 u')\,C/\nu_1=0, \qquad \text{(ii)}$$

represent two planes containing respectively 1 and 3_1, unless $A:B:C=\lambda:\mu:\nu$, or $\lambda_1:\mu_1:\nu_1$, and these planes coincide if $A\lambda'/\lambda=C\lambda'_1/\nu_1$, $B\mu'/\mu=A\mu'_1/\lambda_1$, and $C\nu'/\nu=B\nu'_1/\mu_1$ can hold simultaneously, which is the case since $\lambda\mu\nu=\lambda'\mu'\nu'$ and $\lambda_1\mu_1\nu_1=\lambda'_1\mu'_1\nu'_1$.

It follows that $1, 3_1, 5_2$ lie in the same plane, for 1 and 3_1, 3_1 and 5_2, 5_2 and 1 intersect.

539. *To find the equations of the 45 triple tangent planes.*

6 are such as $u=0$, $u'=0$, &c.,

9 .... $\lambda u+\lambda' u'=0$, $\mu v+\nu' w'=0$, for each root,

6 .... 13_15_2, 1_235_1, (i) or (ii)

6 .... 2_146_2, 24_26_1,

making $6+3\times 9+6+6\equiv 45$.

540. Let uu' denote the line of intersection of $u=0$, $u'=0$, uu' is intersected by 10 lines, which lie by pairs in 5 planes, viz. uv', uw'; vu', wu'; $1, 2$; $1_1, 2_1$; $1_2, 2_2$; 1 is intersected by uu', 2; vv', 6; ww', 4; $3_1, 5_2$; $3_2, 5_1$; which classifies the 135 points of intersection.

541. Consider an arrangement of 12 lines such as

$$\begin{array}{cccccc}1, & 3, & 5, & 2_1, & 4_1, & 6_1,\\ 1_2, & 3_2, & 5_2, & 2, & 4, & 6,\end{array}$$

any line in the top row, such as 5, intersects every line in the bottom row, except 5_2 the one beneath it. Such a combination is called a *double sixer*, of which there are 36, 3 such as that given above,

$$\begin{array}{rcccccc}27\text{ such as} & uu', & vv', & 5, & 3, & 4_1, & 4_2,\\ & 6, & 2, & uv', & vu', & 1_2, & 1_1,\\ \text{and 6 such as} & uu', & vv', & ww', & 3, & 3_1, & 3_2,\\ & vw', & wu', & uv', & 1, & 1_1, & 1_2.\end{array}$$

XXXV.

(1) Prove that the tangent plane to the surface $xyz=a^3$ forms with the coordinate planes a tetrahedron of constant volume.

(2) If tangent planes be drawn at every point of the curve of intersection of the surface $a(yz+zx+xy)=xyz$, with a sphere whose centre is at the origin, shew that the sum of the three intercepts on the axes will be the same for all.

(3) Find the points on the surface $(x^2+y^2+z^2+c^2-a^2)^2=4c^2(x^2+y^2)$ the normals at which make angles α, β, γ with the axes, and the loci of points for which (i) γ is constant, (ii) α is equal to β.

(4) A surface is given by the elimination of a between the equations $F(x, y, z, a)=0$ and $f(x, y, z, a)=0$; shew that the direction-cosines of the normal at a point (x, y, z) are in the ratio

$$F'(x)f'(a)-f'(x)F'(a):F'(y)f'(a)-f'(y)F'(a):F'(z)f'(a)-f'(z)F'(a).$$

(5) Prove that the projection on the plane of xy of the normals to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, at points whose distance from that plane is $c\cos\alpha$, touch the curve $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}\sin^{\frac{2}{3}}\alpha$.

(6) Find the tangent cone at the origin to the surface

$$(x^2+y^2+ax)^2-(c^2-a^2)(x^2+y^2)=0;$$

and shew that as a diminishes and ultimately vanishes, the tangent cone contracts, and ultimately becomes an infinitely thin cylinder, and as a increases up to c, it expands, and finally becomes two coincident planes. Find the singular tangent plane; and give a construction for the surface.

(7) Prove that the condition that the surface $ax^3+by^3+cz^3=1$ should cut the tangent plane at a point (α, β, γ) in a curve having a double point with two branches at right angles is $a^2\alpha^4(b\beta+c\gamma)+b^2\beta^4(c\gamma+a\alpha)+c^2\gamma^4(a\alpha+b\beta)=0$.

(8) Shew that the asymptotic surface of $z(x+y)^2-ax^2+bz^2=0$ is a parabolic cylinder.

(9) Find the asymptotic planes and the asymptotic surface of the conicoid $ax^2+by^2+cz^2=2x$.

(10) From different points of the straight line $x/a=y/b$, $z=0$, asymptotic lines are drawn to the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$; shew that they will all lie in the plane $x/a-y/b=\pm\sqrt{2}\,z/c$.

XXXVI.

(1) Prove that the surface $(ax^2+by^2+cz^2)^2-3(ax^2+by^2)-cz^2+\frac{1}{4}=0$, has two conical points, and two singular tangent planes.

(2) Prove that, for surfaces generated by a line which is always parallel to a fixed plane, the line of striction is the locus of points at which the normals are parallel to that plane. Hence, shew that the lines of striction of the paraboloid $x^2/a^2-y^2/b^2=z/c$ lie in the planes $x/a^2\pm y/b^2=0$.

(3) Shew that the coordinate planes are the three singular asymptotic planes of the surface $xyz=a^3$.

(4) Shew that the surface whose equation is $z(x^2+y^2)=ax^2+by^2$ is a scroll, and give the form of the sections by planes perpendicular to the axes of x and y; and shew that there is an evanescent cylindrical asymptote.

(5) Discuss the form of the surfaces $z(x+y)^2-a(x^2-y^2)\pm b^2z=0$ at an infinite distance.

(6) If tangent planes be drawn to the surface $(x+y+z-c)^2=4(yz+zx+xy)$ at all points where it is met by the surface $4xyz+c(x+y+z-c)^2=0$, the sum of the intercepts on the axes of coordinates will be constant.

(7) If a series of straight lines, generating a surface, be described according to a law such that the shortest distance between two consecutive lines is of a degree superior to the first, it will be at least of the third.

(8) A generating line PQ of $xyz = a(x^2 + y^2)$ meets the axis of z in P, prove that the tangent plane at Q meets the surface in a hyperbola which passes through P. Also, as Q moves along the generator, prove that the tangent to the hyperbola at P generates a plane.

(9) An ellipsoid stands on a horizontal plane with its least axis vertical, find the locus of a luminous point which casts a circular shadow. Shew that it is a hyperbola, and that the radius of the circular shadow is independent of the mean axis of the ellipsoid.

(10) Prove that the torse circumscribing the conicoids $x^2/a + y^2/b + z^2/c = 1$ and $x^2/a' + y^2/b' + z^2/c' = 1$ is the envelope of the family of conicoids represented by $x^2/(a\cos^2\theta + a'\sin^2\theta) + y^2/(b\cos^2\theta + b'\sin^2\theta) + z^2/(c\cos^2\theta + c'\sin^2\theta) = 1$.

XXXVII.

(1) The torse which passes through the two circles $x^2 + y^2 = a^2$, $z = 0$, and $x^2 + z^2 = c^2$, $y = 0$, passes also through the rectangular hyperbola whose equations are $z^2 - y^2 = a^2c^2/(a^2 - c^2)$ and $x = 0$.

(2) Obtain a construction for the form of the surface

$$(x^2 - z^2)(x^2 + 3y^2 - z^2 + 9a^2)^3 = \{6a^2(x^2 + y^2 - z^2) + 4a^4\}^2,$$

and shew that it has a hyperbolic conjugate line in the plane of zx.

(3) If the cone of asymptotic directions have a double side, shew that the surface will generally touch the plane at infinity, and that the section by this plane will have its inflexional tangents in the intersection with the tangent planes at the double side of the cone.

(4) Shew that the conicoid which determines the inflexional asymptotes of the surface, whose equation is $x^4 - y^2z^2 - 2a^2yz = 0$, is a hyperboloid of one or two sheets, the latter giving imaginary asymptotes.

(5) The envelope of the polar plane of a fixed point with respect to a system of confocals is a torse. Prove this, and shew that the torse touches the six tangent planes to any one of the confocals at the points where the normals to that confocal through the fixed point meet that confocal.

(6) The edge of regression of the imaginary developable, circumscribed to a system of confocals, projects orthogonally upon any principal plane of the system into the evolute of the focal conic in that plane.

(7) Shew that on the cubic surface $xyz + 2 = x + y + z$, there lie nine straight lines, three on an ordinary triple tangent plane and three double lines which are generating lines of the tangent cone at the singular point.

Also shew that, when $lx + my + nz = p$ is a tangent plane to the surface, $p = 2\{(mn)^{\frac{1}{2}} + (nl)^{\frac{1}{2}} + (lm)^{\frac{1}{2}}\}$.

(8) A chord of a conicoid is intersected by the normal at a given point of the surface, the product of the tangents of the angles subtended at the point by the two segments of the chords being invariable.

Prove that, O being the given point P, P' the intersections of the normal with two such chords in perpendicular planes containing the normal, the sum of the reciprocals of OP, OP' is invariable.

(9) A straight line intersects at right angles the arc of a fixed circle, and turns about the tangent with half the angular velocity of the point of contact round the circle.

Prove that the surface so generated intersects itself on a straight line, and find the tangent planes at any point of this line.

Shew that the line of striction is a plane curve, whose plane is inclined to the plane of the circle at an angle $\tan^{-1}2$.

(10) Prove that the line of striction crosses the generator of the hyperboloid $ax^2 + by^2 + cz^2 = 1$, which is defined by the parameter α, Art. 213, at a point for which z is given by $\{a + b - 2c + (b - a) \cos 2\alpha\} z \sqrt{(-c)} = (b - a) \sin 2\alpha$.

Prove that the greatest distance from the principal elliptic section to which the line of striction can run is given by $\tan^2\alpha = (b - c)/(a - c)$.

(11) Prove that the developable, which is the envelope of the polar planes of a fixed point P with respect to a system of confocal conicoids, meets Q the polar plane of P with respect to one of the confocals in a line, whose polar line with respect to the same conicoid is perpendicular to Q; and that these polar lines generate the quadric cone, six of whose generators are the normals at P to the three confocals through P, and the three lines through P parallel to their axes.

XXXVIII.

(1) O is a fixed point, P a point such that the polar planes with respect to a given conicoid are at right angles; shew that the locus of P is the plane diametral to all chords of the conicoid perpendicular to the polar plane of O.

(2) If two hyperboloids have two common generating lines of the same system, they will have two other common generators and touch each other in four points.

(3) If two arbitrary points be taken on each of four straight lines meeting in a point, the only conicoids which can be described through the eight points will be cones or combinations of planes.

(4) A conicoid passes through the sides of a skew quadrilateral $ABCD$, shew that the polar plane of the centre of gravity of the tetrahedron $ABCD$ is parallel to AC and BD.

(5) The surface $l/x + m/y + n/z + r/w = 0$ has a tangent cone at each of the angular points of the fundamental tetrahedron. Shew that any two of these cones have a common tangent plane, and also a common plane section containing the edge opposite to their common generating line, the six plane sections meeting in a point.

(6) The surface $lxz + mxw + nyz + ryw = 0$, of which AB, CD are generating lines, will be a paraboloid, if $l + r = m + n$, and the straight line joining the middle points of AB, CD will lie on the surface if $l + m + n + r = 0$.

(7) The surface $lxy = mzw$ passes through the edges BC, CA, AD, DB; i. Find the points in CA, DB at which the tangent planes are parallel, and thence shew that the centre is on the line joining the middle points of AB, CD. ii. If that line meet the surface in P, Q, shew that the tangent planes at P, Q will be parallel to AB and CD.

(8) The equation of a conicoid referred to tetrahedral coordinates is

$$lyz + mzx + nxy + l'xw + m'yw + n'zw = 0,$$

find the equations of the tangent planes at the angular points A, B, C, D of the fundamental tetrahedron. i. If Aa, Bb intersect, Cc, Dd will also intersect. ii. If these four planes form a tetrahedron $abcd$, shew that Aa, Bb, Cc, Dd will meet in a point, when $ll' = mm' = nn'$.

(9) Prove that the surface whose equation is

$$yz/mn + zx/nl + xy/lm + xw/lr + yw/mr + zw/nr = 0$$

cannot be a ruled surface, and that it will be an elliptic paraboloid if

$$l^2 + m^2 + n^2 + r^2 = mn + nl + lm + lr + mr + nr.$$

Shew that the tangent planes at A, B, C, D to the surface intersect the opposite faces of the fundamental tetrahedron in straight lines which all lie in the plane $x/l + y/m + z/n + w/r = 0$.

CHAPTER XIX.

RECIPROCAL POLARS.

542. In the eighth chapter it has been shewn that a dual interpretation can be given to an equation involving unknown coordinates, according as these are looked upon as coordinates of a point or plane, and in these dual results the point and plane may be said to correspond. In this chapter we give other methods of obtaining duplicate results by shewing how two systems of planes and points can be so connected that when a point or plane in one system is given, a single definite plane or point is determined in the other system, the *point* in one system and *plane* in the other being said to *correspond.*

543. The most general analytical machinery for effecting this is to refer each of the two systems of planes and points to its own coordinates, for example, let P and Π be two points in the two systems, which we will call O and Ω, and let them be denoted by (x, y, z), (ξ, η, ζ), each with reference to a set of coordinate axes, chosen arbitrarily for the two systems; the systems are then connected by supposing the two sets of coordinates to have a fixed relation of the form

$$u \equiv (a_1x + b_1y + c_1z + d_1)\,\xi + (a_2x + \ldots)\,\eta + (a_3x + \ldots)\,\zeta + a_4x + \ldots = 0,$$

which may be also written in the form

$$u \equiv (a_1\xi + a_2\eta + a_3\zeta + a_4)\,x + (b_1\xi + \ldots)\,y + (c_1\xi + \ldots)\,z + d_1\xi + \ldots = 0,$$

the coefficients a_1, b_1, &c. being constant.

The required correspondence between the two systems is established by this means, for, take a point Π in the system Ω, ξ, η, ζ are then constant, and $u = 0$ is the equation of a definite plane p, and the point Π and plane p correspond in such a manner that the coordinates of Π and every point in p satisfy simultaneously the equation of condition $u = 0$. Similarly, when any point P is taken in the system O, a plane ϖ corresponds to it in Ω, and if P be a point in p, ϖ must pass through Π.

544. If we take any plane in the system Ω, the ratios

$$a_1x + \ldots : a_2x + \ldots : a_3x + \ldots : a_4x + \ldots$$

are known, and the coordinates x, y, z of a point corresponding to

the plane determined. Or we may take three arbitrary points Π, Π', Π'' in Ω, to which three planes correspond, and the point of intersection of these three planes is the point corresponding to the plane $\Pi\Pi'\Pi''$, since its coordinates and those of each of the points satisfy the fundamental equation $u = 0$.

Take two points $\Pi\Pi'$ in Ω, and p, p' the corresponding planes in O, then $\Pi\Pi'$ a definite line in Ω determines in O a definite line (p, p') the intersection of the planes p and p', and the two lines $\Pi\Pi'$ and (p, p') correspond in the sense that to every point in either corresponds a plane through the other.

545. *Reciprocal polyhedrons.* Consider any system of points P_1, P_2, ... as the angles of a polyhedron in the system O, the corresponding planes ϖ_1, ϖ_2... in the system Ω will be the planes of the faces of another polyhedron.

If P_1, P_2, ...P_n be the n angles of a polygonal face whose plane p corresponds to the point Π in Ω, the planes of the corresponding faces of the polyhedron in Ω, viz. ϖ_1, ϖ_2... will all pass through the solid angle Π which corresponds to p. Thus we have two polyhedrons, one in each system, which have the reciprocal properties that each angular point of one corresponds to a face of the other, that the edges are corresponding lines, and that the number of sides in any face of one is equal to the number of faces in the corresponding solid angle of the other.

546. *Reciprocal surfaces.* If the planes ϖ_1, ϖ_2... in Ω touch a given surface Σ, the corresponding points P_1, P_2, ... in O will lie on some surface S. Suppose that three of these planes ϖ_1, ϖ_2, ϖ_3 move up towards coincidence, their common point will be ultimately a point on Σ, the corresponding points P_1, P_2, P_3 will move up towards coincidence on S, and the plane passing through them will be a tangent plane to S, at the point corresponding to the ultimate position of ϖ_1.

Hence, Σ and S are reciprocal surfaces in the sense that each may be generated from the other, either as the locus of points corresponding to tangent planes to the other, or as enveloped by planes corresponding to points on the other.

These reciprocal properties of the two surfaces follow also from the last Article by treating the surfaces as the limits of polyhedrons when the faces are diminished, while their number is increased indefinitely.

547. If one of the surfaces, as Σ, be of the n^{th} degree, it will be met by an arbitrary straight line in n points, but the reciprocal surface S will have n tangent planes passing through the reciprocal of the arbitrary straight line, it will therefore be of the n^{th} class.

Hence, since conicoids are both of the second degree and second class, it follows that the reciprocal of a conicoid is a conicoid.

548. *Reciprocal of a Cone.* If S be a cone with vertex V, since all the tangent planes pass through V, all the points of the reciprocal figure Σ will lie on a curve in the plane which corresponds to V, and since each tangent plane to S has an infinite number of points of contact which lie in a straight line passing through V, there will be at each point of Σ an infinite number of tangent planes passing through a tangent of the plane curve, being drawn as it were to a flat surface nearly coincident with the plane of the curve, and bounded by the curve itself.

NOTE. Consider the cone as the limit of either of the hyperboloids of which it is the asymptotic cone, the reciprocal of one of the hyperboloids will be ultimately a flat ellipsoid, and the other a flat hyperboloid having the curve corresponding to the cone as the boundary.

549. *Reciprocal of a Torse.* If S be a torse, every tangent plane will have an infinite number of points of contact lying on a straight line, and every point of Σ will have an infinite number of tangent planes intersecting in one straight line. Σ will therefore in this case be a tortuous curve, whose tangent lines will correspond to the generating lines of S. It follows that the torse generated by the tangent lines to Σ will correspond to the edge of regression of S.

550. *Reciprocal of a plane section of a surface.* Since every point of the section of S lies on the surface and also on a fixed plane p, the corresponding plane must touch the surface Σ, and also pass through the point Π which corresponds to p; the reciprocal of the section must therefore be the cone enveloping Σ, the vertex of which is Π. The line corresponding to the tangent at any point of the section will be a side of the cone.

If the cutting plane be a tangent plane the reciprocal of the section by this plane, which will have a multiple point at the point of contact, will be a cone whose vertex is a point on Σ, and the tangent plane to Σ at the vertex will touch this cone along two generating lines which will correspond to the tangents at the multiple point of the plane section.

COR. *Reciprocal of a pole and its polar plane with respect to a conicoid S.* If Σ be the reciprocal of S, the plane and point corresponding to the pole and polar plane, will be a polar plane and its pole with respect to Σ.

551. *Reciprocal of a multiple point.* Since there are an infinite number of tangent planes at such a point each touching a cone, in the reciprocal surface there will be an infinite number of points of contact lying in a curve on the tangent plane.

And, if the tangent cone be two planes, the tangent plane will be a double tangent plane, having two points of contact.

552. By referring to Art. 394, it will be seen that the relation between the coordinates of a point and those of any point in its

polar plane with reference to a conicoid is of the kind given above, Art. 543, and the following geometrical method of establishing reciprocal relations between points and planes is a particular case of the preceding.

553. *Auxiliary Conicoid.* Let a conicoid be chosen, fixed in position and magnitude, then corresponding to every point P there is a definite plane U, viz. the polar plane with respect to the fixed conicoid; and, by Art. 264, the polar plane of any point in U passes through P; or, by the definition given in Art. 280, if PQ cut the conicoid in Q, Q', and the polar plane U in R, $PQ : PQ' :: QR : RQ'$, $\therefore P$ and R are each in the polar plane of the other.

The conicoid employed as the machinery for fixing the correspondence of points and planes is called the *auxiliary conicoid*, and the reciprocal of a surface with respect to it is called the *polar reciprocal* of the surface.

554. *Reciprocation with respect to a point.* If, as a particular case of the auxiliary conicoid, we take a sphere, whose radius is R, and its centre S, and SY be drawn perpendicular to the polar plane of a point P, then $SY.SP = R^2$; hence, we may reject the idea of the sphere, and speak of reciprocating with respect to a point, the plane corresponding to a point, and *vice versâ*, being determined by the equation given above.

The centre of the auxiliary sphere is called the *origin of reciprocation.*

If the auxiliary conicoid be not specified it is always supposed to be a sphere, and any change in the radius of the sphere not altering the species of the reciprocal surface, but only its dimensions, one surface is said to be the polar reciprocal of the other with respect to the point which is the centre of the sphere.

555. The construction for the reciprocal of any surface with respect to a point S is as follows. Let fall a perpendicular SY on any tangent plane to the surface, and in SY or SY produced take a point P, such that $SP.SY$ is constant, the locus of P is the reciprocal surface.

Or, the reciprocal construction may be made, viz. in SP take SY, such that $SP.SY$ is constant, then the envelope of the plane through Y perpendicular to SP is the reciprocal surface.

556. *Equation of the reciprocal with respect to a point of a given surface.*

Let the equation of the surface be $F(x, y, z) = 0$, (1), (α, β, γ) the origin of reciprocation S, the equation of a tangent plane may be written $l(x-\alpha) + m(y-\beta) + n(z-\gamma) = p$, and if ρ be the distance of the corresponding point (ξ, η, ζ), $p\rho = R^2$;

$$\therefore\ (\xi-\alpha)(x-\alpha) + (\eta-\beta)(y-\beta) + (\zeta-\gamma)(z-\gamma) = R^2, \quad (2)$$

and, if (x, y, z) be the point of contact of the tangent plane,

$$(\xi-\alpha)/U=(\eta-\beta)/V=(\zeta-\gamma)/W. \tag{3}$$

The equation of the reciprocal is found by eliminating x, y, and z from the four equations (1), (2), and (3).

557. *Species of the polar reciprocal of a conicoid.*

The reciprocal of a ruled surface must be a ruled surface, since to every straight line on one must correspond a straight line on the other. Hence the reciprocal of a hyperboloid of one sheet or a hyperbolic paraboloid is a paraboloid or hyperboloid as the centre of the auxiliary conicoid is or is not on the surface; for, when it is on the surface, its polar plane, which is a tangent plane of the reciprocal conicoid, is at an infinite distance.

The reciprocal A' of an umbilical surface A, that is, of an ellipsoid, hyperboloid of two sheets, or an elliptic paraboloid, will be an ellipsoid, an elliptic paraboloid, or a hyperboloid of two sheets, according as the centre of the auxiliary conicoid is within, upon, or without the surface A. For, if the centre be without the surface, the conical envelope of A with the centre as vertex will be real, and its reciprocal will be a curve on the plane at infinity, every point of which corresponds to a tangent plane of A, therefore A' has a real plane section at infinity, and must be a hyperboloid of two sheets, since it is not a ruled surface.

If the centre be on the surface A, the conical envelope will become a tangent plane, and the plane at infinity a tangent plane to A'.

If the centre be within A the conical envelope will be imaginary, and there will be no real points at infinity on A'.

558. *Centre of the reciprocal conicoid.*

Every plane through the centre has its pole at an infinite distance, therefore every point in the plane corresponding to the centre has for its polar plane a plane passing through the centre of the auxiliary conicoid; hence, the polar plane of the centre of the auxiliary conicoid with respect to the given conicoid corresponds to the centre of the reciprocal.

This can also be deduced from the consideration that the centre is the vertex of the real or imaginary asymptotic cone.

559. *Reciprocal of a conicoid with respect to a point.*

The reciprocal polar of a sphere with respect to a point is a surface of revolution, about the transverse axis, of which the point is the focus, and the line joining the point and the centre of the sphere the axis; for, taking any plane through this line, the section of the reciprocal polar by this plane is a conic of which the point is the focus and the line before mentioned the major axis. The reciprocal of the polar of the point, with respect to the sphere, will

be the centre, and the reciprocal of the centre, the directrix plane of the surface of revolution, exactly as in two dimensions. Properties of conicoids of revolution having a common focus may be immediately obtained in this manner. These are, however, generally at once deducible from the corresponding properties of plane curves. It is shewn, Art. 361, that the enveloping cone from a point on a focal conic of a conicoid is a right cone; if we take such a point for the origin of reciprocation S, the point corresponding to any tangent plane to the conical envelope will be at an infinite distance in the perpendicular from S to the tangent plane, and will therefore be in the direction of a generating line of the asymptotic cone of the reciprocal; this asymptotic cone is therefore also a right cone, and the reciprocal surface a surface of revolution. This result is of course true whether the asymptotic cone employed in the proof be real or impossible. Conversely, the reciprocal polar of a surface of revolution with respect to a point will be a conicoid of which the point is a focus.

Hence, from a sphere may be obtained, by successive reciprocations, any of the umbilical conicoids, but, as before shewn, the ruled surfaces cannot be obtained in this manner.

For any position of the origin of reciprocation S consider the cone, enveloping the conicoid, of which S is the vertex, the points which correspond to the tangent planes common to this cone and the conicoid are at an infinite distance, therefore the asymptotic cone of the polar reciprocal of the conicoid is a cone reciprocal to the enveloping cone whose vertex is S. Hence the principal axes of the polar reciprocal are parallel to those of the enveloping cone.

The analytical proof of all these propositions is included in the following articles.

560. *Reciprocal of a given conicoid with respect to a point.*

Let the equation of the given conicoid U be $x^2/a + y^2/b + z^2/c = 1$, (α, β, γ) the origin of reciprocation S, (ξ, η, ζ) any point in the reciprocal surface, ρ its distance from S, R^2/ρ the perpendicular upon the corresponding plane, (l, m, n) the direction of ρ, so that $\xi - \alpha = l\rho$, &c.,

$$\therefore\ l(x-\alpha) + m(y-\beta) + n(z-\gamma) = R^2/\rho$$

is the equation of a tangent plane to U, and the perpendicular upon it from the centre of U is $l\alpha + m\beta + n\gamma + R^2/\rho$,

$$\therefore\ l^2a + m^2b + n^2c = (l\alpha + m\beta + n\gamma + R^2/\rho)^2, \text{ Art. 256,}$$

and the equation of the reciprocal conicoid is

$$a(\xi-\alpha)^2 + b(\eta-\beta)^2 + c(\zeta-\gamma)^2 = \{\alpha(\xi-\alpha) + \beta(\eta-\beta) + \gamma(\zeta-\gamma) + R^2\}^2.$$

561. If the conicoid be a sphere, radius a, and the origin S of reciprocation be in the axis of x, at a distance c from the centre of the sphere, transferring the origin to S the equation becomes, $a^2(\xi^2 + \eta^2 + \zeta^2) = (c\xi + R^2)^2$,

$$\text{or}\quad (a^2 - c^2)\{\xi - cR^2/(a^2-c^2)\}^2 + a^2(\eta^2 + \zeta^2) = a^2R^4/(a^2 - c^2),$$

or, if $c = a$, $\eta^2 + \zeta^2 = 2\xi R^2/a + R^4/a^2$, the reciprocal is therefore a prolate spheroid, paraboloid of revolution, or hyperboloid of revolution of two sheets as $c <= \text{or} > a$, the eccentricity being c/a, and focus S.

562. *Centre of the reciprocal conicoid.* The centre (ξ_0, η_0, ζ_0) is given by three equations such as

$$a(\xi_0 - a) = \alpha\{\alpha(\xi_0 - \alpha) + \beta(\eta_0 - \beta) + \gamma(\zeta_0 - \gamma) + R^2\} = a\sigma_0, \text{ suppose};$$

$$\therefore\ \sigma_0 = \sigma_0(\alpha^2/a + \beta^2/b + \gamma^2/c) + R^2,$$

and is positive or negative as S is within or without U.

Let r_0 be the distance of the centre from S, then

$$r_0 = \pm\sigma_0\sqrt{(\alpha^2/a^2 + \beta^2/b^2 + \gamma^2/c^2)}.$$

The equation of the polar plane of S with respect to U being $\alpha x/a + \beta y/b + \gamma z/c = 1$, the perpendicular p_0 from S on this plane is

$$\pm(1 - \alpha^2/a - \beta^2/b - \gamma^2/c)/\sqrt{(\alpha^2/a^2 + \beta^2/b^2 + \gamma^2/c^2)};$$

$\therefore\ r_0 p_0 = R^2$, and r_0 is directed perpendicular to the polar plane, hence the centre is the point corresponding to this plane.

563. *Principal axes of the reciprocal conicoid.* The equation of Art. 560 becomes, when the origin is transferred to the centre,

$$(a - \alpha^2)\xi^2 + \ldots - 2\beta\gamma\eta\zeta - \ldots = \sigma^2_0(1 - \alpha^2/a - \beta^2/b - \gamma^2/c) = R^4/(1 - \alpha^2/a - \beta^2/b - \gamma^2/c).$$

When the surface is referred to its principal axes, the equation is

$$s_1\xi^2 + s_2\eta^2 + s_3\zeta^2 = R^4/(1 - \alpha^2/a - \beta^2/b - \gamma^2/c), \qquad (1)$$

where, since, in Art. 411, $\lambda = a$, $\mu = b$, and $\nu = c$, s_1, s_2, s_3 are the three real roots of $\alpha^2/(a - s) + \beta^2/(b - s) + \gamma^2/(c - s) = 1$, (2), $a - s_1$, $a - s_2$, $a - s_3$ are therefore the squares of the primary axes of the confocals through S.

The asymptotic cone has its sides parallel to those of the reciprocal cone of the conical envelope of U with vertex S.

The expression $b'c' - (a - s)a'$ in Art. 415 becomes $(a - s)\beta\gamma$, therefore the direction-cosines are in the ratio $\alpha/(a - s) : \beta/(b - s) : \gamma/(c - s)$, that is, the principal axes are in the directions of the three normals to the confocals through (α, β, γ).

564. *Origin of reciprocation a point on one of the focal conics of an ellipsoid.*

i. If S be on the modular focal curve of an ellipsoid, $\alpha^2/(a - c) + \beta^2/(b - c) = 1$, $\gamma = 0$, two of the values of s in (2) are c, and the third s_3 is $a + b - c - \alpha^2 - \beta^2$, but $\alpha^2 + \beta^2 < a - c$; $\therefore\ s_3 > b$, and (1) becomes $c(x^2 + y^2) + s_3 z^2 = +$, $\therefore$ the reciprocal is an oblate spheroid.

ii. If S be on an umbilical focal conic $\beta = 0$, and $\alpha^2/(a - b) - \gamma^2/(b - c) = 1$, $s_1 = a - b + c - \alpha^2 - \gamma^2$, $s_2 = b = s_3$, if S be within the ellipsoid, $\alpha^2 + \gamma^2 > a - b < a + c - b$; $\therefore\ s_1 < c$ and positive, (1) becomes $s_1 x^2 + b(y^2 + z^2) = +$, and the reciprocal is a prolate spheroid.

If S be without, $\alpha^2 + \gamma^2 > a + c - b$ and s_1 is negative, $\therefore$ (1) becomes $-s_1 x^2 - b(y^2 + z^2) = +$, and the reciprocal is a hyperboloid of revolution of two sheets about the transverse axis.

565. *Examples of the method of reciprocal polars.*

In the subjoined lists, the theorems to be proved by the method of reciprocals are placed first, and side by side with them the simpler theorems from which they may be deduced.

i. If two conicoids have one common enveloping cone, they will also have another.	If two conicoids have one common plane section, they will also have another.
ii. Any straight line through a point is divided harmonically by a conicoid, and the polar plane of the point with respect to the conicoid.	Any straight line through a point is divided into two equal parts by two parallel tangent planes to a conicoid, and the parallel plane through the centre.

iii. If three conicoids have one common enveloping cone, the second enveloping cones of those surfaces, taken two and two, will have their vertices in the same straight line.

If three conicoids have one common plane section, the second planes of intersection of the surfaces, taken two and two, will intersect in the same straight line.

iv. Two conicoids, each touching another conicoid along a plane curve will have two common enveloping cones, whose vertices lie on the same straight line with the poles of the planes of contact.

Two conicoids, each touching another along a plane curve, will themselves intersect in two plane curves, whose planes pass through the line of intersection of the planes of contact.

v. Two conicoids, touching each other at two points, will have two common enveloping cones, whose vertices lie on the line of intersection of the tangent planes at those points.

Two conicoids, touching each other at two points, will have two plane curves of intersection passing through those points.

vi. If a conicoid touch seven given planes, the locus of its centre will be a plane.

If a conicoid pass through seven fixed points, the polar plane of any other fixed point will pass through a fixed point.

vii. Two cones, having a common vertex, and a common focal line, cannot have more than two real common tangent planes.

Two circles, lying in the same plane, cannot have more than two real common points.

viii. If two prolate surfaces of revolution have a common focus, their points of intersection will lie on plane sections.

Two spheres have two common enveloping cones.

ix. If, in a prolate surface of revolution, a cone be described with a focus of the surface as vertex, and a plane section as base, it will be a cone of revolution.

Any enveloping cone of a sphere is a cone of revolution.

x. If two prolate surfaces of revolution have a common focus, their planes of intersection will pass through the line of intersection of their directrix planes.

The vertex of a common enveloping cone of two spheres lies on the line joining their centres.

xi. If two tetrahedrons be such that each angular point of one is the pole of a face of the other with respect to a given conicoid, the lines joining corresponding angular points will be generators of one conicoid circumscribing both tetrahedrons.

If two tetrahedrons be such that each angular point of one is the pole of a face of the other with respect to a given conicoid, the lines of intersection of corresponding faces will lie on one conicoid which will touch the faces of both tetrahedrons.

xii. If a paraboloid of revolution be described passing through a given ellipse, and a right circular cylinder be described also passing through the ellipse, the axis of the cylinder will be parallel to that of the paraboloid, and will pass through the pole of the plane of the ellipse.

If a sphere be inscribed in a right circular cone, the section of the cone made by any plane touching it will have the point of contact as a focus, and its directrix will lie in the plane of contact of the sphere and cone.

xiii. If with any point on the focal hyperbola of an ellipsoid as focus be described a paraboloid of revolution enveloping the ellipsoid, the axis of this paraboloid will be parallel to the generating lines of one of the right circular cylinders which envelope the ellipsoid, and the axis of this cylinder will pass through the pole of the plane of contact of the ellipsoid and paraboloid.

If a sphere be inscribed in a conicoid of revolution, the section of the conicoid by any plane touching the sphere will have the point of contact for a focus, and the corresponding directrix will lie on the plane of contact of the sphere and conicoid.

Reciprocating this last with respect to an arbitrary point, we obtain the following proposition, which is due, we believe, to M. Chasles.

If two conicoids touch each other along a plane curve, and a tangent plane be drawn to one of them at an umbilic, the section of the other made by this plane will have the umbilic for a focus, and its corresponding directrix in the plane of contact of the two conicoids.

Let O be the origin of reciprocation, the surface corresponding to the paraboloid will pass through O, Art. 557; also O will be a point on its focal curve, Art. 559; O will then be an umbilic, Art. 349. The tangent plane at O will correspond to the point at infinity on the paraboloid, and therefore to the vertex of the right circular cylinder; hence to the right circular cylinder will correspond a plane section of the reciprocal of the ellipsoid by the tangent plane at O. Since the cylinder is a right circular cylinder, the corresponding curve will be a conic whose focus is O, and since the axis of the cylinder passes through the pole of the plane of contact of the ellipsoid and paraboloid, the directrix of the corresponding conic will lie in the plane of contact of the two corresponding surfaces.

XXXIX.

(1) Prove that the reciprocal of a circle with respect to a point is a cone of which one of the focal lines passes through the origin of reciprocation.

(2) Prove that the reciprocal of a conic section with respect to a focus is a right circular cylinder.

(3) Shew, by reciprocals, that the focal lines of a cone pass through the foci of the sections of the cone made by planes perpendicular to the focal lines.

(4) If a conicoid touch the faces of the fundamental tetrahedron $ABCD$ in a, b, c, d; shew that if Aa, Bb intersect each other, Cc and Dd will also intersect each other.

(5) The reciprocal polar of the surface $ax^2 + by^2 + cz^2 = 1$ with respect to the surface $2a'yz + 2b'zx + 2c'xy = 1$, is

$$(b'z + c'y)^2/a + (c'x + a'z)^2/b + (a'y + b'x)^2/c = 1.$$

(6) Shew that the surfaces, whose equations are $xyz = a(x^2 + y^2)$ and $z(x^2 + y^2) = cxy$, are reciprocals of one another with respect to the origin, where ac is the constant of reciprocation.

(7) If a series of straight lines be drawn through a point O, such that the straight lines reciprocal to them, with respect to a given conicoid, are respectively perpendicular to them, these straight lines will lie on a cone of the second degree, and the reciprocal straight lines will be tangent lines to a parabola.

(8) If three cones of the second degree have a common vertex, and a common focal line, the lines of intersection of the common tangent planes to them, taken two and two, will lie in a plane.

(9) The reciprocal polar of the hyperboloid $xy = kzw$ with respect to the conicoid $x^2 + y^2 = z^2 + w^2$ is $kxy = zw$.

XL.

(1) If two hyperboloids of revolution of two sheets, or two prolate spheroids, have a common focus, and equal minor axes, they will have a common enveloping cylinder, one of whose focal lines will pass through the common focus.

(2) If BC, CA, AB be three chords of a conicoid, each of which subtends a right angle at a fixed point S, the plane ABC will touch a prolate conicoid of revolution, of which S is a focus, and the polar plane of S with regard to the given conicoid the corresponding directrix plane.

(3) If three tangent planes to a cone of the second degree intersect in three straight lines VP, VQ, VR, and if P, Q, R be points such that QR, RP, PQ each subtends a right angle at a fixed point O, the plane PQR will envelope a conicoid of revolution.

(4) The reciprocal polar of the conicoid

$$mnyz + nlzx + lmxy + lrxw + mryw + nrzw = 0,$$

with respect to the auxiliary conicoid $ax^2 + by^2 + cz^2 + dw^2 = 0$,

$$\text{is } a^2x^2/l^2 + \ldots - abxy/lm - \ldots = 0.$$

(5) Prove that the equation of the reciprocal of the anchor ring with respect to the centre is

$$\{(c^2 - a^2)(x^2 + y^2) - a^2z^2 + R^4\}^2 = 4c^2R^2(x^2 + y^2).$$

Shew that the angle of the conical tangent at the multiple point of the reciprocal is the supplement of the angle of the cone containing all the points of contact of the two singular tangent planes of the anchor ring.

(6) The reciprocal polar of the conicoid

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 1,$$

with respect to a sphere, radius R, and centre (α, β, γ) is, with the notation of Art. 391,

$$\Delta\{\alpha(x - \alpha) + \beta(y - \beta) + \gamma(z - \gamma) + R^2\}^2 = A(x - \alpha)^2 + \ldots + 2A'(y - \beta)(z - \gamma) + \ldots,$$

(7) From the reciprocal of the last problem, obtain the equations of the focal curves of the given conicoid in the form

$$\{Ayz + x(A'x - B'y - C'z)\}/(\Delta yz - A') = \ldots = \ldots.$$

(8) Determine the foci of a central conicoid of revolution from the consideration that the reciprocal with respect to a focus is a sphere.

Shew that α one of the coordinates of a focus is given by the equation

$$\Delta^2\alpha^4 - (2A - B - C)\Delta\alpha^2 + (A - B)(A - C) - A'^2 = 0.$$

CHAPTER XX.

CLUSTERS OF CONICOIDS. CONICOIDS AND SPHERES REFERRED TO A TETRAHEDRON. TANGENTIAL EQUATION OF A SPHERE.

566. In this chapter the properties of the clusters of conicoids are discussed, which have been mentioned in Arts. 441 and 449; we have also applied the method of tetrahedral coordinates to conicoids satisfying given conditions, and in particular to spheres inscribed in and circumscribed about a tetrahedron.

The statement of the reciprocal propositions and the interpretation of the equations as tangential equations in four-point coordinates has generally been left to the student; the reciprocal propositions being obtained by the substitution of *plane* for *point*, *point* for *plane*, *polar plane* for *pole*, and *vice versâ*, conicoids inscribed in the same *torse* for conicoids having a common *curve of intersection*, &c.

567. *Cluster of conicoids passing through eight given points.*

Let $U=0$, $V=0$ be the equations of two particular conicoids U and V which pass through the eight points, then $\lambda U+\mu V=0$ will be satisfied for all points of the curve of intersection of U and V, on which the eight points must lie; and, by choosing properly the ratio $\lambda:\mu$, the particular conicoid of the cluster can be represented which satisfies any ninth condition, such as touching a given plane, or passing through a ninth point not on the curve of intersection of U and V.

568. *To find the cones of the cluster of conicoids having the same base passing through eight arbitrary points.*

If $\lambda U+\mu V=0$ be the equation of one of the cones of the cluster, the value of the ratio $\lambda:\mu$ is found by equating the discriminant to zero, or, using the notation of Art. 392, $H(\lambda U+\mu V)=0$, from which generally there are four values of the ratio, that is, there are four cones of the cluster.

Since the values of $\lambda:\mu$ are the same by whatever system of coordinates the surfaces U and V are represented, the mutual ratios of the coefficients of the biquadratic are invariants of the system of two conicoids $U=0$ and $V=0$. The equation for determining $\lambda:\mu$ is written by Salmon $\Delta\lambda^4+\Theta\lambda^3\mu+\Phi\lambda^2\mu^2+\Theta'\lambda\mu^3+\Delta'\mu^4=0$, and he shews that Θ will vanish whenever it is possible to inscribe in V a tetrahedron which shall be self-conjugate with regard to U, and that Θ' will vanish whenever it is possible to find a tetrahedron

self-conjugate with regard to U whose faces touch V. This is proved by referring U to a self-conjugate tetrahedron, so that

$$U = ax^2 + \beta y^2 + \gamma z^2 + \delta w^2,$$
$$V = ax^2 + \ldots + 2a'yz + \ldots + 2a''xw + \ldots;$$
$$\therefore\ \Theta = a\beta\gamma\delta + b\gamma\delta a + c\delta a\beta + da\beta\gamma,$$

which vanishes when a, b, c, d all disappear,

$$\text{and } \Theta' = aP + \beta Q + \gamma R + \delta S,$$

where $P = 0$ is the condition that the face BCD of the tetrahedron may cut the surface V in two straight lines real or imaginary.

Φ vanishes when the edges of a self-conjugate tetrahedron, with respect to either, touches the other.

569. *The conicoids of a cluster passing through eight given points, or having a common curve of intersection, have the same self-conjugate tetrahedron.*

The equation of the polar of any point (x, y, z, w) with regard to the conicoid $\lambda U + \mu V = 0$ is $(\lambda dU/dx + \mu dV/dx)\xi + \ldots = 0$, and the vertex of one of the four cones passing through the base is given by $\lambda_1 dU/dx + \mu_1 dV/dx = 0$, $\lambda_1 dU/dy + \mu_1 dV/dy = 0$, &c., hence the polar of the vertex of this cone with respect to any conicoid of the cluster is $(\lambda\mu_1 - \mu\lambda_1)(\xi dU/dx + \eta dU/dy + \ldots) = 0$, and is the same for every conicoid, and in particular for the other three cones. Also, the polar of a point with respect to any cone passes through the vertex of that cone, it follows therefore that the polar of the vertex of each of the four cones with respect to any conicoid of the cluster is the plane passing through the vertices of the three remaining cones.

When the cluster is referred to the common self-conjugate tetrahedron, the equation of each member is of the form

$$lx^2 + my^2 + nz^2 + rw^2 = 0,$$

and the polar of (x, y, z, w) is

$$lx\xi + my\eta + nz\zeta + rw\omega = 0;$$

$\therefore\ \omega = 0$ is the polar of $A(0, 0, 0, 1)$.

NOTE. If in particular cases there be not four cones, the argument fails. For example, if the base be a straight line and a cubic curve, there will be only two cones, and the conicoids cannot have a common self-conjugate tetrahedron.

570. *If a cluster of conicoids pass through eight given points, the polars of any other given point with respect to every one of the cluster will intersect in a fixed straight line.*

Let $P = 0$, $Q = 0$ be the polars of the point with respect to $U = 0$ and $V = 0$, then $\lambda P + \mu Q = 0$ will be the polar with respect to $\lambda U + \mu V = 0$, and for all values of $\lambda : \mu$ the polars pass through the line $P = 0$, $Q = 0$.

To find whether the given point can be so situated that the polar plane is fixed for all the cluster, refer the cluster to the common self-conjugate tetrahedron, the polar of (x', y', z', w') with respect to any conicoid of the cluster given by the equation

$$\lambda(ax^2+by^2+cz^2+dw^2)+\mu(a'x^2+b'y^2+c'z^2+d'w^2)=0$$

$$\text{is } \lambda(ax'x+by'y+\ldots)+\mu(a'x'x+b'y'y+\ldots)=0,$$

this polar will be fixed for all the conicoids when

$$ax'/a'x'=by'/b'y'=\ldots.$$

Hence, there are generally only four such points, viz. the vertices of the self-conjugate tetrahedron.

571. *If a conicoid pass through eight given points the pole of any given plane will lie in a cubic curve.*

Referring the cluster to the common self-conjugate tetrahedron $ABCD$, let the equation of the given plane be $lx+my+nz+rw=0$, this must be identical with $(\lambda a+\mu a')x'x+\ldots=0$, (x', y', z', w') being its pole with respect to $\lambda u+\mu v=0$;

$$\therefore\ (\lambda a+\mu a')x'/l=(\lambda b+\mu a')y'/m=\ldots; \qquad (1)$$

by eliminating $\lambda:\mu$, we shew that the pole lies on the intersection of the two cones

$$(b'c-bc')l/x+(c'a-ca')m/y+(a'b-ab')n/z=0,$$

$$\text{and } (b'd-bd')l/x+(d'a-da')m/y+(a'b-ab')r/w=0,$$

which have a common generating line CD, not part of the locus of the pole, since it does not satisfy the equation (1).

COR. 1. The locus of the centres of conicoids passing through eight points is a particular case, the fixed plane being at an infinite distance.

COR. 2. There will be four planes whose poles with respect to all the conicoids of the cluster will be the same, viz. the faces of the self-conjugate tetrahedron.

572. *Number of conicoids passing through eight points and touching a given plane.*

Let the equation of any conicoid through the eight points, referred to the common self-conjugate tetrahedron, be $(\lambda a+\mu a')x^2+(\lambda b+\mu b')y^2+\ldots=0$, and $px+qy+rz+sw=0$ that of the given plane, then. if (x', y', z', w') be the point of contact, $(\lambda a+\mu a')x'/p=(\lambda b+\mu b')y'/q=\ldots$; whence

$$p^2/(\lambda a+\mu a')+q^2/(\lambda b+\mu b')+r^2/(\lambda c+\mu c')+s^2/(\lambda d+\mu d')=0,$$

which is a cubic for determining $\lambda:\mu$; there are, therefore, generally three such conicoids.

When $p=0$, one value of $\lambda:\mu$ is given by $\lambda a+\mu a'=0$, the corresponding conicoid being one of the four cones passing through the eight points, its vertex being a point in the given plane, which satisfies the condition of tangency, but is not in general a proper tangent plane. In this case there are only two conicoids which satisfy the conditions.

If the given plane pass through two of the vertices only one, and, if three, no conicoid can be described as required.

573. *Number of conicoids passing through eight points and touching a given straight line.*

Let $\lambda U + \mu V = 0$, (1) be the equation of any conicoid satisfying the required condition, and let the equations of the line be

$$(x-f)/l = (y-g)/m = (z-h)/n = (w-k)/r = \rho.$$

Counting these equations with (1), the condition that the quadratic in ρ, which gives the points of intersection of the given line with the conicoid, may have equal roots, is a quadratic in $\lambda : \mu$. Hence, two conicoids satisfy the required conditions.

574. When the eight points lie in two planes, and no three of the points lie in one straight line, the general equation of the cluster is $\lambda U + \mu LM = 0$, where $L=0$, $M=0$ are the equations of the two planes; in this case the base of the cluster is two conics which intersect at C, D in the line of intersection of the planes.

The tangent plane at C to any conicoid of the cluster is the plane containing the tangents to the two conics forming the base. Hence, all the conicoids have double contact with each other at the extremities of their common chord.

If $M=L$, the base of the cluster will be two coincident conics, and every tangent to U at a point in the plane L will meet the base in two consecutive points and be a tangent to each conicoid of the cluster $\lambda U + \mu L^2 = 0$, therefore all the conicoids touch each other at every point of the base; in particular, they have a common tangent cone.

575. The propositions relating to the cluster represented by the equation $\lambda U + \mu LM = 0$, which are stated in the following articles, can be proved analytically by choosing the fundamental tetrahedron $ABCD$ such that ABC, ABD are the tangent planes at C and D to one of the conicoids of the cluster. Since the plane $w=0$ intersects the conicoid in two straight lines, real or imaginary, passing through C, the equation must reduce to $ax^2+by^2+2c'xy=0$, when $w=0$, and similarly for the plane $z=0$;

$$\therefore\ U \equiv ax^2 + by^2 + 2c'xy + 2c''zw,$$

and the equation of any conicoid of the cluster will be $U+2kxy=0$, also that of the polar with respect to it of a point (x', y', z', w') will be

$$\{ax' + (c'+k)y'\}\,x + \{by' + (c'+k)x'\}\,y + c''(w'z + z'w) = 0. \quad (1).$$

When $M=0$ gives the plane at infinity, $\lambda U + \mu LM = 0$ is the general equation of conicoids similar to U, intersecting in one plane curve, the other common curve being at an infinite distance, through which the conical asymptotes of all the conicoids pass. In particular, spheres have a common imaginary circle at infinity.

When both $L=0$ and $M=0$ give planes at infinity, the poles of these planes are the same for all the conicoids represented by $\lambda U + \mu LM = 0$, which therefore are concentric.

576. *All the conicoids of the cluster have common tangent planes at the two points where the conics forming the base cross one another.*

At C $x'=0$, $y'=0$, and $w'=0$; $\therefore$ $w=0$ is the tangent plane, and similarly for D. This appears geometrically, since the tangent plane at C is the plane containing the tangents to both conics.

577. *Of the conicoids forming the cluster whose base is two conics, two are cones whose vertices lie in the polar line of the intersection of the planes of the conics.*

For, $H(u+2kxy)=0$; $\therefore$ $(c'+k)^2=ab$, and the equations of the two cones are $(x\sqrt{a}\pm y\sqrt{b})^2+2c''zw=0$, the vertices dividing AB externally and internally in the ratio $\sqrt{a}:\sqrt{b}$; also, AB, being the intersection of tangent planes at C and D, is the polar of CD.

578. *To find the points for which the polar planes are the same for all conicoids of the cluster.*

In order that this may be the case, the equation (1), Art. 575, must be the same for all values of k, hence

i. $x'=0$, $y'=0$, and the fixed polar plane is $w'z+z'w=0$, so that the polar plane of any point P in CD passes through AB, and with P divides CD harmonically.

ii. $z'=0$, $w'=0$, and $ax'/y'=by'/x'$.

In this case the points are the vertices of the two cones of the cluster, and each of the fixed polar planes contains CD and passes through the vertex of the other.

579. *When a cluster of conicoids has two common plane sections, the polar plane of a given point passes through a fixed straight line, and the pole of a given plane lies on a plane conic.*

Taking the fundamental tetrahedron as in Art. 575, the polar plane of a given point (x', y', z', w') for any conicoid of the cluster having the equation (1) of that article, contains the line whose equations are

$$ax'x+by'y+c''(w'z+z'w)=0 \text{ and } y'x+x'y=0,$$

which intersects CD in P, where $w'z+z'w=0$, and a plane through AB and the given point intersects CD in Q where $w'z-z'w=0$, therefore $CPDQ$ is a harmonic range.

For the pole (x', y', z', w') of a plane $lx+my+nz+rw=0$,

$$\{ax'+(c'+k)y'\}/l=\{by'+(c'+k)x'\}/m=c''w'/n=c''z'/r;$$

$$\therefore\ r(ax'^2-by'^2)=c''z'(lx'-my') \text{ and } rw'=nz',$$

which gives a conic passing through the vertices of the two cones of the cluster, Art. 577, and a point on CD which with the given plane divides CD harmonically.

The locus of the centre, the pole of $x+y+z+w=0$, is a conic passing through the middle point of CD.

580. *When two conicoids have double contact, their curve of intersection breaks up into two conics, generally; in particular cases into a cubic curve and a straight line.*

CD being the line joining the points of contact, ABC, ABD the common tangent planes, the equations of the two conicoids are

$$ax^2 + by^2 + 2c'xy + 2c''zw = 0,$$
$$ax^2 + \beta y^2 + 2\gamma' xy + 2\gamma'' zw = 0,$$

and the curve of intersection lies in the two planes

$$(ax^2 + by^2 + 2c'xy) / c'' = (ax^2 + \beta y^2 + 2\gamma' xy) / \gamma''.$$

The exception is when CD is a generating line of both conicoids; in this case choosing the tetrahedron such that ACD, BCD are tangent planes at C and D, the two equations are of the form

$$ax^2 + by^2 + 2a'yz + 2c'xy + 2a''xw = 0,$$

and, eliminating yz or xw from these equations, we obtain two cones, which have a common generator CD; hence, the curve of intersection of the two surfaces breaks up into a straight line and a cubic curve.

581. *General form of the equation of a conicoid passing through seven given points.*

Take $U = 0$, $V = 0$, $W = 0$ as the equations of three particular conicoids, which satisfy the condition, and have not a common curve of intersection. The equation $\lambda U + \mu V + \nu W = 0$ will be the general equation required. For it is satisfied whenever U, V, W simultaneously vanish, so that it represents a conicoid passing through the seven points, also it involves two arbitrary constants, by means of which it can be made to satisfy two other conditions, and therefore it represents any conicoid passing through the seven points.

Since the equations $U = 0$, $V = 0$, $W = 0$ determine eight points, any conicoid which passes through seven fixed points will necessarily pass through an eighth whose position may be determined from the seven. Hence, if the two extra conditions be that a conicoid passes through two points, one of which is the eighth mentioned above, the nine points will not be sufficient to identify the conicoid.

For example, if the seven points be angular points of a parallelepiped, and the three surfaces corresponding to U, V, and W be each a pair of parallel planes, any conicoid through the seven angular points will also pass through the eighth.

582. *If a conicoid pass through seven given points, the polar plane of any other given point will pass through a fixed point.*

Let $P = 0$, $Q = 0$, $R = 0$ be the polar planes of the given point with respect to the three surfaces U, V, W, then $\lambda P + \mu Q + \nu R = 0$ is that with respect to any conicoid through the seven points. This plane passes through a fixed point which is the intersection of the three polar planes with respect to U, V, and W.

If the fixed point be the vertex of any quadric cone passing through the seven points, the polar plane with respect to any

conicoid through the seven points will pass through a fixed straight line; for, taking the surface $U=0$ to be the cone, of which the fixed point is the vertex, the polar $P=0$ is indeterminate in position and the polar $\mu Q+\nu R=0$ always passes through the straight line in which $Q=0$ and $R=0$ intersect.

583. *When a conicoid passes through seven given points to find the locus of the pole of a given plane.*

Taking as the tetrahedron of reference one in which the face BCD is the given plane, the equations determining the pole with respect to the conicoid $\lambda U+\mu V+\nu W=0$ are

$$\lambda dU/dy+\mu dV/dy+\nu dW/dy=0, \text{ \&c.}$$

Hence the locus of the pole is a cubic surface

$$\begin{vmatrix} dU/dy, & dV/dy, & dW/dy \\ dU/dz, & dV/dz, & dW/dz \\ dU/dw, & dV/dw, & dW/dw \end{vmatrix}=0.$$

The locus of the centre, which is the pole of $x+y+z+w=0$, is obtained by eliminating λ, μ, and ν from the equations

$$\lambda dU/dx+\mu dV/dx+\nu dW/dx=\lambda dU/dy+\ldots=\lambda dU/dz+\ldots=\lambda dU/dw+\ldots.$$

584. *To find a general form of the equation of a conicoid passing through seven points, six of which lie by threes on two non-intersecting straight lines.*

The two straight lines lie altogether on the conicoid, and if a straight line be drawn through the seventh point intersecting the other two lines, three points on this line, and therefore the whole line, will lie on the conicoid. Take these lines as the edges AB, CD, BC of the fundamental tetrahedron, these edges, and therefore the seven points, lie in the three pairs of planes $xz=0$, $xw=0$, $yw=0$, and the general equation is $\lambda xz+\mu xw+\nu yw=0$.

585. *In the conicoid of the last article the pole of a given plane lies in a fixed plane.*

Let $lx+my+nz+rw=0$ be the given plane, (x', y', z', w') its pole with respect to the conicoid $\lambda xz+\mu xw+\nu yw=0$, we shall then have

$$(\lambda z'+\mu w')/l=\nu w'/m=\lambda x'/n=(\mu x'+\nu y')/r;$$

$\therefore$ $lx'+my'-nz'-rw'=0$ is the equation of a fixed plane in which the pole lies.

Cor. The centres of all the conicoids lie in the plane $x'+y'=z'+w'$, which is parallel to the two edges AB, CD and passes through the centre of gravity of the tetrahedron, and the four points bisecting the other edges.

586. *Number of conicoids which pass through seven given points and touch two given planes.*

Let the given planes be $x=0$, $y=0$; at the point of contact with the first

$$\lambda dU/dy+\mu dV/dy+\nu dW/dy=0,\ \lambda dU/dz+\ldots=0, \text{ and } \lambda dU/dw+\ldots=0;$$

eliminating y, z, w from these, the eliminant is of the third degree in λ, μ, and ν. Similarly, the condition of touching $y=0$ leads to an equation of the third degree in λ, μ, and ν, and the final equation for determining $\lambda:\mu$ will be of the ninth degree. There are, therefore, generally nine conicoids satisfying the required conditions.

Similarly, it can be shewn that four conicoids through the seven points touch two given straight lines, and that six touch a plane and a straight line.

587. *Equation of a sphere circumscribing the fundamental tetrahedron.*

Let x', y', z' be the triangular coordinates of a point P in the plane of the triangle ABC, $x, y, z, 0$ the tetrahedral coordinates

$$x' = \Delta PBC / \Delta ABC = \text{vol.} DPBC / \text{vol.} DABC = x;$$

for the circle circumscribing ABC, $a^2y'z' + b^2z'x' + c^2x'y' = 0$. Hence the equation of the circumscribing sphere must give, when $w = 0$, $a^2yz + b^2zx + c^2xy = 0$, and similarly for $x = 0$, $y = 0$, $z = 0$, the equation of the sphere is therefore

$$a^2yz + b^2zx + c^2xy + a'^2xw + b'^2yw + c'^2zw = 0.$$

588. *General equation of a sphere.*

Since all spheres intersect the plane at infinity in the same circle, the equation

$$U \equiv (px + qy + rz + sw)(x + y + z + w) - a^2yz + \ldots - a'^2xw + \ldots = 0$$

represents a sphere, and is the general equation of a sphere since it contains four disposable constants.

In this form of the equation of the sphere, if the coordinates of any point $P(x', y', z', w')$ be substituted in the left side of the equation, the result is the rectangle r_1r_2, where r_1, r_2 are the distances from P to the points of intersection with any line through P.

For $(x - x')/\lambda = (y - y')/\mu = (z - z')/\nu = (w - w')/\rho = r/\sigma$ are the equations of any line through P, where $\lambda + \mu + \nu + \rho = 0$,

$$\text{and } a^2\mu\nu + \ldots + a'^2\lambda\rho = -\sigma^2, \text{ Art. 103},$$

$$\therefore U' + Pr + r^2 = 0, \text{ where } U' = r_1r_2.$$

589. *To find the radius of the sphere circumscribing the fundamental tetrahedron.*

Let R be the radius of the sphere whose equation is

$$a^2yz + \ldots + a'^2xw + \ldots \equiv f(x, y, z, w) = 0.$$

By the last Article, if (x_0, y_0, z_0, w_0) be the centre,

$$r_1r_2 = -R^2, \text{ and } f'(x_0) = f'(y_0) = f'(z_0) = f'(w_0) = 2f(x_0, y_0, z_0, w_0) = 2R^2;$$

$$\therefore 2R^2 \begin{vmatrix} 0, & c^2, & b^2, & a'^2, & 1 \\ c^2, & 0, & a^2, & b'^2, & 1 \\ b^2, & a^2, & 0, & c'^2, & 1 \\ a'^2, & b'^2, & c'^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} + \begin{vmatrix} 0, & c^2, & b^2, & a'^2 \\ c^2, & 0, & a^2, & b'^2 \\ b^2, & a^2, & 0, & c'^2 \\ a'^2, & b'^2, & c'^2, & 0 \end{vmatrix} = 0,$$

$$\text{or } 4R^2\{b'^2c'^2a^2 + c'^2a'^2b^2 + a'^2b'^2c^2 + a^2b^2c^2 - a^2a'^2(b^2 + b'^2 + c^2 + c'^2 - a^2 - a'^2) - \ldots\}$$
$$= a^4a'^4 + b^4b'^4 + c^4c'^4 - 2b^2b'^2c^2c'^2 - 2c^2c'^2a^2a'^2 - 2a^2a'^2b^2b'^2. \quad (1)$$

With centre D describe a sphere cutting DA, DB, DC in a, b, c, and draw am perpendicular to the side of the spherical triangle abc, then, by the two right-angled triangles, $\cos ab = \cos am \cos bm$ and $\cos ac = \cos am \cos cm$, whence

$$\sin^2 am \sin^2 bc = 1 - \cos^2 bc - \cos^2 ca - \cos^2 ab + 2\cos bc \cos ca \cos ab.$$

Let V be the volume of the tetrahedron, then $6V = b'c' \sin bc \times a' \sin am$, and, writing $b'^2 + c'^2 - a^2$ for $2b'c' \cos bc$, we can shew that the coefficient of $4R^2$ is $-144V^2$.

Also, if $2s = aa' + bb' + cc'$, the right side of the equation (1) is

$$-16s(s-aa')(s-bb')(s-cc'),$$

$$\therefore \; R = \sqrt{\{s(s-aa')(s-bb')(s-cc')\}}/6V.$$

590. *Equation of a conicoid touching the faces of the fundamental tetrahedron.*

Let the equation of the conicoid be

$$ax^2 + by^2 + cz^2 + dw^2 + 2a'yz + 2b'zx + 2c'xy + 2a''xw + \ldots = 0.$$

In order that the conicoid may touch the face $w = 0$,

$$ax^2 + \ldots + 2a'yz + \ldots \tag{1}$$

must be the product of two linear factors, since the intersection with $w = 0$ is two real or imaginary generators of the conicoid;

$$\therefore \; abc - aa'^2 - bb'^2 - cc'^2 + 2a'b'c' = 0, \tag{2}$$

and if $\cos\alpha = -a'/\sqrt{(bc)}$, $\cos\beta = -b'/\sqrt{(ca)}$, $\cos\gamma = -c'/\sqrt{(ab)}$,

$$1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma = 0;$$

$\therefore$ α, β, γ are the angles of a triangle, and (1) becomes

$$\{x\sqrt{a} - (y\sqrt{b}\cos\gamma + z\sqrt{c}\cos\beta)\}^2 + (y\sqrt{b}\sin\gamma - z\sqrt{c}\sin\beta)^2.$$

By treating the other planes in the same way, if

$$\cos\alpha' = -a''/\sqrt{(ad)}, \quad \cos\beta' = -b''/\sqrt{(bd)}, \quad \cos\gamma' = -c''/\sqrt{(cd)},$$

$x = 0$ being a tangent plane gives $\beta' + \gamma' + \alpha = \pi$; $y = 0$ and $z = 0$ give $\alpha' + \gamma' + \beta = \pi$ and $\alpha' + \beta' + \gamma = \pi$;

$$\therefore \; 2\alpha' + \beta + \gamma - \alpha = \pi = \beta + \gamma + \alpha;$$

$\therefore$ $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \gamma$; hence, the equation of the inscribed conicoid is, writing ξ, η, ... for $x\sqrt{a}$, $y\sqrt{b}$, ...,

$$\xi^2 + \eta^2 + \zeta^2 + \omega^2 - 2\cos\alpha(\eta\zeta + \xi\omega) - 2\cos\beta(\zeta\xi + \eta\omega)$$
$$- 2\cos\gamma(\xi\eta + \zeta\omega) = 0, \text{ where } \alpha + \beta + \gamma = \pi.$$

The tetrahedral coordinates of the points of contact with the face ABC are given by

$$x\sqrt{a}/\sin\alpha = y\sqrt{b}/\sin\beta = z\sqrt{c}/\sin\gamma.$$

591. *To find the equation of a sphere inscribed in the fundamental tetrahedron.*

The equation of the sphere being of the form

$$(lx + my + nz + rw)(x + y + z + w) - a^2yz - \ldots - a'^2xw - \ldots = 0,$$

we have to find l, m, n, and r.

Let P be the point of contact with the face opposite to A, and let R be the radius of the sphere, O its centre; with centre D let a spherical triangle be drawn whose angular points a', b', c' are in

the edges DA, DB, DC; DO will pass through the centre of the circle inscribed in $a'b'c'$, whose radius is ρ, where $\tan\rho = R/DP$; by Art. 588 $r = DP^2$; if, then, α, β, γ be the angles of the solid angle D, and $2s = \alpha + \beta + \gamma$,

$$\tan\rho\,\sqrt{\sin s} = \sqrt{\{\sin(s-\alpha)\sin(s-\beta)\sin(s-\gamma)\}};$$

$$\therefore\ r\sin(s-\alpha)\sin(s-\beta)\sin(s-\gamma) = R^2\sin s,\ \text{similarly for } l,\ m,\ n.$$

592. *General tangential equation of a sphere.*

Let x, y, z, w be the tetrahedral coordinates of the centre of the sphere, R its radius, p, q, r, s the point coordinates of any tangent plane of the sphere; then, by Arts. 109 and 112,

$$xp + yq + zr + ws = \pm R;$$

also, by Art. 112,

$$p^2/p_0^2 + q^2/q_0^2 + \ldots - 2\cos(AD)\,qr/q_0r_0 - \ldots = 1;$$

hence, in a homogeneous form, the required equation is

$$(xp + yq + zr + ws)^2 = R^2\,(p^2/p_0^2 + \ldots - 2\cos(AD)\,qr/q_0r_0 - \ldots).$$

593. *Tangential equation of a sphere touching the faces of the fundamental tetrahedron.*

For the inscribed sphere the tetrahedral coordinates of the centre are R/p_0, R/q_0, R/r_0, R/s_0; therefore the equation reduces to

$$\cos^2\tfrac{1}{2}(AD)\,qr/q_0r_0 + \ldots + \cos^2\tfrac{1}{2}(BC)\,ps/p_0s_0 + \ldots = 0.$$

For the escribed sphere touching ABC, we must write $-R/s_0$ for R/s_0, and the equation becomes

$$\cos^2\tfrac{1}{2}(AD)\,qr/q_0r_0 + \ldots - \sin^2\tfrac{1}{2}(BC)\,ps/p_0s_0 - \ldots = 0.$$

If p', q', r', s' be the coordinates of any tangent plane to these spheres $p'/p_0 + q'/q_0 + r'/r_0 \pm s'/s_0 = 1$, and the equation of the point of contact with the inscribed sphere will be

$$\{\cos^2\tfrac{1}{2}(CD)\,q'/q_0 + \cos^2\tfrac{1}{2}(BD)\,r'/r_0 + \cos^2\tfrac{1}{2}(BC)\,s'/s_0\}\,p/p_0 + \ldots = 0.$$

There will also be three spheres which touch two faces on the side of their subtending angles, and two on the opposite sides, the equations are found by writing two of the tetrahedral coordinates R/p_0, R/q_0, &c. negative.

594. *The tangential equation of any surface is the equation, referred to tetrahedral coordinates, of the reciprocal of that surface with respect to the surface $x^2 + y^2 + z^2 + w^2 = 0$.*

Let p, q, r, s be the four-point coordinates of a tangent plane to a surface whose equation is $F(x, y, z, w) = 0$ in tetrahedral coordinates, (x'', y'', z'', w'') being the point of contact;

$$px + qy + rz + sw = 0$$

is therefore the equation of the tangent plane, and p, q, r, s are determined by

$$p/F'(x'')=q/F'(y'')=r/F'(z'')=s/F'(w'').$$

But if (x', y', z', w') be the pole of that plane with respect to $x^2+y^2+z^2+w^2=0$, the equation of the plane will be

$$x'x+y'y+z'z+w'w=0, \text{ whence } x'/F'(x'')=y'/F'(y'')=\ldots;$$

hence, by eliminating x'', y'', z'', w'', we obtain the equation of the reciprocal of the given surface $\phi(x', y', z', w')=0$, as well as the tangential equation of the given surface $\phi(p, q, r, s)=0$ since the elimination is the same.

Hence, if any proposition with respect to surfaces be proved by the use of tetrahedral coordinates, the reciprocal proposition may be deduced from a different interpretation of the same equations, namely, by considering them throughout as relations between the point coordinates of the tangent planes, *i.e.* as the tangential equations of surfaces.

XLI.

(1) Prove that the equations of two conicoids cannot both be obtained in the form of Art. 569, if they have a common generating line.

(2) Two conicoids, each of which has plane contact with a third, intersect each other in plane curves.

(3) If three conicoids have a common plane section, the other planes in which, taken two and two, they intersect will meet in one straight line.

(4) If A, B, C, D be the vertices of the four cones of the second degree, which can be described through the curve of intersection of two conicoids, the triangle BCD will be a conjugate triad of the section made by its plane of the cone whose vertex is A.

(5) The number of paraboloids, which can be drawn through eight given points, is, in general, three.

(6) A cluster of conicoids have a common curve, shew that the polars of a fixed line lie in a hyperboloid of one sheet.

(7) Shew that a sphere can be described touching the edges a, a'; b, b'; c, c' of a tetrahedron, if $a \pm a' = b \pm b' = c \pm c'$, the ambiguities being independent.

(8) If of eight given points six lie by threes on two non-intersecting straight lines, shew that no cones can be described through the eight points; but that there is an infinite number of points, lying on two straight lines, which have their polar planes, with respect to any conicoid containing the eight points, fixed.

(9) Shew that only one conicoid can be described containing two given non-intersecting straight lines, and touching three given planes.

XLII.

(1) Four cones of the second degree can be drawn, each containing the locus of the centre of a conicoid passing through eight given points; and having their vertices coincident with the vertices of the cones on which lie the eight given points.

(2) If of seven given points six lie by threes on two non-intersecting straight lines, shew that the remaining line of intersection of any two paraboloids passing through the seven points will be a fixed straight line at infinity.

(3) If a conicoid be described containing the edges AB, BC, CD of a tetrahedron, the pole of the plane bisecting the edges AB, CD, AC, BD will lie on the plane bisecting the edges AB, CD, AD, BC.

(4) From the form $U = L^2$ deduce that when two conicoids have plane contact, the tangent plane at the umbilic of one cuts the other in a conic of which the umbilic is a focus.

(5) The equation of a conicoid referred to tetrahedral coordinates is

$$ax^2 + by^2 + cz^2 + dw^2 + 2a'yz + 2b'zx + 2c'xy + 2a''xw + 2b''yw + 2c''zw = 0;$$

shew that, if the conicoid break up into two planes, their line of intersection meets the plane $w = 0$, where $x/(bc - a'^2) = y/(a'b' - cc') = z/(a'c' - bb')$.

(6) When a system of conicoids passes through seven arbitrarily chosen points, shew that there is no point such that its polar with respect to each of the conicoids is a fixed plane. But, if the seven points be such that the polar plane of a point A will be fixed for all the conicoids, the seven points must lie on four straight lines passing through A.

(7) Prove that if a sphere can be inscribed so as to touch all the edges of the fundamental tetrahedron internally, the three lines joining the points of contact with the opposite edges will be concurrent.

(8) The conicoid $lx^2 + my^2 + nz^2 + rw^2 = 0$ will be a paraboloid if

$$l^{-1} + m^{-1} + n^{-1} + r^{-1} = 0,$$

and will be elliptic or hyperbolic as $lmnr$ is negative or positive. If this condition be satisfied, and a, b, c be the middle points of DA, DB, DC, a', b', c' of BC, CA, AB, the paraboloid will touch the planes $b'c'a$, $c'a'b$, $a'b'c$, and also the planes $bcb'c'$, $cac'a'$, $aba'b'$; and the points of contact of the former will be the angular points of a tetrahedron whose faces will intersect the corresponding faces of $ABCD$ in four lines lying in one plane; and this plane will pass through the points of contact of the latter three planes.

(9) Shew that four conicoids can in general be described passing through five points in one plane, two other given points not in that plane, and touching two given planes.

XLIII.

(1) Find the twelve centres of similarity of the sphere inscribed in the fundamental tetrahedron and of the three escribed spheres opposite the angles A, B, C of the tetrahedron, shew that they lie by sixes on eight planes.

(2) Shew that the square of the distance between the centres of the spheres inscribed in and circumscribed about the fundamental tetrahedron is

$$R'^2 - R^2 (a^2/q_0 r_0 + a'^2/p_0 s_0 + b^2/r_0 p_0 + \ldots),$$

where R, R' are the radii of the two spheres.

(3) Shew that eight conicoids can in general be described touching three given planes, and passing through four given points, the intersections of the tangent planes at each of which with the corresponding planes containing the points lie in one plane.

(4) If a tetrahedron be self-conjugate with respect to a sphere, shew that the opposite edges are, two and two, at right angles; and that all the plane angles containing one of the solid angles must be obtuse. Shew that this angular point will lie within the sphere, and the three others without, and determine the radius of the sphere.

(5) Prove that, in general, one conicoid can be described, which is self-conjugate with respect to one given tetrahedron, and also with respect to another of which three angular points are given. Shew also that the fourth angular point of the second tetrahedron cannot be in an arbitrary position.

(6) If two conicoids be so related that a tetrahedron can be drawn, whose faces touch one of the conicoids, and two pairs of whose opposite edges lie on the other, an infinite number of tetrahedrons can be so drawn.

(7) If $lx^2 + my^2 + nz^2 + rw^2 = 0$ and $l'x^2 + m'y^2 + n'z^2 + r'w^2 = 0$ be two conicoids related as in the last problem, prove that a similar relation will exist between the conicoids

$$l'x^2 + m'y^2 + n'z^2 + r'w^2 = 0 \text{ and } l'^2x^2/l + m'^2y^2/m + n'^2z^2/n + r'^2w^2/r = 0.$$

(8) Prove that two tetrahedrons may be inscribed in the conicoid $lx^2 + my^2 + nz^2 + rw^2 = 0$, having their faces parallel to the faces of the fundamental tetrahedron, provided that $(l + m + n + r)(l^{-1} + m^{-1} + n^{-1} + r^{-1}) < 4$.

(9) Shew that with the notation of Art. 568 $\Theta = 0$ will vanish when it is possible to find a tetrahedron self-conjugate with regard to V whose faces touch U.

(10) Two conicoids have two common generators of the same system, which are opposite edges of a tetrahedron, one of the remaining pair of opposite edges lies on each of the conicoids, shew that $\Theta = 0$, $\Theta' = 0$ and $\Phi^2 = 4\Delta\Delta'$.

CHAPTER XXI.

TORTUOUS CURVES. CURVATURE. TORTUOSITY.

595. We have already shewn that curves may be considered as the complete or partial intersection of surfaces, but in the investigation of the equations of tangents, osculating planes, &c., we shall also look upon a curve as the locus of points which satisfy more general laws, the algebraical statement of which assumes the form of equations between the coordinates of any point of the curve and variable parameters, the number of equations being two more than the number of parameters.

Instances of the latter mode of representation of a curve occur in dynamical problems, in which the curve is defined by equations between the coordinates of the position of a particle and the time of its arrival at that position.

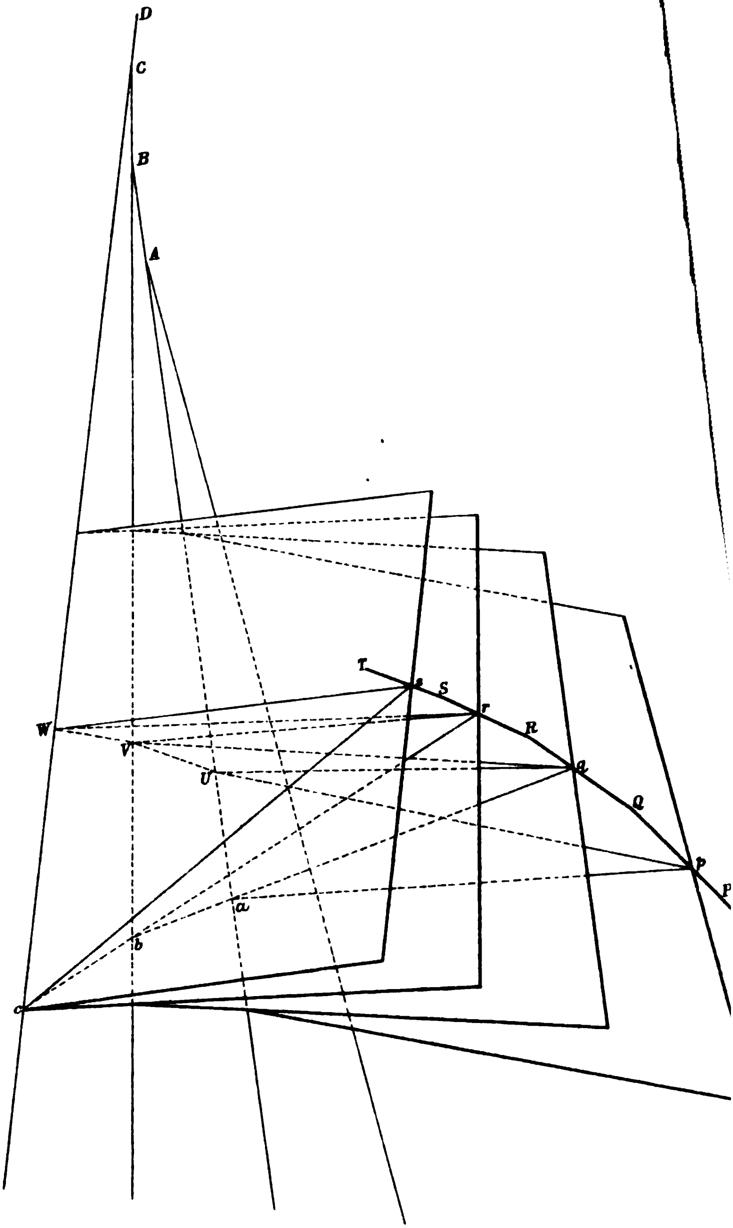
If the parameters were eliminated from the equations connecting the coordinates and parameters, the result would be two final equations which would be the equations of two surfaces whose complete or partial intersections would be the curve in question.

596. If the coordinates of any point on a curve can be expressed as functions of a single parameter t, so that for each value of t there is a single value of each coordinate, the curve is called *unicursal*.

597. As an example of an unicursal curve, we may take the helix, which is generated by the uniform motion of a point along a generating line of a right cylinder as the generating line revolves with uniform angular velocity about the axis of the cylinder.

If we take the axis for the axis of z, and the axis of x through the generating point at any initial time, θ the angle through which the generating line has revolved when the point has moved through a space z on the generating line, we have for the coordinates of the point, a being the radius of the cylinder, $x = a\cos\theta$, $y = a\sin\theta$, $z = na\theta$; here θ is the variable parameter, and the curve is the intersection of the surfaces $x^2 + y^2 = a^2$, and $y = x\tan(z/na)$.

598. In order to explain the terms employed in the examination of curves which are not plane, we shall consider such curves as the limits of polygons whose sides are indefinitely small; and we observe that the plane which contains any two consecutive sides of the polygon of which the curve is the limit does not generally contain the next side; the property, that the plane in which the curvature is taking place at any point changes as the point changes, is represented by calling the curve *tortuous* and the measure of the corresponding property *tortuosity*.



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599. *Osculating Plane.* The plane containing two sides of the polygon of which a tortuous curve is the limit is in its ultimate position an *osculating plane* of the curve.

600. *Normal Plane.* Any side of the polygon in its limiting position is, when produced, a tangent to the curve, and a plane drawn perpendicular to the tangent through the point of contact is a *normal plane*, being the locus of all the normals at the point.

601. *Principal Normal.* The particular normal which lies in the osculating plane is called the *principal normal.*

602. *Binormal.* The normal which is perpendicular to the osculating plane is called the *binormal*, being perpendicular to two elements of the curve.

603. *Polar Developable.* Let an equilateral polygon be inscribed in a curve, of which consecutive sides are PQ, QR, RS, ST, and let p, q, r, s be the middle points of these sides.

Let Aap, Bbq, Ccr be planes perpendicular to these sides, forming the polygon $ABCD$ by their intersections.

If the sides PQ, QR, ... be diminished indefinitely, their directions are ultimately those of tangents to the curve, the planes Aap, Bbq, ... are ultimately normal planes to the curve, the planes PQR, QRS, ... are osculating planes, and the surface generated by the plane elements Aab, Bbc, Ccd, ... is ultimately the developable surface enveloped by the normal planes of the curve, of which $ABCD$ ... is ultimately the edge of regression.

The developable surface enveloped by the normal planes was called by Monge the *Polar Developable.*

604. *Circle of Curvature.* A circle can be described containing the points P, Q, R; when the sides are indefinitely diminished, this circle lies in the osculating plane, and its curvature may be taken as the measure of curvature of the curve in the osculating plane. Let the plane PQR meet Aa in U, and let pU, qU be joined; then since PQ is perpendicular to the plane Apa, it is perpendicular to pU; similarly QR is perpendicular to qU; U is therefore the centre of the circle through PQR. Therefore the centre of the circle of curvature is the point of intersection of two consecutive normal planes and the osculating plane.

605. *Polar Line.* Draw pa, qa to any point in Aa, then, since $Pp = Qp$, a is equally distant from P and Q, and similarly from Q and R, and therefore from every point in the circle of curvature. The line of intersection of two consecutive normal planes is called by Monge the *Polar Line.*

606. *Angle of Contingence.* The angle pUq, which is equal to the angle between the two consecutive sides PQ, QR of the polygon,

is ultimately equal to the angle between two consecutive tangents, and is called the *angle of contingence.*

607. *Sphere of Curvature.* Any point in Aa is equally distant from P, Q and R; also any point in Bb is equally distant from Q, R and S; therefore their point of intersection is equally distant from the four points P, Q, R, S.

Hence, it follows that a sphere can be described whose centre is B, and which contains the four points P, Q, R, S, this sphere is ultimately the sphere which has the closest possible contact with the curve, since no sphere can be made to pass through more than four arbitrary points, it is therefore called the *sphere of curvature;* the locus of its centre is the edge of regression of the polar developable.

608. *Evolutes.* If a be *any* point in the intersection of the planes normal to PQ, QR, at their middle points p, q, ap and aq will be equal and it is clear that they will also make equal angles with Aa. Produce qa to meet Bb in b; then a string, placed in the position bap, would remain in that position if subject to tension, since the tensions of the portions ab, ap resolved parallel to Aa would be equal, and, if its extremity were then moved from p to q it would occupy the position baq. Similarly, if rb be produced to c in Cc, and if sc be produced to d in Dd. If we proceed to the limit, it follows that a string may be stretched upon the polar developable in such a manner that the free end, starting from any point in the curve, would describe the curve, if the string were unwrapped from the surface so that the part in contact with the surface remained stationary. The portion in contact lies on a curve called the *evolute.*

Also, since the position of the line pa is arbitrary, the curve which is the limit of a, b, c, d,... will change its position according to the position of a, hence the number of evolutes is infinite.

609. *Locus of Centres of Circular Curvature not an Evolute.* Since qU will not, if produced, pass through V, because qU and qV include an angle in the same normal plane, the locus of the centres of circular curvature is not one of the evolutes.

610. *Angle of Torsion.* The plane pUq perpendicular to AUa contains the sides PQ, QR, and the plane qVr perpendicular to BVb contains the sides QR, RS, and, since qU, qV are perpendicular to the line of intersection QR of the two planes, the angle UqV is their angle of inclination.

This angle, which is ultimately the angle between consecutive osculating planes, is called the *angle of torsion.*

Also, since a circle goes round $BVUq$, the angles UqV and UBV are equal, and the angle of torsion of the curve PQR,...... is equal

to the angle of contingence of the edge of regression of the polar developable.

611. *Rectifying Developable.* If through every point of a curve a plane be drawn perpendicular to the corresponding principal normal, these planes will envelope a torse on which the curve will be a geodesic line, since its osculating plane will contain the normal to the surface at every point, as will be shewn in the chapter on geodesics; if, therefore, the torse be developed into a plane, the curve will be developed into a straight line. On account of this property the torse is called the *rectifying developable.*

612. *Rectifying Line.* The line of intersection of two consecutive planes, enveloping the rectifying developable, is called the *rectifying line* for any point of the curve, being the line about which the curve must turn at that point in order to become straight, when the torse is developed into a plane.

It may be observed that the rectifying line is not generally coincident with the binormal, which is the normal perpendicular to the osculating plane.

In the figure at p. 251 the surface whose edge of regression is the limit of ABC... is the rectifying surface to the curve which is the limit of abc.... Aa is the rectifying line at a, and the binormal does not coincide with the rectifying line unless pa be perpendicular to Aa, or a be the centre of curvature of the involute of abc....

613. If the polygon $PQRS$... were transformed into a plane polygon by turning the portion $QRST$... through the angle of torsion VqU about QR, and the portion RST... about RS through the corresponding angle of torsion, any side ST in the new position in the plane of PQR would be inclined to PQ at an angle equal to the sum of the inclinations of the sides taken in order, and estimated in the same direction.

Proceeding to the limit, we see that if, as a point moves along a tortuous curve, at every position which the point assumes the curve be turned about the tangent line through the angle of torsion, the curve will be replaced by a plane curve, such that the inclination of the tangents at the starting point and any other point will be the sum of all the angles of contingence; if, therefore, ε be taken for the angle between the tangents in the plane curve, $d\varepsilon$ will be the angle of contingence corresponding to the extremity of the arc traversed by the moving point.

614. *Rate of Torsion.* The rate per unit of length of arc at which the osculating plane twists about the tangent line at any point, called the *rate of torsion*, is measured by the limit of the ratio of the angle of torsion to the arc at the extremities of which the osculating planes are taken.

If, as we pass from PQ to QR, see figure, p. 251, QR be turned in the plane PQR so that PQR is a straight line, and the plane QRS

be then turned through the angle VqU, the process being repeated along the whole of a given arc, the perimeter will become rectified, and the inclination of the last to the first position of the plane containing two elements will be the sum of all angles such as VqU between the extremities of the arc so rectified.

Proceeding to the limit, it follows that, if osculating planes be taken along the curve, and the elements of the arc be rectified in each osculating plane in order, the angle between the first and final positions of the osculating plane when the curve is so rectified will be the sum of the angles of torsion throughout the arc.

If, therefore, τ be this angle, $d\tau$ will be the angle of torsion, corresponding to the point at which the last osculating plane is drawn.

615. *Integral and Average Curvature.** The *integral curvature* of any portion of a curve is the angle through which the tangent will have turned as we pass from one extremity to the other, the *average curvature* is its whole curvature divided by its length.

Let a sphere of unit radius have its centre at a fixed point, and let radii be drawn parallel to the tangents to the curve at successive points, the length of the curve traced on the sphere by the extremities of the radii measures the integral curvature of the portion of the curve considered, and the average curvature is the integral curvature divided by the length of the curve.

616. *Integral and Average Tortuosity.* These are respectively the angle through which the osculating plane has turned in passing from end to end of any portion of a curve, and this angle divided by the length of the arc considered.

On the sphere discussed in the last article let a curve be described by the poles of the great circles which are tangents to the curve which measures the integral curvature, the length of this curve measures the integral tortuosity, and this length divided by the length of the arc of the tortuous curve the average tortuosity.

Tangents.

617. *Tangent to a curve at a given point.*

Let s, $s + \Delta s$ be the lengths measured along the arc of a curve from a given point to the points P and Q, whose coordinates are x, y, z and $x + \Delta x, y + \Delta y, z + \Delta z$, and let $c =$ chord PQ.

As Q approaches to and ultimately coincides with P, the chord PQ and arc Δs become equal, PQ is the direction of the tangent at P, and the direction cosines of PQ, viz. $\Delta x/c$, $\Delta y/c$, $\Delta z/c$ become ultimately dx/ds, dy/ds, dz/ds.

* Thomson and Tait, *Nat. Phil.* Art. 12.

The equations of the tangent are therefore

$$(\xi - x)/dx = (\eta - y)/dy = (\zeta - z)/dz.$$

Also since $c^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$,

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2.$$

i. Let the equations of the curve be given in terms of a variable parameter θ, in the form

$$x = \phi(\theta),\ y = \psi(\theta),\ z = \chi(\theta),$$

then $dx : dy : dz = \phi'(\theta) : \psi'(\theta) : \chi'(\theta)$,

and the equations of the tangent at a point corresponding to θ are

$$(\xi - x)/\phi'(\theta) = (\eta - y)/\psi'(\theta) = (\zeta - z)/\chi'(\theta).$$

ii. Let the equations be those of surfaces containing the curve $F(\xi, \eta, \zeta) = 0$, and $G(\xi, \eta, \zeta) = 0$.

Then, at any point P of the curve,

$$F'(x)\,dx + F'(y)\,dy + F'(z)\,dz = 0,$$

and $G'(x)\,dx + G'(y)\,dy + G'(z)\,dz = 0$;

whence the equations of the tangent PQ may be written

$$F'(x)(\xi - x) + F'(y)(\eta - y) + F'(z)(\zeta - z) = 0,$$

and $G'(x)(\xi - x) + G'(y)(\eta - y) + G'(z)(\zeta - z) = 0$,

which equations represent analytically the fact that the tangent to the curve lies in the tangent plane to each surface at the common point P.

iii. If the surfaces, the intersection of which gives the curve, be cylindrical surfaces whose sides are parallel to the two axes of z and y, and their equations be $\eta = f(\xi)$, $\zeta = \phi(\xi)$, the equations of the tangent will be $\eta - y = f'(x)(\xi - x)$, $\zeta - z = \phi'(x)(\xi - x)$.

These equations are the analytical representation of the fact that the projections of the tangent to the curve on the coordinate planes of xy, zx are the tangents to the respective projections of the curve, which is obviously true, since the projections of P and Q have their ultimate coincidence simultaneously with that of P and Q.

618. *To find the directions of the branches of the curve of intersection of two surfaces at a multiple point of the curve.*

The equations of the surfaces being

$$F(\xi, \eta, \zeta) = 0, \text{ and } G(\xi, \eta, \zeta) = 0,$$

and (x, y, z) being a multiple point P on the curve, let

$$(\xi - x)/\lambda = (\eta - y)/\mu = (\zeta - z)/\nu = r \qquad (1)$$

be the equations of a line through P; the points in which this line meets the surfaces are given by the equations

$$\left.\begin{aligned} F(x + \lambda r,\ y + \mu r,\ z + \nu r) &= 0 \\ \text{and } G(x + \lambda r,\ y + \mu r,\ z + \nu r) &= 0 \end{aligned}\right\},$$

there are an infinite number of directions which give two values of r equal to zero, since the curve has a multiple point at P; therefore the two equations

$$\lambda F'(x)+\mu F'(y)+\nu F'(z)=0, \qquad (2)$$

$$\text{and } \lambda G'(x)+\mu G'(y)+\nu G'(z)=0, \qquad (3)$$

must be one or both identically satisfied, or else they must not be independent equations.

i. If only one of the equations (2) and (3) be identically satisfied; suppose this to be (2), then (x, y, z) will be a multiple point on the surface $F(\xi, \eta, \zeta)=0$; and, if this be a double point, the line (1) must be one of the tangents whose directions are given by

$$\left(\lambda\frac{d}{dx}+\mu\frac{d}{dy}+\nu\frac{d}{dz}\right)^2 F(x, y, z)=0; \qquad (4)$$

and, since it lies in the tangent plane to $G(\xi, \eta, \zeta)=0$, the equations (3) and (4) give the directions of the two tangent lines, which are the intersections of the conical tangent to the first surface with the tangent plane to the second; and, similarly, for higher degrees of multiplicity.

ii. If (2) and (3) be both identically satisfied, the line (1) will be in any of the directions of common tangents to $F(\xi, \eta, \zeta)=0$ and $G(\xi, \eta, \zeta)=0$; the directions are, therefore, given by

$$\left(\lambda\frac{d}{dx}+\mu\frac{d}{dy}+\nu\frac{d}{dz}\right)^s F(x, y, z)=0$$

$$\text{and } \left(\lambda\frac{d}{dx}+\mu\frac{d}{dy}+\nu\frac{d}{dz}\right)^t G(x, y, z)=0,$$

where s and t are the degrees of multiplicity of the multiple points of the two surfaces at (x, y, z).

iii. If neither (2) nor (3) be identically satisfied, but the two equations be identical so as to be satisfied by an infinite number of values of $\lambda : \mu : \nu$, there will be a surface $AF+BG=0$, which will pass through the intersection of $F=0$ and $G=0$, for which $\left(\lambda\frac{d}{dx}+\mu\frac{d}{dy}+\nu\frac{d}{dz}\right)(AF+BG)=0$ will be identically satisfied, if $-B/A$ be the value of the equal ratios $F'(x)/G'(x)$, $F'(y)/G'(y)$, and $F'(z)/G'(z)$.

In this case, therefore, $\lambda : \mu : \nu$ is determined by one of the equations (2), (3), and

$$\left(\lambda\frac{d}{dx}+\mu\frac{d}{dy}+\nu\frac{d}{dz}\right)^2 (AF+BG)=0.$$

If in any of these cases two values of $\lambda : \mu : \nu$ be equal, there will be either a point of osculation or a cusp on the curve.

619. As an example of case iii in the last Article, suppose we wish to find the directions of the tangents at the point $(a, 0, 0)$ in the curve of intersection of the hyperboloid and hyperbolic paraboloid, whose equations are

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1, \text{ and } y^2/l - z^2/l' = 2(x - a).$$

At this point the surfaces have a common tangent plane, whose equation is $x = a$; the third surface, on which $(a, 0, 0)$ is a multiple point, is in this case the cone $(x-a)^2 + (a^2/b^2 + a/l)y^2 - (a^2/c^2 + a/l')z^2 = 0$, and the directions of the tangents to the curve are given by $(a^2/b^2 + a/l)\mu^2 - (a^2/c^2 + a/l')\nu^2 = 0$, and $\lambda = 0$.

620. *Normal plane of a curve at a given point.*

The normal plane being perpendicular to the tangent to the curve, its equation is $(\xi - x)\,dx + (\eta - y)\,dy + (\zeta - z)\,dz = 0$.

621. *To find the edge of regression of the polar developable of a curve.*

The edge of regression is the locus of the intersection of three consecutive normal planes to the curve.

The equation of the normal plane at (x, y, z) is

$$(\xi - x)\,dx + (\eta - y)\,dy + (\zeta - z)\,dz = 0, \qquad (1)$$

that of the normal plane at a consecutive point is found by writing in this equation $x + dx$ for x, &c., the line of intersection of the two normal planes will lie in the plane

$$(\zeta - x)\,d^2x + (\eta - y)\,d^2y + (\zeta - z)\,d^2z - (dx)^2 - (dy)^2 - (dz)^2 = 0. \quad (2)$$

Again, writing $x + dx$ for x, &c., we obtain a plane in which the line of intersection of the second and third normal planes lies,

$$(\xi - x)\,d^3x + (\eta - y)\,d^3y + (\zeta - z)\,d^3z - 3\,(dx\,d^2x + dy\,d^2y + dz\,d^2z) = 0, \quad (3)$$

and the coordinates of the point of the edge of regression satisfy these three equations. If we eliminate x, y, z from the equations (1), (2), (3) and the equations of the curve, we shall obtain the two equations of the edge of regression.

The line of which (1) and (2) are the equations is Monge's polar line, which is the axis of the osculating circle.

The point given by the three equations (1), (2), (3) is the centre of spherical curvature corresponding to the point (x, y, z) of the curve.

622. *To find the differential coefficient of the arc referred to polar coordinates.*

Transforming to polar coordinates

$$x = r \sin\theta \cos\phi = \rho \cos\phi,$$
$$y = r \sin\theta \sin\phi = \rho \sin\phi,$$
$$z = r \cos\theta, \qquad \rho = r \sin\theta\,;$$

hence if x' be written for $dx/d\theta$, &c.,

$$x'^2 + y'^2 = \rho'^2 + \rho^2\phi'^2,$$
$$z'^2 + \rho'^2 = r'^2 + r^2,$$
$$\therefore s'^2 = r'^2 + r^2 + r^2 \sin^2\theta\, \phi'^2.$$

The equation is easily obtained geometrically by observing that ultimately $(\Delta s)^2 = (\Delta r)^2 + (r\Delta\theta)^2 + (r\sin\theta\Delta\phi)^2$.

Also, if p be the perpendicular from the pole upon the tangent, and ψ the angle between r and the tangent, $p = r\sin\psi$,

$$\text{and } \Delta s/\Delta r = \sec\psi \text{ ultimately, } \therefore\ (ds/dr)^2 = r^2/(r^2 - p^2).$$

Osculating Plane.

623. *Equation of the osculating plane.*

The osculating plane may be considered as the plane which passes through three consecutive points, whose coordinates are x, y, z; $x + dx, \ldots$ and $x + 2dx + d^2x, \ldots$.

Let the equation of the osculating plane be

$$A(\xi - x) + B(\eta - y) + C(\zeta - z) = 0; \quad \therefore Adx + Bdy + Cdz = 0,$$

$$\text{and } A(2dx + d^2x) + B(2dy + d^2y) + C(2dz + d^2z) = 0,$$

$$\text{or } Ad^2x + Bd^2y + Cd^2z = 0,$$

hence the equation is

$$\begin{vmatrix} \xi - x, & \eta - y, & \zeta - z \\ dx, & dy, & dz \\ d^2x, & d^2y, & d^2z \end{vmatrix} = 0.$$

It may be noted that the equations of the tangent and osculating plane are of the same form, whether the axes be rectangular or oblique.

624. It should be observed with respect to the notation used above that if x, y, z be supposed given as functions of t, and we take points corresponding to values t, $t + \tau$, $t + 2\tau$, which is the same as making t the independent variable, the values of x for $t + \tau$ and $t + 2\tau$ are

$$x + \frac{dx}{dt}\tau + \frac{d^2x}{dt^2}\frac{\tau^2}{2} + \ldots \text{ and } x + \frac{dx}{dt}2\tau + \frac{d^2x}{dt^2}\frac{(2\tau)^2}{2} + \ldots;$$

and if the first be written $x + \Delta x$, the second will be

$$x + \Delta x + \Delta(x + \Delta x) \text{ or } x + 2\Delta x + \Delta^2 x;$$

hence if d be written for Δ, when τ is indefinitely diminished, $dx = \dfrac{dx}{dt}\tau$ and $d^2x = \dfrac{d^2x}{dt^2}\tau^2$ ultimately.

625. As an exercise the student should find the equation of the osculating plane, considered as given by any of the following definitions:

i. As a plane containing a tangent and a point indefinitely near the point of contact.

ii. As a plane containing a tangent and parallel to a consecutive tangent.

iii. As a plane which has a closer contact with the curve than any other plane.

In employing the definition ii. he may shew that the shortest distance between the tangents at the extremity of any arc ds is generally of the order of ds^3.

626. *Direction cosines of the binormal.*

The direction cosines of the binormal, which is perpendicular to the osculating plane, are in the ratio

$$dyd^2z - dzd^2y : dzd^2x - dxd^2z : dxd^2y - dyd^2x,$$

and the sum of the squares of these expressions

$$= \{(dx)^2+(dy)^2+(dz)^2\}\{(d^2x)^2+(d^2y)^2+(d^2z)^2\}-(dxd^2x+dyd^2y+dzd^2z)^2$$

$$= (ds)^2\{(d^2x)^2+(d^2y)^2+(d^2z)^2\}-(dsd^2s)^2,$$

$$\text{since } (ds)^2=(dx)^2+(dy)^2+(dz)^2,$$

hence the direction cosines of the binormal are

$$\pm\frac{dy\,d^2z - dz\,d^2y}{ds\,\{(d^2x)^2+(d^2y)^2+(d^2z)^2-(d^2s)^2\}^{\frac{1}{2}}}, \text{ \&c.}$$

If the pivot of the hands of a watch, with its face in the osculating plane, be supposed placed at the centre of curvature of the curve so that the extremities of the hands move in the direction in which ds is measured, the $+$ sign must be used, when the direction of the pivot is chosen as the positive direction of the binormal. In fact, if the pivot make an acute angle with the axis of x, it is evident that, when s is measured in the direction of the motion of the hands, dz/dy will increase with s, and $dyd^2z - dzd^2y$ will be positive.

627. *To find the condition that an osculating plane may be stationary, and that a curve whose equations are given may be plane.*

The condition that an osculating plane may be stationary, or that the osculating planes at two consecutive points on the curve may coincide, will be satisfied for any point (x, y, z), if the osculating plane at that point contain a fourth point, for which the value of x is $x+3dx+3d^2x+d^3x$.

If we write this value and the corresponding values of y and z for ξ, η and ζ in the equation of the osculating plane, we shall obtain the equation

$$(dy\,d^2z - dz\,d^2y)\,d^3x + (dz\,d^2x - dx\,d^2z)\,d^3y + (dx\,d^2y - dy\,d^2x)\,d^3z = 0$$

as the condition required.

If this condition hold at every point of the curve, the curve will be plane, and the equation of its plane will be that of the osculating plane.

628. When the curve is given by means of the equations of two surfaces of which it is the intersection, the calculations required for the determination of the osculating plane may be conveniently conducted as follows:

629. *To find the osculating plane of the curve of intersection of two surfaces, whose equations are given.*

Let $\phi(x, y, z) = 0$, $\phi'(x, y, z) = 0$ be the given equations, then, using the notation of Art. 461,

$$Udx + Vdy + Wdz = 0, \qquad U'dx + V'dy + W'dz = 0.$$

Let D, E, F denote the determinants $\left\| \begin{matrix} U, & V, & W \\ U', & V', & W' \end{matrix} \right\|$;

$$\therefore\ dx/D = dy/E = dz/F = k \text{ suppose,}$$

$$\text{whence } d^2y = kdE + Edk, \quad d^2z = kdF + Fdk;$$

$$\therefore\ dyd^2z - dzd^2y = k^2(EdF - FdE),$$

hence the equation of the osculating plane is

$$(EdF - FdE)(\xi - x) + \ldots = 0.$$

630. ***Equation of the osculating planes in terms of the equations of the tangent planes to the surfaces.***

Employing the notation of the preceding article, we see that

$$DU + EV + FW = 0;$$

$$\therefore\ UdD + VdE + WdF + DdU + EdV + FdW = 0,$$

$$\text{and}\quad dU = dx\frac{dU}{dx} + dy\frac{dU}{dy} + dz\frac{dU}{dz}$$

$$= k\left(D\frac{d}{dx} + E\frac{d}{dy} + F\frac{d}{dz}\right)\frac{d\phi}{dx} = k\frac{d}{dx}\Gamma(\phi),$$

if Γ denote the operation in the brackets, in the performance of which D, E, F are considered constant;

$$\therefore\ DdU + EdV + FdW = k\Gamma^2(\phi),$$

$$\text{hence}\quad UdD + VdE + WdF = -k\Gamma^2(\phi),$$

$$\text{similarly}\quad U'dD + V'dE + W'dF = -k\Gamma^2(\phi');$$

$$\therefore\ EdF - FdE = k\{U\Gamma^2(\phi') - U'\Gamma^2(\phi)\},$$

and the equation of the osculating plane becomes

$$\Gamma^2(\phi')\{U(\xi - x) + V(\eta - y) + W(\zeta - z)\} = \Gamma^2(\phi)\{U'(\xi - x) + \ldots\}.$$

631. ***To find the osculating plane of the intersection of two concentric and coaxial conicoids.***

Let the equations be $ax^2 + by^2 + cz^2 = 1$, and $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$, (1)

$$D = 4(b\gamma - c\beta)yz = Ayz, \quad E = Bzx, \quad F = Cxy,$$

$$EdF - FdE = E^2d(F/E) = BCz^2x^2d(y/z)$$

$$= BCx^2(zdy - ydz) = kBCx^2(Ez - Fy),$$

$$\text{and by (1)}\quad Ez - Fy = 4(\alpha - a)x;$$

$$\therefore\ EdF - FdE = 4kBC(\alpha - a)x^2,$$

and the equation of the osculating plane is

$$A^{-1}(\alpha - a)x^3(\xi - x) + B^{-1}(\beta - b)y^3(\eta - y) + C^{-1}(\gamma - c)z^3(\zeta - z) = 0, \quad \text{Art. 629,}$$

which may be reduced to

$$\frac{BC}{(\beta - b)(\gamma - c)}x^3\xi + \frac{CA}{(\gamma - c)(\alpha - a)}y^3\eta + \frac{AB}{(\alpha - a)(\beta - b)}z^3\zeta + 1 = 0.$$

632. Or, by the method of Art. 630, since $\frac{1}{2}\Gamma^2(\phi) = D^2a + E^2b + F^2c$, the equation may be written

$$(D^2\alpha + E^2\beta + F^2\gamma)(ax\xi + by\eta + cz\zeta - 1) - (D^2a + E^2b + F^2c)(\alpha x\xi + \beta y\eta + \gamma z\zeta - 1) = 0,$$

betw the coefficient of $\xi = \frac{1}{4}(E^2C - F^2B)x$

of ds^3. $= \frac{1}{4}BC(Ez - Fy)x^2 = BC(\alpha - a)x^3$, as before.

633. *To find the condition for a stationary osculating plane of the curve of intersection of two surfaces.*

The equation of an osculating plane is

$$(EdF - FdE)(\xi - x) + \ldots = 0,$$

the line of intersection of this plane with the next consecutive osculating plane is in the plane

$$(Ed^2F - Fd^2E)(\xi - x) + \ldots - (EdF - FdE)\,dx - \ldots = 0;$$

the last three terms are identically zero, since $dx = kD$, and in order that the two osculating planes should coincide,

$$\frac{Ed^2F - Fd^2E}{EdF - FdE} = \frac{Fd^2D - Dd^2F}{FdD - DdF} = \frac{Dd^2E - Ed^2D}{DdE - EdD},$$

which are clearly equivalent to one distinct equation; and each of the fractions is equal to $\dfrac{d^2D\,(Ed^2F - Fd^2E) + \ldots}{d^2D\,(EdF - FaE) + \ldots}$, the numerator of which vanishes;

$$\therefore\ d^2D\,(EdF - FdE) + \ldots = 0.$$

Principal Normal.

634. *To find the equations of the principal normal at any point of a curve.*

In this and the following articles, where s is the independent variable, x', y', z' and x'', y'', z'' will be written for the first and second differential coefficients of x, y, z.

The principal normal is in the osculating plane and perpendicular to the tangent, it is therefore perpendicular to two lines whose direction-cosines are respectively proportional to $y'z''-z'y''$, $z'x''-x'z''$, $x'y''-y'x''$, and x', y', z'. But identically,

$$x''(y'z'' - z'y'') + y''(z'x'' - x'z'') + z''(x'y'' - y'x'') = 0,$$

and, since $\quad x'^2 + y'^2 + z'^2 = 1, \qquad x''x' + y''y' + z''z' = 0;$

therefore the direction-cosines of the principal normal are proportional to x'', y'' and z'', hence its equations are

$$(\xi - x)/x'' = (\eta - y)/y'' = (\zeta - z)/z''.$$

635. *If from any point in a curve equal distances be measured along the curve and its tangent, the limiting position of the line joining the extremities of these distances is the principal normal.*

From the point (x, y, z) let equal distances σ be measured along the curve and the tangent to the points Q, T; the coordinates of Q are

$$x + x'\sigma + \tfrac{1}{2}(x'' + \epsilon)\,\sigma^2, \text{ \&c.},$$

and those of T, $x + x'\sigma$, &c., ϵ vanishing in the limit. The equations of QT are

$$(\xi - x - x'\sigma)/(x'' + \epsilon) = (\eta - y - y'\sigma)/(y'' + \epsilon') = (\zeta - z - z'\sigma)/(z'' + \epsilon'');$$

therefore the limiting position of QT is the principal normal.

Cauchy proposed, as a *definition* of the principal normal at any point, the limiting position of the line joining the points on the curve and tangent, whose distances from the point of contact measured along the curve and tangent respectively are equal, by which means the definition was made independent of the osculating and normal planes.

Measure of Curvature.

636. *To find the radius of curvature at any point of a tortuous curve.*

The reciprocal of the radius of curvature is the measure of curvature, or the rate per unit of length at which the tangent to the curve changes its direction. If ρ be the radius of curvature at a point P, and $d\varepsilon$ be the angle of contingence corresponding to the arc ds, $\rho = ds/d\varepsilon$.

Draw Op, Oq of unit length through the origin parallel to the tangents at P and Q the extremities of the arc ds, join pq; then, since the plane pOq is parallel to the osculating plane, pq, which is ultimately perpendicular to Op, is parallel to the principal normal.

The coordinates of p are x', y', z', and those of q are ultimately $x'+dx'$, $y'+dy'$, $z'+dz'$;

$$\therefore\ pq^2 = (dx')^2 + (dy')^2 + (dz')^2, \text{ and } pq = d\varepsilon \text{ ultimately};$$

$\therefore\ \rho^{-2} = x''^2 + y''^2 + z''^2$, and the direction cosines of the binormal are $\rho\,(y'z'' - z'y'')$, &c.

637. The direction of the radius of curvature and the coordinates of the centre of curvature may be found by projections as follows:—Since pq is ultimately in the direction of the radius of curvature, let l, m, n be its direction-cosines; project $OpqO$ on the axis of x; $\therefore\ x' + l d\varepsilon - (x'+dx') = 0$; $\therefore\ l = x''ds/d\varepsilon = x''\rho$, and similarly for m and n. Also, if C be the centre of curvature at P, the projection of OPC on $Ox = x + l\rho = x + \rho^2 x''$, hence the coordinates of C are $x + \rho^2 x''$, $y + \rho^2 y''$, $z + \rho^2 z''$.

638. The student should, as an exercise, find the radius and centre of curvature, when the latter is considered as the point of intersection of two consecutive normal planes and the osculating plane.

Measure of Tortuosity.

639. *To find the measure of tortuosity of a tortuous curve.*

Let l, m, n and $l+dl$, $m+dm$, $n+dn$ be the direction cosines of the binormals at two points P, Q, whose distance along the curve is ds. Draw unit lengths Op, Oq parallel to the two binormals, $pq = d\tau$ is the angle between the osculating planes, and $d\tau/ds$, the rate at which the osculating plane turns round the tangent line per unit of arc, is the measure required, which we shall call σ^{-1}. Now, since the coordinates of p and q are l, m, n and $l+dl$, $m+dm$, $n+dn$, $pq^2 = (dl)^2 + (dm)^2 + (dn)^2$; $\therefore\ \sigma^{-2} = l'^2 + m'^2 + n'^2$, where $l = \rho\,(y'z'' - z'y'')$ &c. σ is sometimes called the radius of torsion at P, but it is better to look upon σ^{-1} as the measure of tortuosity.

640. The measure of tortuosity may be expressed in another form.

Since $l : m : n :: X : Y : Z$, where $X = dy\,d^2z - dz\,d^2y$, and similar expressions for Y and Z,

$$ldx + mdy + ndz = 0, \quad ld^2x + md^2y + nd^2z = 0;$$

$$\therefore\ dl\,dx + dm\,dy + dn\,dz = 0, \text{ and } dl.l + dm.m + dn.n = 0;$$

$$\therefore\ \frac{dl}{m\,dz - n\,dy} = \frac{dm}{n\,dx - l\,dz} = \frac{dn}{l\,dy - m\,dx} = \frac{dl\,d^2x + dm\,d^2y + dn\,d^2z}{lX + mY + nZ}$$

$$= -\frac{l\,d^3x + m\,d^3y + n\,d^3z}{lX + mY + nZ} = -\frac{Xd^3x + Yd^3y + Zd^3z}{X^2 + Y^2 + Z^2},$$

and $(m\,dz - n\,dy)^2 + \ldots = (l^2 + m^2 + n^2)\,\{(dx)^2 + (dy)^2 + (dz)^2\} - (l\,dx + \ldots)^2 = ds^2$;

$$\therefore\ \frac{1}{\sigma} = \frac{Xd^3x + Yd^3y + Zd^3z}{X^2 + Y^2 + Z^2}.$$

If there be no tortuosity, or the curve be plane, $Xd^3x + Yd^3y + Zd^3z = 0$ at every point of the curve, as in Art. 627.

Geometrical Interpretations.

641. Saint Venant observes* that, if we take three consecutive points P, Q, R for which $\xi = x$, $x + dx$, $x + 2dx + d^2x$ respectively, the projections of PQ, QR upon the axis of x will be dx, $dx + d^2x$; and if the parallelogram $PQRM$ be completed, by projecting the sides of the triangle PQM in order, since $PM = QR$, we shall have $dx +$ projection of $QM - (dx + d^2x) = 0$; therefore d^2x is the projection of QM.

642. In the general case, if a figure be drawn in which dx, d^2x, dy, d^2y, are all positive, the projection of twice the triangle PQM on the plane of xy will be easily seen to be

$$dxd^2y + dydx - dy\,(dx + d^2x) = dxd^2y - dyd^2x.$$

Again, if Mm be drawn perpendicular to PQ, $PQ = ds$, and ultimately $mQ = QR - PQ = d^2s$;

$$\therefore\ Mm^2 = QM^2 - Qm^2 = (d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2,$$

$$\text{and } \frac{1}{\rho^2} = \text{limit of } \left(\frac{Mm}{PQ^2}\right)^2 = \frac{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2}{(ds)^4}.$$

If we make s the independent variable, this implies that $QR = PQ$, in which case QM will bisect the angle PQR and be ultimately in the direction of the principal normal, the direction-cosines of which will be as $d^2x : d^2y : d^2z$.

The radius of the circle circumscribing the triangle PQR will be PQ^2 / QM; hence, if ρ be the radius of curvature, $\rho^{-2} = x''^2 + y''^2 + z''^2$.

643. *Osculating plane, binormal and curvature of the helix.*

In the case of the helix, Art. 597,

$$\begin{aligned} dx &= -a\sin\theta\,d\theta, & d^2x &= -a\cos\theta\,(d\theta)^2 \\ dy &= a\cos\theta\,d\theta, & d^2y &= -a\sin\theta\,(d\theta)^2, \\ dz &= na\,d\theta, & d^2z &= 0; \end{aligned}$$

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* *Journal de l' Ecole Pol.*, vol. 18.

$$dy\,d^2z - dz\,d^2y = \quad na^2 \sin\theta\,(d\theta)^3,$$
$$dz\,d^2x - dx\,d^2z = -\,na^2 \cos\theta\,(d\theta)^3,$$
$$dx\,d^2y - dy\,d^2x = \quad a^2\,(d\theta)^3;$$

hence the equation of the osculating plane is

$$(\xi - x)(n\sin\theta) + (\eta - y)(-n\cos\theta) + \zeta - z = 0, \text{ or } n(\xi y - \eta x) + a(\zeta - z) = 0.$$

This plane contains the point $(0, 0, z)$, and therefore the radius of the cylinder which passes through the point (x, y, z), and this radius is the principal normal.

The direction cosines of the binormal will be $\sin\alpha\sin\theta$, $-\sin\alpha\cos\theta$, and $\cos\alpha$, if $\alpha = \tan^{-1}n$ be the pitch of the screw.

The measures of curvature and tortuosity are respectively $\cos^2\alpha/a$ and $\sin\alpha\cos\alpha/a$.

644. *To find the radius of curvature of a curve which is the intersection of two surfaces whose equations are given; and to express it in terms of the radii of curvature of the normal sections of the two surfaces and the angle between them, the plane of each section containing the tangent to the curve.*

Employing the same notation as in Arts. 629, 630, and 640.

$$X = dy\,d^2z - dz\,d^2y = k^3(EdF - FdE) = k^3\{U\Gamma^2(\phi') - U'\Gamma^2(\phi)\},$$

and if ρ be the radius of curvature,

$$(ds)^6/\rho^2 = X^2 + Y^2 + Z^2 = k^6[(U^2 + V^2 + W^2)\{\Gamma^2(\phi')\}^2$$
$$-2(UU' + VV' + WW')\Gamma^2(\phi)\Gamma^2(\phi') + (U'^2 + V'^2 + W'^2)\{\Gamma^2(\phi)\}^2],$$
$$\text{and } (ds)^2 = k^2(D^2 + E^2 + F^2);$$
$$\therefore\ \frac{1}{\rho^2} = \frac{(U^2 + V^2 + W^2)\{\Gamma^2(\phi')\}^2 - \dots}{(D^2 + E^2 + F^2)^3}.$$

Let ω be the angle between the tangent planes to the surfaces at (x, y, z), and let $P^2 \equiv U^2 + V^2 + W^2$;

$$\therefore\ UU' + VV' + WW' \equiv PP'\cos\omega, \text{ and } D^2 + E^2 + F^2 \equiv P^2P'^2\sin^2\omega;$$
$$\therefore\ \frac{1}{\rho^2} = \frac{P^2\{\Gamma^2(\phi')\}^2 - 2PP'\cos\omega\,.\,\Gamma^2(\phi')\,.\,\Gamma^2(\phi) + P'^2\{\Gamma^2(\phi)\}^2}{P^2P'^2\sin^2\omega\,(D^2 + E^2 + F^2)^2}. \qquad (1)$$

As an example of the use of the preceding formulæ, we shall obtain the radii of curvature r, r' of the normal sections by replacing the equation of the surface $\phi' = 0$ by the equation $\phi'_1 = lx + my + nz - p = 0$ of a normal plane; in which case, if D_1, E_1, F_1, Γ_1 be the corresponding values of D, E, F, Γ, $\Gamma_1^2(\phi'_1) = 0$, and, since the normal plane contains the tangent to the curve,

$$D_1 : E_1 : F_1 = dx : dy : dz = D : E : F;$$

hence, since $\omega = \frac{1}{2}\pi$, we obtain from (1)

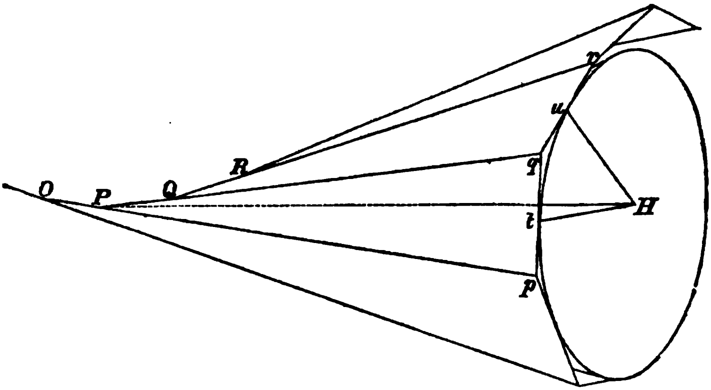
$$\frac{1}{r^2} = \frac{\{\Gamma^2(\phi)\}^2}{P^2(D_1^2 + E_1^2 + F_1^2)^2} = \frac{\{\Gamma^2(\phi)\}^2}{P^2(D^2 + E^2 + F^2)^2},$$
$$\text{whence } \frac{1}{\rho^2} = \frac{1}{\sin^2\omega}\left(\frac{1}{r^2} + \frac{1}{r'^2} - \frac{2\cos\omega}{rr'}\right),$$

a result which will be obtained in the next chapter by Meunier's theorem.

645. *To find the vertical angle of the osculating cone of a curve.*

Let pOo, qPp, rQq be three consecutive planes which become ultimately the osculating planes of a curve $OPQR$: these planes intersect in P.

Take P as the vertex of a circular cone which touches each of the planes, this cone is, in the limit, the *osculating cone* of the curve at P. Let PH be its axis, op, pq, qr the sections of the three planes made by a plane perpendicular to the axis, and t, u the points of contact with pq, qr.



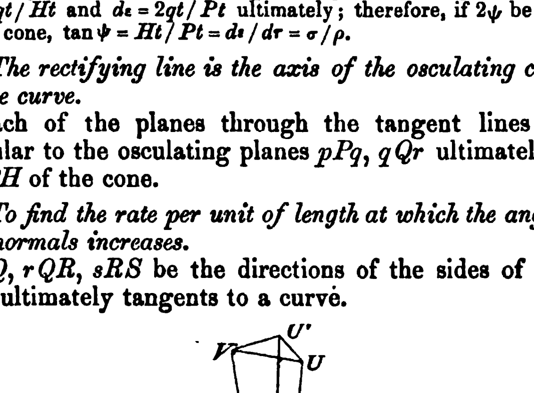
Draw tH, uH perpendicular to the planes pPq, qQr; then the angle tHu will be the angle of torsion, and pPq the angle of contingence, and we shall have $d\tau = 2qt / Ht$ and $d\epsilon = 2qt / Pt$ ultimately; therefore, if 2ψ be the vertical angle of the cone, $\tan\psi = Ht / Pt = d\epsilon / d\tau = \sigma / \rho$.

646. *The rectifying line is the axis of the osculating cone at any point of the curve.*

For each of the planes through the tangent lines PQ, QR perpendicular to the osculating planes pPq, qQr ultimately contains the axis PH of the cone.

647. *To find the rate per unit of length at which the angle between principal normals increases.*

Let PQ, rQR, sRS be the directions of the sides of a polygon which are ultimately tangents to a curve.



In the planes PQr, rRs respectively draw QU, QV perpendicular to rQR, sRS, these are ultimately in the direction of consecutive principal normals.

Draw QU' in the plane rRs, perpendicular to RQr, so that

UQU' is the angle between the consecutive osculating planes $= d\tau$, and $U'QV$ is the angle between consecutive tangents $= d\epsilon$.

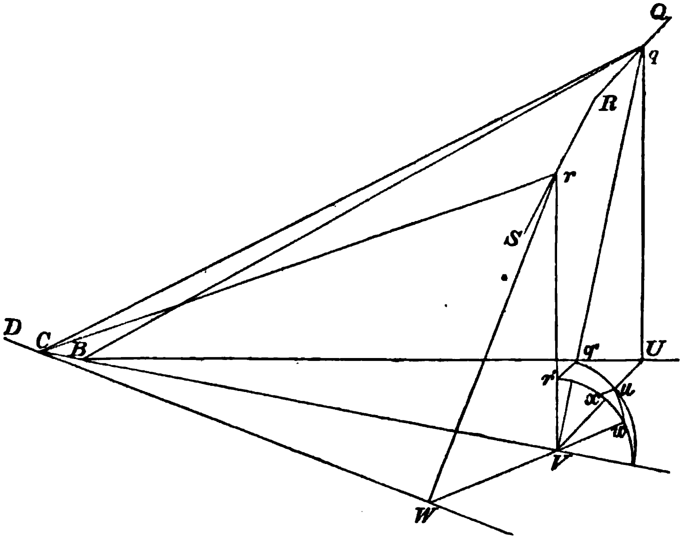
Let $d\chi$ be the angle between the lines QU, QV, and let these lines and QU' meet a sphere of radius unity, whose centre is in Q, in U, V, U', the angle $VU'U$ is a right angle; therefore, ultimately,

$$VU^2 = VU'^2 + U'U^2, \text{ or } (d\chi)^2 = (d\tau)^2 + (d\epsilon)^2,$$

$d\chi/ds$ the rate per unit of length of the increase of the angle between principal normals is the reciprocal of what we shall call the radius of *screw*; if κ be this radius, $\kappa^{-2} = \sigma^{-2} + \rho^{-2}$.

It may be noticed that $d\tau = d\chi \cos\psi$ and $d\epsilon = d\chi \sin\psi$, which represents that the angular motion round the rectifying line, which is normal to the plane UQV, is the resultant of two angular motions round the tangent line and the binormal which produce the rectification.

648. *To find the angle of contingence, and the element of the arc, of the locus of the centres of curvature of a given curve.*



Let QR, RS be sides of an equilateral polygon which are ultimately tangents to the given curve; let BV be the intersection of planes perpendicular to QR, RS through q, r their middle points; BV is therefore ultimately a tangent to the edge of regression of the polar developable; let BU, CW be the tangents preceding and succeeding BV.

From q draw qU, qV perpendicular to BU and BV, join rV, which will be perpendicular to BV, and draw rW perpendicular to CW.

If WV be produced to w, UVw and UV will ultimately be the angle of contingence $d\epsilon'$ and element ds' required.

Let R be the radius of spherical curvature at q, and ϕ the angle which it makes with the polar line BU. Since circles can be circumscribed about the quadrilaterals $BVUq$, $CWVr$,

$$qVU = qBU = \phi, \text{ and } VBU = VqU = d\tau, \text{ ultimately,}$$
$$\text{also } rVw = rCW = rCV + VCW = \phi + d\phi + d\tau, \text{ ultimately.}$$

With V as centre and radius unity describe a sphere, and let q', r', u, w be the projections of q, r, U, W upon it, and draw ux perpendicular to $r'w$, then $q'r' = d\iota$, $xr' = uq' = \phi$, $ux = d\iota\cos\phi$, and $wu^2 = ux^2 + xw^2$ ultimately;

$$\therefore\ (d\iota')^2 = (d\iota\cos\phi)^2 + (d\phi + d\tau)^2.$$

By drawing a diameter through V to the circle $BVUq$, it is easily seen that $VU = R\sin VBU$, and therefore that $d\iota' = Rd\tau$.

649. *To find the radius of spherical curvature.*

If in the figure $BVUq$, UM be drawn perpendicular to QV, $VM = d\rho$ ultimately; $\therefore\ (Rd\tau)^2 = (\rho d\tau)^2 + (d\rho)^2$;

$$\therefore\ R^2 = \rho^2 + \sigma^2 (d\rho/ds)^2, \text{ also } Rd\tau\cos\phi = VM = d\rho;$$

therefore the distance between the centres of circular and spherical curvature $= R\cos\phi = d\rho/d\tau = \sigma\, d\rho/ds$.

650. *To find expressions for the radius of curvature of the edge of regression of the polar developable of a curve.*

These are readily obtained by a method suggested by Routh,* which can be explained by the last figure.

Considering the curve BCD, U and V are the feet of the perpendiculars from q on the tangents to the curve; and substituting the corresponding letters in the known plane formulæ $p + d^2p/d\psi^2$, $r\,dr/dp$, we obtain two expressions for the radius of curvature of the edge of regression, viz. $\rho + d^2\rho/d\tau^2$ and $R\,dR/d\rho$.

From the expression $dp/d\psi$ for the distance of the foot of the perpendicular from the point of contact, we obtain $R\cos\phi = d\rho/d\tau$, as in Art. 649.

It will repay the student to read the paper referred to.

651. *To expand the coordinates of any point of a curve in terms of the arc, when the axes of coordinates are the tangent, principal normal, and binormal at the point from which the arc is measured.*

Let Ox be the tangent at O, Oy the principal normal, Oz the binormal, the planes of yz, zx, xy being the normal, rectifying, and osculating planes, and let s be the distance of a point (x, y, z) from O, measured along the arc.

Then, at the origin, $x' = 1$, $y' = 0$, $z' = 0$; $\rho x'' = 0$, $\rho y'' = 1$, $\rho z'' = 0$, these quantities being the direction cosines of the tangent and principal normal. If α be the angle made by the tangent at (x, y, z) with the tangent Ox, $x' = \cos\alpha$;

$$\therefore\ x'' = -\alpha'^2\cos\alpha - \alpha''\sin\alpha = -\rho^{-2};$$

since at O, $d\alpha$ is the angle of contingence, and $\alpha = 0$.

Again, if β be the angle made by the principal normal at (x, y, z) with Oy, $\rho y'' = \cos\beta$, $\therefore\ \rho' y'' + \rho y''' = -\beta'\sin\beta$; therefore, at O, $\rho'/\rho + \rho y''' = 0$; and if γ be the angle made by the binormal at (x, y, z) with the principal normal at the origin,

$$\rho\,(z'x'' - x'z'') = \cos\gamma;$$
$$\therefore\ \rho'\,(z'x'' - x'z'') + \rho\,(z'x''' - x'z''') = -\gamma'\sin\gamma;$$

* *Quarterly Journal*, vol. VII. p. 42.

hence, at O, $\rho z''' = \gamma' \sin\gamma = \sigma^{-1}$, since $d\gamma$ is the angle of torsion; therefore, by Maclaurin's theorem,

$$x = s - \frac{s^3}{6\rho^2}, \quad y = \frac{s^2}{2\rho} - \frac{s^3}{6\rho^2}\frac{d\rho}{ds}, \quad z = \frac{s^3}{6\rho\sigma}.$$

652. *To find the angle between two consecutive principal normals.*

The direction cosines of the principal normal at (x, y, z), a point near the origin, are as $-s/\rho : 1 : s/\sigma$, and the secant of the angle between this normal and that at the origin is $\sqrt{\{1 + s^2(\rho^{-2} + \sigma^{-2})\}}$; therefore, the angle required is ultimately $s\sqrt{(\rho^{-2} + \sigma^{-2})}$; $\therefore\ \kappa^{-2} = \rho^{-2} + \sigma^{-2}$, where κ is the radius of screw, as in Art. 647.

653. *To find the shortest distance between two consecutive principal normals and its position.*

The equations of the principal normal at (x, y, z) are approximately

$$-\rho(\xi - x) = s(\eta - y) = \sigma(\zeta - z),$$

and its shortest distance from the principal normal at the origin is the perpendicular distance from the origin of its projection upon the plane of zx, the equation of which is $\rho(\xi - s) + \sigma\zeta = 0$, hence the shortest distance is

$$\rho s / \sqrt{(\rho^2 + \sigma^2)} = \kappa s/\sigma.$$

If the line on which the shortest distance lies meet the axis of y at a distance r from the origin, the equations of the line will be $\xi/\rho = \zeta/\sigma$ and $\eta = r$, and this line will intersect the line $-\rho(\xi - s) = s\eta = \sigma\zeta$;

$$\therefore\ (\rho - r)/r = \rho^2/\sigma^2; \quad \therefore\ \frac{\rho - r}{\rho^2} = \frac{r}{\sigma^2} = \frac{\rho}{\rho^2 + \sigma^2}.$$

Hence the shortest distance divides the radius of curvature into two segments which are in the duplicate ratio of the radii of curvature and torsion.

654. *To find the angle between the rectifying line and the tangent at any point.*

The tangent plane to the rectifying developable at any point contains the tangent and binormal, and its normal is the principal normal whose direction cosines are in the ratio $-s/\rho : 1 : s/\sigma$, retaining only the principal terms.

Therefore the rectifying line is the ultimate intersection of the planes $\eta = 0$, and $s\rho^{-1}(\xi - x) + \eta - y + s\sigma^{-1}(\zeta - z) = 0$. Hence the tangent of the angle made with the tangent to the curve at O is σ/ρ.

655. *To find the element of the arc of the locus of the centres of curvature.*

If (ξ, η, ζ) be the coordinates of the centre of curvature at (x, y, z), neglecting s^2, $-(\xi - x)\rho/s = \eta - y = (\zeta - z)\sigma/s = \rho + \rho' s$, and the element of the arc is ultimately $\sqrt{\{\xi^2 + (\eta - \rho)^2 + \zeta^2\}} = s\sqrt{\{\rho'^2 + (\rho/\sigma)^2\}}$. The direction cosines of the tangent to the locus are as $\xi : \eta - \rho : \zeta = 0 : \rho' : \rho/\sigma$.

656. *To find the axis and pitch of the helix which has the same curvature and tortuosity as a curve at a given point.*

Let a be the radius of the cylinder and α the pitch, then, referred to the tangent, principal normal, and binormal, the coordinates corresponding to an arc s are, by transformation of coordinates, $x = a\cos\alpha\sin(s\cos\alpha/a) + s\sin^2\alpha$, $y = a - a\cos(s\cos\alpha/a)$, $z = -a\sin\alpha\sin(s\cos\alpha/a) + s\sin\alpha\cos\alpha$, and, equating these coordinates to those of the curve as far as s^3,

$$s - s^3\cos^4\alpha/6a^2 = s - s^3/6\rho^2, \quad s^2\cos^2\alpha/2a = s^2/2\rho - s^3\rho'/6\rho^2,$$

$$s^3\cos^3\alpha\sin\alpha/6a^2 = s^3/6\rho\sigma; \quad \therefore\ a = \rho\cos^2\alpha = \sigma\sin\alpha\cos\alpha;$$

$$\therefore\ \cos\alpha/\sigma = \sin\alpha/\rho = (\rho^2 + \sigma^2)^{-\frac{1}{2}} \text{ and } a = \rho\sigma^2/(\rho^2 + \sigma^2);$$

hence the axis, whose inclination to the binormal is $\alpha \tan^{-1}(\rho/\sigma)$ lies along the shortest distance between consecutive principal normals.

Also, if along a curve and the osculating helix equal small arcs be measured from the point of contact and on the same side of it, the distance between the ends of these arcs will be ultimately $s^3\rho'/6\sigma^2$.

657. DEF. The line of greatest slope on a surface is the curve which at every point is inclined at a greater angle to a given plane than any other line drawn through that point on the surface.

If the given plane be horizontal, the bed of a shallow brook on a hill side will be a line of greatest slope which the water will have selected for its course.

658. *To find the equations of the line of greatest slope on a given surface.*

Let $F(\xi, \eta, \zeta) = 0$ be the equation of the surface, l, m, n the direction cosines of the given plane.

The equation of the tangent plane at any point (x, y, z) is

$$U(\xi - x) + V(\eta - y) + W(\zeta - z) = 0.$$

The direction cosines of the line in which this plane cuts the given plane are proportional to $mW - nV$, $nU - lW$, $lV - mU$. Of all the tangent lines drawn through (x, y, z), that line which has the greatest inclination to the given plane is perpendicular to the line of intersection.

Hence the differential equation

$$(mW - nV)\,dx + (nU - lW)\,dy + (lV - mU)\,dz = 0,$$

with the equation of the surface, determines all the lines of greatest slope which can be drawn on the surface, the constant introduced in the integration being determined for any particular curve by some point through which it passes.

If the given plane be the plane xy, since $l = 0$, $m = 0$, the equations of the line of greatest slope will be

$$Vdx - Udy = 0 \text{ and } F(x, y, z) = 0,$$

perpendicular to the lines of level $Udx + Vdy = 0$.

659. *To find the lines of greatest slope on the cone $ax^2 + by^2 = cz^2$, the plane xy being the plane of reference.*

The differential equation is, in this case, $bydx - axdy = 0$;

$$\therefore\ b\log x - a\log y = \log C, \text{ or } x^b = Cy^a.$$

The lines of greatest slope are the intersection of the cone with the cylinders represented by this equation; and it may be observed that no generating line, except those in the principal planes, is a line of greatest slope, unless the cone be a right cone.

Four-point System.

660. *Equation of a curve of double curvature.*

If p, q, r, s be the four-point coordinates of any plane, the equation of a point is $Lp + Mq + Nr + Rs = 0$ (1), and we have seen, Art. 131, that if L, M, N, R involve one variable t in the first degree, the locus of all the points which can be obtained by giving to t values from $-\infty$ to $+\infty$ is a straight line.

Let L, M, N, R be any functions of a single variable t, we can show that the locus of points corresponding to all values of the variable is a curve line; for, if the locus be cut by any plane, and the coordinates of the plane be substituted for p, q, r, s in the equation of the point, the resulting equation will determine a series of values of the variable t, which will correspond to the points in which the locus is intersected by the plane; and, by shifting the plane, we obtain a continuous series of such points which form the different portions of the curve line, of which (1) may therefore be considered to be the equation.

If L, M, N, R be rational and integral functions of t, not having a common factor, one at least being of the n^{th} degree, any plane will determine n values of t, real or imaginary, and therefore meet the curve in n points, hence the curve will be of the n^{th} degree.

We may observe that, in order to be sure that the curve is of the n^{th} degree, it must not be possible to make any substitution of a new variable so as to diminish the degree, while the functions remain rational.

661. If the curve which is the locus of $Mq + Nr + Rs = 0$ be traced on the fundamental plane BCD, every point in the curve which is the locus of (1) will lie on a line joining A with a point of this curve, that is, on the surface of a cone whose vertex is A and guiding curve $Mq + Nr + Rs = 0$.

662. If $L = a_0 + a_1t + a_2t^2 + \ldots$, $M = b_0 + b_1t + b_2t^2 + \ldots$, ...,

and $a_0p + b_0q + c_0r + d_0s = a'p'$, $a_1p + b_1q + c_1r + d_1s = b'q'$, ...,

the equation of the curve may be written $a'p' + b'tq' + c't^2r' + d't^3s' + \ldots = 0$.

This reduced equation shews that a curve of the first degree is the straight line joining the points $p' = 0$, $q' = 0$.

Also that a curve of the second degree is a plane curve, the plane containing the three points $p' = 0$, $q' = 0$, $r' = 0$.

Again, by the preceding article, a curve of the third degree, which is not necessarily a plane curve, lies on two cones whose vertices are A' and D', and whose guiding curves are the conics traced on $B'C'D'$ and $A'B'C'$, whose equations are $b'q' + c'tr' + d't^2s' = 0$ and $a'p' + b'tq' + c't^2r' = 0$, which have a common generating line $A'D'$.

663. *To find the equation of the tangent to a curve.*

Let $f(t) \equiv Lp + Mq + Nr + Rs = 0$ be the equation of the curve, and let t_0 determine any point P in the curve, $t_0 + \tau$ a point Q adjacent to it, whose equation will be $f(t_0 + \tau) \equiv f(t_0) + \tau f'(t_0) + \frac{1}{2}\tau^2 f''(t_0) + \ldots = 0$.

The straight line whose equation is $f(t_0) + \mu f'(t_0) = 0$ is the equation of the tangent at P, since, when Q moves up to P and ultimately coincides with it, the straight line ultimately passes through Q.

The distance between adjacent points in the tangent and curve is of the order τ^2, generally, Art. 123.

If $f''(t_0) = 0$, the distance is of the order τ^3, and the curve, which in ordinary cases lies on the same side of the tangent on each side of the point of contact, in this case lies on opposite sides, or there is a point of inflexion in the osculating plane.

The equation $f(t_0) + \mu f'(t_0) + \frac{1}{2}\mu^2 f''(t_0) = 0$ is the equation of a conic which has a contact of the second order.

A double point occurs when L, M, N, R retain the same ratio for two values of t.

664. *To find the equation of the osculating plane at any point of a curve.*

The plane whose equation is $f(t_0) + \mu f'(t_0) + \nu f''(t_0) = 0$ is the plane which passes through the points $f(t_0) = 0$, $f'(t_0) = 0$, and $f''(t_0) = 0$, and therefore coincides with the limiting position of the plane which passes through three contiguous points of the curve; the equation is therefore the equation required

The distance of a point $f(t_0 + \tau) = 0$ from the osculating plane is ultimately $c\tau^3 f'''(t_0)$, or the curve generally lies on opposite sides of the osculating plane in passing through the point of contact.

Singularities of Curves and Torses.

665. A tortuous curve and the torse of which it is the edge of regression may be considered in connexion with one another; and we may expect that singularities in one will have corresponding singularities in the other. Cayley, in *Liouville's Journal*, tom. x., and in the *Cambridge and Dublin Mathematical Journal*, vol. v., and Salmon, in the same place in the latter journal, have investigated equations among the number of such singularities; and Salmon has proceeded to shew how curves of double curvature may be classified by the consideration of the number of apparent double points in the curves.

We shall introduce the student to some of the methods employed, and leave him to consult the papers referred to, if he desire to enter more fully into the subject.

666. A tortuous curve may be considered as the locus of a system of points, or as the envelope of a system of straight lines, and the corresponding torse as the locus of the system of lines, or as the envelope of a system of planes.

We may consider the three systems of points, lines, and planes as connected in the following ways.

i. If the system of points be supposed given, each line of the second system joins two consecutive points of the given system, and each plane contains three consecutive points of the same system.

ii. If the system of planes be supposed given, each line of the second system is the intersection of two consecutive planes of the given system, and each point of the first system is the intersection of three consecutive planes of the given system.

iii. If the system of lines be supposed given, each point of the first system is the intersection of two consecutive lines of the given system, and each plane contains two consecutive lines of the same system.

667. The following terms will be employed:

A line through two points denotes a line joining *any* two arbitrary points of the system of points.

A line in two planes denotes the line of intersection of *any* two planes of the system of planes.

A point in two lines is the intersection of *any* two lines of the system of lines which intersect.

A plane through two lines is the plane containing *any* two lines which intersect.

A stationary plane is a singular plane which contains four consecutive points, or in three consecutive lines, and occurs when two consecutive planes coincide.

A stationary point is a singular point which lies in four consecutive planes, or in three consecutive lines, and occurs when two consecutive points coincide.

Any plane not belonging to the system contains a certain number of *lines in two planes*.

Any point not belonging to the system lies in a certain number of *lines through two points*.

Any plane contains a certain number of *points in two lines*.

Any point lies in a certain number of *planes through two lines*.

These four numbers will be denoted by l, λ, p, and ϖ, respectively, and the numbers of stationary planes and points by s and σ.

λ is the number of *apparent double points*, since a line drawn from the eye in any position to the points where the curve seems to cross itself is a line through two points.

668. The *degree* of a curve is the number of points in which it intersects an arbitrary plane.

The *class* of a curve is the number of osculating planes which contain an arbitrary point.

The *degree* of the torse of which the curve is the edge of regression is the number of points in which it meets an arbitrary straight line.

The *class* of the torse is the number of tangent planes which can be drawn through an arbitrary point.

The *rank* of the system is the number of lines of the system which intersect an arbitrary straight line.

Hence the rank of the system is the same as the degree of the surface, and the class of the curve is the same as the class of the surface.

669. Singularities in tortuous curves are by these considerations made to depend upon the singularities of plane curves.

Let a given plane intersect the torse, the plane curve is thus connected with the lines and planes of the system as follows.

Every point of the plane curve is in a line of the system, every tangent to the plane curve is the intersection of the cutting plane with a plane of the system.

A straight line in the cutting plane meets the torse in r points, if r be the degree of the torse; therefore the degree of the curve of intersection is r.

A point in the cutting plane lies in n tangent planes to the torse if n be the class of the torse, hence n tangent lines to the curve of intersection can be drawn through the point; therefore the class of the plane curve is n.

When the cutting plane has a *point in two lines* the plane curve has a double point, since the curve of intersection may be supposed generated by the intersection of lines of the system with the cutting

plane, and the generating line in this twice passes through the same point.

When the cutting plane has a *line in two planes*, it may be seen similarly that the plane curve has a double tangent, since it is the envelope of the intersection of planes of the system with the cutting plane.

At the points in which the cutting plane meets the tortuous curve, which is the edge of regression of the torse, two lines of the system meet the plane curve in the same point, and the plane containing the two lines intersects the cutting plane in a tangent to the curve, this point is therefore a cusp in the plane curve.

For every stationary plane, two consecutive planes coincide, and therefore two tangents to the plane curve coincide, or there is a point of inflexion.

If m be the degree of the curve of double curvature, the plane curve is of the degree r, and of the class n; it has p double points, l double tangents, m cusps, and s points of inflexion.

Hence the formulæ for plane curves give three independent equations among these numbers.

These formulæ are given by Salmon* in the following forms:

If μ be the degree of a plane curve, ν its class, δ the number of double points, κ of cusps, τ of double tangents, ι of points of inflexion,

$$\nu = \mu(\mu-1) - 2\delta - 3\kappa, \quad \iota - \kappa = 3(\nu - \mu),$$
$$\text{and } 2(\tau - \delta) = (\nu - \mu)(\nu + \mu - 9);$$
$$\text{whence } n = r(r-1) - 2p - 3m, \quad s - m = 3(n - r),$$
$$\text{and } 2(l - p) = (n - r)(n + r - 9).$$

670. If, instead of considering the system in connexion with a plane, which intersects the developable surface in a curve, we consider it in connexion with a point, which is made the vertex of a conical surface whose guiding curve is the tortuous curve, we shall obtain other relations among the number of singularities, which can be connected with those of a plane section of the conical surface.

Every plane through the vertex cuts the curve in m points, corresponding to which are m generating lines of the cone; also a plane which cuts the cone meets the curve in m points; therefore the line of intersection of the two planes contains m points on the curve, the plane section of the cone is therefore of the m^{th} degree.

Again, r tangent lines meet the straight line joining any point in the cutting plane with the vertex of the cone, and hence r tangent planes to the cone can be drawn through the point in the cutting plane; therefore there are r tangents to the plane section, which can be drawn through the point, that is, the class of the plane section is r.

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* *Higher Plane Curves*, Art. 82, 3rd edit.

The vertex lies in λ *lines through two points*, hence there are λ double generating lines of the cone, or λ double points on the plane section.

The vertex lies in ϖ *planes through two lines*, hence there are ϖ double tangent planes to the cone, and therefore ϖ double tangent lines to the plane section.

Each of the n planes of the system which pass through the vertex contains three consecutive points on the edge of regression, and therefore corresponds to three generating lines, hence there are n points of inflexion of the plane section.

For every stationary point, two consecutive points coincide; therefore the cone has a cuspidal edge, and therefore the number of cusps of the plane section is σ.

Hence, the plane section of the cone is of the degree m and of the class r; and it has λ double points, ϖ double tangents, σ cusps, and n points of inflexion.

Thus, three more independent equations are obtained,

$$r = m(m-1) - 2\lambda - 3\sigma, \quad n - \sigma = 3(r - m),$$
$$\text{and } 2(\varpi - \lambda) = (r - m)(r + m - 9).$$

671. As an exercise, the values of μ, ν, &c. should be obtained: i. when the plane section employed in Art. 669 is one of the planes in the system, and ii. when the vertex of the cone of Art. 670 is a point of the system.

i. The plane section becomes two straight lines and a curve whose degree is $r-2$, and class $n-1$, and the number of points of inflexion s, whence, by the formulæ of Art. 669, $\delta = p - 2r + 3$, $\kappa = m - 3$, $\tau = l - n + 2$, which numbers should be also accounted for separately, for example, by considering that every plane of the system passes through three consecutive points of the system, and that every point where the curve part of the section is met by the straight part counts for two double points.

ii. For the section of the cone $\mu = m - 1$, $\nu = r - 2$, $\iota = n - 3$, $\kappa = 6$, $\delta = \lambda - m + 2$, $\tau = \varpi - 2r + 8$.

672. If a cone be described whose guiding curve is a given curve of double curvature, λ *lines through two points* pass through the vertex and determine a double side of the cone.

The two points through which any *line through two points* passes may be either distinct or coincident, as in the case of a multiple point of the curve; to an eye placed at the vertex of the cone two different branches will in both cases *appear* to intersect, but will actually intersect only in the latter case; and in the case of intersection the actual intersection will take place for all positions of the vertex. The sum of the *apparent* and *actual* double points is λ.

Salmon has employed the number of these double points to construct a classification of the curves which are the complete or partial intersections of two surfaces of given degrees; in which the distinctions are made according to the number of points in which the surfaces touch, and the nature of the contact where they do touch.

XLIV.

(1) The equations of the tangent to the curve of intersection of the surfaces $ax^2 + by^2 + cz^2 = 1$ and $bx^2 + cy^2 + az^2 = 1$ are

$$\frac{x(\xi - x)}{ab - c^2} = \frac{y(\eta - y)}{bc - a^2} = \frac{z(\zeta - z)}{ac - b^2}.$$

Shew that the tangent line at the point $x = y = z$ lies in the plane

$$(a - b)x + (b - c)y + (c - a)z = 0.$$

(2) The equations of a sphere and cylinder being $x^2 + y^2 + z^2 = 4a^2$ and $x^2 + z^2 = 2ax$, prove that the equations of the tangent to the curve of intersection at the point (f, g, h) are $(f - a)x + hz = af$ and $gy + ax = a(4a - f)$, and that the equation of the normal plane is $x/f - y/g = (1 - a/f)(z/h - y/g)$.

(3) Shew that the equation of the normal plane at any point (f, g, h) on the curve defined by the equations $x^2/a + y^2/b + z^2/c = 1$, $x^2 + y^2 + z^2 = d^2$ is

$$xa(b - c)/f + yb(c - a)/g + zc(a - b)/h = 0.$$

(4) A curve is traced on a right cone so as always to cut the generating line at the same angle; shew that its projection on the plane of the base is an equiangular spiral.

(5) Prove that the curve $x^2/a^2 + y^2/b^2 = 1$, $y/x = \tan(z/c)$ is unicursal, and shew that the cosine of the angle between the osculating planes at two points in which the curve meets the principal planes of the cylinder is

$$ab/\sqrt{\{(a^2 + c^2)(b^2 + c^2)\}}.$$

(6) P, Q are two near points on a curve, QM is perpendicular to the tangent at P, QN to the osculating plane at P; shew that ultimately $QN : PQ - PM ::$ radius of curvature at P : radius of torsion. Shew also that the shortest distance between the tangents at P and Q is ultimately $\frac{1}{2}QN$.

(7) If a curve in space be defined by the equations $x = 2a\cos t$, $y = 2a\sin t$, $z = bt^2$, prove that the radius of circular curvature will be

$$2a^{-1}(a^2 + b^2t^2)^{\frac{3}{2}}(a^2 + b^2 + b^2t^2)^{-\frac{1}{2}}.$$

(8) When the radius of curvature is a maximum or minimum, the tangent to the locus of the centres of curvature is perpendicular to the radius of curvature.

(9) If ρ be the radius of curvature of a curve, then that of its projection on a plane inclined at an angle α to the osculating plane is $\rho\sec\alpha$ if the plane be parallel to the tangent, and $\rho\cos^2\alpha$ if it be parallel to the principal normal.

XLV.

(1) Shew that the intersection of the surfaces $z(x + z)(x - a) = a^3$ and $z(y + z)(y - a) = a^3$ consists of plane curves, and find the equations of the normal planes at any point in each of the curves.

(2) Prove that if every pair of consecutive principal normals to a curve intersect, the curve must be plane; and find $f(\theta)$ so that the curve, whose coordinates are given by $x = a\cos\theta$, $y = b\sin\theta$, $z = f(\theta)$, may be plane.

(3) If a string be unwound from a helix so that the straight portion is a tangent to it, shew that any point on the string will describe the involute of a circle.

(4) Prove that the locus of the centres of curvature of a helix is a similar helix; and find the condition that it shall be traced on the same cylinder.

(5) Prove that the equation of the polar developable of the helix is

$$x \cos\theta + y \sin\theta + a \tan^2\alpha = 0,$$

$$\text{where } x^2 + y^2 = \tan^2\alpha \{a^2 \tan^2\alpha + (z - a\theta \tan\alpha)^2\},$$

and that its edge of regression is a helix on a cylinder whose radius is $a \tan^2\alpha$.

(6) Prove that the angle between the shortest distance of tangents at two consecutive points and the binormal at one of them is equal to half the corresponding angle of torsion.

(7) When the polar surface of a curve is developed into a plane, prove that the curve itself degenerates into a point on the plane; and if r, p be the radius vector and perpendicular on the tangent to the developed edge of regression of the polar surface drawn from this point, prove that, ρ, σ and s referring to the points on the original curve, $\sqrt{(r^2 - p^2)} = \sigma\, d\rho / ds$.

(8) Find the equation of the projection on the plane of xy of the lines of greatest slope on the surface $z = x + \frac{1}{2}c \log(x^2 + y^2)/a^2$, the plane of xy being horizontal.

XLVI.

(1) Prove that the ratio of the radius of curvature of a curve traced on a torse to that of the curve which it becomes when the torse is developed into a plane is equal to the cosine of the angle between the tangent plane to the torse and the osculating plane of the curve.

(2) The paraboloid whose equation is $ax^2 + by^2 = 4z$ has traced upon it a curve, every point of which is the extremity of the latus rectum of the parabolic section through the axis of z; shew that the tangent to the curve traces upon the plane of xy the curves whose equations are $r \sin 2\theta = \pm 2 (a - b)^{-1}$.

(3) A curve is traced on a sphere of radius a, and cuts all the meridians at an angle β, shew that the radius of curvature at the point whose co-latitude and longitude are θ, ϕ is $a \sin\theta\, (\sin^2\theta + \sin^2\beta \cos^2\theta)^{-\frac{1}{2}}$; test by the cases $\beta = 0$ and $\beta = \frac{1}{2}\pi$.

(4) Prove that a system of helices having the same pitch can be intersected orthogonally by another system of helices.

(5) A point moves on an ellipsoid $ax^2 + by^2 + cz^2 = 1$ so that its direction of motion always passes through the perpendicular from the centre of the ellipsoid on the tangent plane at the point; shew that the curve traced out by the point is given by the intersection of the ellipsoid with the surface $x^{b-c}y^{c-a}z^{a-b} = \text{const.}$

(6) Shew that the shortest distance of tangents at the extremities of a small arc δs of a helix, whose pitch is α and radius of cylinder a, is $\dfrac{\sin\alpha \cos^3\alpha}{12a^2} (\delta s)^3$.

(7) If a tortuous curve be projected on a plane, the normal to which is inclined at angles α, β to the tangent and binormal at any point, the curvature of the projection will be to that of the curve as $\cos\gamma : \sin^3\alpha$.

(8) If ρ^{-1}, σ^{-1} be the measures of curvature and tortuosity at any point of a curve in space; ρ_1, σ_1 similar quantities at the corresponding point of the locus of the centre of spherical curvature; then $\sigma\sigma_1 = \rho\rho_1$.

(9) If the measures of curvature and tortuosity of a curve be constant at every point, the curve will be a helix traced on a cylinder.

(10) Prove that the angle between the radius of the osculating sphere and the edge of regression of the polar surface is equal to the angle between the radius of the osculating circle and the locus of the centre of curvature.

XLVII.

(1) If the osculating plane at every point of a curve pass through a fixed point, prove that the curve will be plane.

Hence prove that the curve of intersection of the surfaces whose equations are $x^2 + y^2 + z^2 = a^2$ and $x^4 + y^4 + z^4 = \frac{1}{2}a^4$ are circles of radius a.

(2) A straight line is drawn on a plane, which is then wrapped on a cone; shew that the radius of curvature of the curve on the cone varies as the cube of the distance from the vertex.

(3) A curve is drawn on a right circular cylinder, so that its osculating plane at any point meets the normal section through the generator at that point in a line making a constant angle with the generator; prove that the curve will become a parabola when the cylinder is unwrapped.

(4) In any curve in which the difference between the radii of absolute and spherical curvature is constant, the arcs of the loci of the centres of absolute and spherical curvature, measured between corresponding points, are equal.

(5) A circle of radius a is traced on a rectangular piece of paper, which is then folded so as to become a cylinder of radius b, shew that, if ρ be the radius of curvature at any point of the curve which the circle becomes,

$$\rho^{-2} = a^{-2} + b^{-2} \cos^4(s/a,)$$

s being the distance of the point from a certain fixed point of the curve, measured along the arc.

(6) Prove the following equations of the lines of greatest slope on the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, placed in any position,

$$p\frac{dx}{ds} + a^2\frac{d\psi}{dx} = 0, \quad p\frac{dy}{ds} + b^2\frac{d\psi}{dy} = 0, \quad p\frac{dz}{ds} + c^2\frac{d\psi}{dz} = 0,$$

where ψ is the inclination of the normal to the vertical at the point (x, y, z) and p the perpendicular on the tangent plane at that point.

(7) A curve is given by $x = 4a\cos^3\theta$, $y = 4a\sin^3\theta$, $z = 3c\cos^2\theta$, prove that it has four cusps. Find the equation of the normal plane at any point, and shew that the locus of the centre of the sphere of curvature has also four cusps. Find the condition that the locus of the centre of the sphere of curvature may be similar to the original curve.

CHAPTER XXII.

CURVATURE OF SURFACES. NORMAL SECTIONS. INDICATRIX. DISTRIBUTION OF NORMALS. SURFACE OF CENTRES. INTEGRAL CURVATURE. LINES OF CURVATURE. UMBILICS.

673. In this chapter we shall examine the curvature of surfaces, and explain how the amount of curvature may be estimated.

If the student will consider the simpler surfaces with which he is familiar, such as a sphere, an ellipsoid, or a hyperboloid of one sheet, he will have examples of the kind of flexure which may take place at an ordinary point of any surface.

Any point of a sphere or a pole of a prolate or of an oblate spheroid is an example of a point of a surface at which the curvature is the same for all sections made by planes containing the normal at the point.

Any point of an ellipsoid is an example of a point on a surface at which, if a tangent plane be drawn, the surface in the neighbourhood of the point will lie entirely on one side of the tangent plane; such surfaces are called *synclastic*.

If a tangent plane be drawn at any point of an hyperboloid of one sheet, the surface will intersect the tangent plane, and bend from it partly on one side and partly on the other; such surfaces are called *anticlastic*.

674. Let two planes be drawn through the same tangent line at any point of a sphere, one containing the normal and the other inclined at an angle θ to the normal, the sections made by these planes will have their radii in the ratio $1 : \cos\theta$.

This simple relation between the radii of curvature of a normal and oblique section, containing the same tangent line, will be proved to be true for any surface at an ordinary point.

The student may for an exercise prove it when the tangent line is drawn through the extremity of a principal axis of an ellipsoid parallel to another principal axis.

675. Consider next the curvature of the sections of an ellipsoid whose centre is O, made by planes passing through OA, the normal at A an extremity of one of the principal axes; if AP be one of these sections intersecting the principal section BC, perpendicular to OA, in P, OP and OA will be its semi-axes, and its radius of curvature at $A = OP^2/OA$; also OB^2/OA and OC^2/OA

will be the radii of curvature at A of the principal sections AB and AC, and since OB, OP, OC are in order of magnitude for all positions of OP, we see that, of all the normal sections through A, the two sections which have their curvature a maximum and minimum have their planes perpendicular to each other.

This property of normal sections will be found to hold for any ordinary point of all surfaces.

676. If $POB = \theta$, $\cos^2\theta/OB^2 + \sin^2\theta/OC^2 = OP^{-2}$; hence, if ρ, ρ' be the radii of curvature of the sections AB, AC at A, and R that of the section AP, $\cos^2\theta/\rho + \sin^2\theta/\rho' = R^{-1}$.

This relation between the radii of curvature of the normal sections of least and greatest curvature, called principal sections, and that of any other normal section inclined at an angle θ to one of the principal sections, will also be found to hold for any surface.

677. These three properties which are true for all surfaces will enable us to determine the radii of curvature of all plane sections through any point, when those of any two sections, not containing the same tangent, are known.

Normal Sections.

678. *To find the relation between the radii of curvature of sections made by planes containing the normal at any point of a surface.*

Let the surface be referred to the normal at the point O as the axis of z, and the tangent plane at O as the plane of xy.

The values of p and q at the origin vanish,

$$\therefore\ z = \tfrac{1}{2}(rx^2 + 2sxy + ty^2) + \&c.$$

Let the surface be cut by a plane passing through Oz, and inclined at an angle θ to the plane of zx; at every point of this plane let $x = u\cos\theta$, $y = u\sin\theta$;

$$\therefore\ z = \tfrac{1}{2}(r\cos^2\theta + 2s\cos\theta\sin\theta + t\sin^2\theta)\,u^2\,(1 + \varepsilon),$$

where ε vanishes in the limit.

If R be the radius of curvature of this section,

$$R^{-1} = \text{limit}\ 2z/u^2 = r\cos^2\theta + 2s\cos\theta\sin\theta + t\sin^2\theta.$$

The directions of the normal sections of which the curvature is a maximum or minimum are given by the equation

$$-(r-t)\sin 2\theta + 2s\cos 2\theta = 0.$$

If α be one solution, the rest will be included in the formula $\frac{1}{2}n\pi + \alpha$, hence the sections of maximum and minimum curvature are at right angles.

These sections are called the *principal sections* of the surface at the point considered.

If the planes of the principal sections be taken for the planes of zx and yz, $\alpha = 0$, and therefore $s = 0$, and the expression for the

curvature of any section will become $R^{-1} = r\cos^2\theta + t\sin^2\theta$; let ρ, ρ' be the radii of curvature of the principal sections, then $\rho^{-1} = r$ and $\rho'^{-1} = t$, $\therefore R^{-1} = \cos^2\theta/\rho + \sin^2\theta/\rho'$; also if R, R' be the radii of curvature of any perpendicular normal sections,

$$R^{-1} + R'^{-1} = \rho^{-1} + \rho'^{-1}.$$

These theorems are due to Euler.

The Indicatrix.

679. Euler's theorems and other theorems relating to the curvature of plane sections of surfaces are deduced with great facility by means of a curve called the indicatrix, employed first by Dupin for this purpose.

DEF. The *indicatrix* at any point P of a surface is the section made by a plane parallel to the tangent plane at P and at an infinitely small distance from it.

In cases in which, as in anticlastic surfaces, the curve of intersection extends to any finite distance, the name of indicatrix only applies to the portions of the curve which are infinitely near to P.

680. *At any ordinary point of a surface the indicatrix is a conic.*

Taking the axes as in Art. 677, the equation of the surface is of the form $z = \frac{1}{2}(rx^2 + 2sxy + ty^2) + \&c.$, and by transformation of axes the term involving xy may be made to disappear, so that $2z = ax^2 + by^2 +$ terms of a higher order.

If the surface be cut by a plane parallel to the tangent plane and very near to it, for which $z = h$, in the neighbourhood of the point of contact $2h = ax^2 + by^2$, the indicatrix is therefore a conic whose centre is in the normal.

681. Pendlebury has noticed* that the indicatrix may be, at particular points of some surfaces, of any form, and the number of directions of principal curvature for such points may be any number, in fact, equal to the number of apses in the indicatrix. He gives as an example a surface $x^2 + y^2 = az\phi(y/x)$ generated by a parabola revolving round its axis, its latus rectum increasing or decreasing with the angle through which its plane has revolved; such surface would look like a paraboloid with ridges and furrows radiating from the vertex.

682. *The radius of curvature of a normal section of a surface varies as the square of the corresponding central radius of the indicatrix.*

Let CP be the central radius of the indicatrix which lies in the plane of any normal section whose radius of curvature at O is R; then $2R = \text{limit } CP^2/OC$ (see fig. page 283); therefore, since OC is constant for all normal sections, $R \propto CP^2$.

Hence all theorems in central conics which can be expressed by equations homogeneous in terms of the radii and axes, can be replaced by corresponding theorems in curvature. Euler's theorems

* *Messenger of Mathematics*, vol. I. p. 148.

follow at once; if R, R' be radii of curvature of normal sections inclined to a principal section of a surface at angles θ, θ', such that $\tan\theta\tan\theta' = -\rho'/\rho$, we obtain from known properties of conics the relations $R + R' = \rho + \rho'$ and $RR'\sin^2(\theta' - \theta) = \rho\rho'$.

683. When the indicatrix is an ellipse, the surface is synclastic at the point considered.

At a point of a surface for which the indicatrix is a circle the curvatures of all sections made by planes containing the normal are equal. Such points are *umbilics*.

When the form of the indicatrix is hyperbolic, the surface is anticlastic at the point considered; in this case the radii of curvature of normal sections containing the asymptotes are infinite, such sections pass through the inflexional tangents, and their directions are given by $\tan^2\theta = \rho'/\rho$, ρ, ρ' being the absolute values of the radii of curvature.

In order to deduce theorems from geometrical properties of the hyperbola, it may be necessary to suppose two indicatrices, one on each side of the tangent plane at equal distances from it.

If $\rho' = \rho$ and R be the radius of curvature of a normal section inclined at an angle θ to a principal section, $R\cos 2\theta = \rho$.

When the form is parabolic, the part of the section which is called the indicatrix is two parallel lines which become ultimately, as in the case of a developable surface, two coincident lines.

Such points are called *parabolic points*, sometimes also *cylindrical points*.

684. As an example of a parabolic point, take a point of the cone $x^2/a^2 + y^2/b^2 = z^2/c^2$, at a distance l from the vertex in the generator $x/a = z/c$; transform the axes so that the normal at this point is the axis of z, and the generator the axis of x, the resulting equation of the cone is

$$lz = y^2 ac/2b^2 - zx - z^2(c^2 - a^2)/2ac;$$

let $z = h$ and $a = c\tan\alpha$, then $y^2 = 2b^2h(x + l - h\cot 2\alpha)/ac$, the section by a plane parallel to the tangent plane is therefore a parabola, the distance of whose vertex from the normal at the point considered is $l - h\cot 2\alpha$, and since this remains finite, when h is made indefinitely small, the degeneration into two nearly coincident parallel lines in the neighbourhood of the point is explained. The finite principal radius of curvature is b^2l/ac.

685. *The intersection of two consecutive tangent planes and the line joining the points of contact are parallel to conjugate diameters of the indicatrix.*

Let CP, CD be conjugate semi-diameters of the indicatrix for the point O of a surface; since the tangent plane to the surface at P contains the tangent to the indicatrix at P, its intersection with the tangent plane at O is parallel to CD, and proceeding to the limit, when OC vanishes, the proposition follows.

DEF. Tangent lines at any point of a surface drawn parallel to conjugate diameters of the indicatrix, are called *conjugate tangents*.

COR. It follows from this property of consecutive tangent planes, that *if a torse envelope any surface the directions of the generating lines at any point of the curve of contact are conjugate to the tangents to the curve.*

686. *To find the relation between the radii of curvature of a normal and an oblique section of a surface made by planes passing through the same tangent line.*

Let the tangent line through which the planes are taken be the axis of x, and let θ be the inclination of the planes of the oblique and normal sections through Ox.

The equation of the indicatrix is of the form $2h = ax^2 + 2cxy + by^2$, and where the oblique section cuts the indicatrix $y = h \tan\theta$, therefore xy and y^2 vanish compared with x^2; hence the radius of curvature of the oblique section at O is the limit of $x^2/2h \sec\theta$, and if R, R' be the radii of curvature of the normal and oblique sections $R' = R \cos\theta$; this is Meunier's theorem.

687. Besant* gives the following elegant proof of Meunier's theorem: take a normal and an oblique section at any point of a surface, the two curves of section having the same tangent line, and therefore having two consecutive points in common. In each of the curves take a third consecutive point and describe a sphere through the four contiguous points; the sections of the sphere by the two planes are evidently the circles of curvature of the normal and oblique sections, and the theorem follows immediately.

688. *Radius of curvature of the curve of intersection of two surfaces.*

Let ρ be the radius of curvature of the curve at any point P, r, r' those of the normal sections of the surfaces made by planes containing the tangent at P; let ω be the angle between the planes, and ϕ, $\omega - \phi$ the angles between the osculating plane of the curve at P and the two normal planes.

Now the curvature of the curve is the same as that of the section of either surface by the osculating plane, since they have three consecutive points in common, and by Meunier's theorem,

$$r^{-1} = \rho^{-1}\cos\phi, \quad \text{and} \quad r'^{-1} = \rho^{-1}\cos(\omega - \phi) = r^{-1}\cos\omega + \rho^{-1}\sin\omega\sin\phi;$$

$$\therefore\ \rho^{-2}\sin^2\omega = r^{-2} - 2r^{-1}r'^{-1}\cos\omega + r'^{-2}, \text{ as in Art. 644.}$$

689. In order to see how a surface bends in different directions, starting from a given point, we ought to have a clear notion of the manner in which the normals at points adjacent to the given point are directed.

The indicatrix affords a satisfactory explanation of the mode of distribution.

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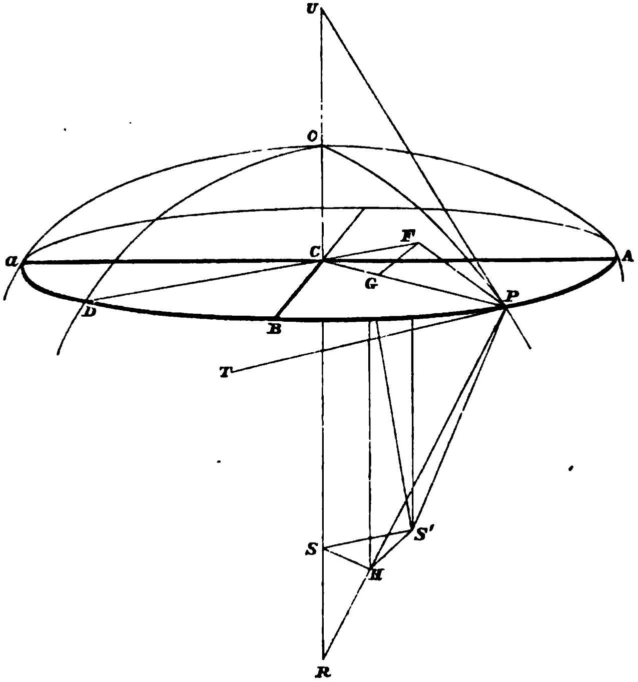
* *Quart. Jour. of Math.*, vol. VI. p. 140.



690. *Normals at consecutive points intersect, when taken along the directions of greatest and least curvature, and not generally when taken in any other direction.*

Let P be any point of the indicatrix for the point O, CD a semi-diameter conjugate to CP, PT a tangent at P, PF perpendicular to CD.

Since the normal PS' to the surface at P is perpendicular to the tangent PT, PF is its projection on the plane of the indicatrix. Hence the normal PS' cannot intersect the normal at O unless CF vanish, which is the case only when P is at one of the extremities of either axis of the indicatrix; that is, when P is in one of the principal sections.

The same kind of argument shews that, in particular cases where the indicatrix is not a conic, the normals still intersect whenever the tangent at P is perpendicular to CP.

691. *When normals at consecutive points do not intersect, to find the direction and magnitude of the shortest distance between them.*

Let SS' be the shortest distance of the normals OS, PS'; CF is its projection on the plane of the indicatrix, the shortest distance is therefore in the direction of the diameter conjugate to CP.

Also, $CF^2 = CP^2 - PF^2$ and $PF.CD = CA.CB$; therefore, if R, R' be the radii of curvature of the sections OP, OD, and ρ, ρ' those of the principal sections, $SS'^2 = CF^2 = CP^2(1 - \rho\rho'/RR')$, Art. 682,

$$\text{where } R' = \rho + \rho' - R = (\cos^2\theta\, \rho'/\rho + \sin^2\theta\, \rho/\rho')\, R, \text{ Art. 678;}$$

692. *To find the point of nearest approach of a normal consecutive to a given normal.*

Draw $S'H$, FG perpendicular to the normal plane SOP, then PH is the projection of PS' on that plane; the tangent PU to the normal section is perpendicular to PS' and $S'H$, and therefore to PH, and R, the intersection of PH and OS, is ultimately the centre of curvature of the normal section OP.

$$\text{Then } CS : CR :: PG : CP :: PF^2 : CP^2;$$

$$\therefore\ OS \text{ or } CS : R :: \rho\rho' : RR' \text{ ultimately;}$$

hence OS is ultimately $= \rho\rho' / R'$, and $OS^{-1} = R\,(\cos^2\theta / \rho^2 + \sin^2\theta / \rho'^2)$.

693. *To find the angle between consecutive normals.*

The angle between the normals at O and P is ultimately

$$\frac{PF}{S'F} = \frac{PF}{CS} = \frac{PF}{R} \cdot \frac{CP}{PG} = \frac{CP}{R} \cdot \frac{CP}{PF} = \frac{CP}{R} \cdot \sqrt{\left(\frac{RR'}{\rho\rho'}\right)}$$
$$= CP\,\sqrt{(\cos^2\theta / \rho^2 + \sin^2\theta / \rho'^2)}.$$

694. We leave to the student the calculation of the shortest distance and its position from the equation of the normal at the point whose coordinates are $r\cos\theta$, $r\sin\theta$, $\frac{1}{2}r^2/R$, viz.

$$\rho\,(\xi - r\cos\theta)/r\cos\theta = \rho'\,(\eta - r\sin\theta)/r\sin\theta = -\zeta + \tfrac{1}{2}r^2/R..$$

The expression for the shortest distance will be found to be

$$\frac{r\sin\theta\cos\theta\,(\rho'^{-1} \sim \rho^{-1})}{\sqrt{(\rho^{-2}\cos^2\theta + \rho'^{-2}\sin^2\theta)}}.$$

695. *All the normals to a surface in the neighbourhood of a point converge to or diverge from two focal lines at right angles to one another.*

The equation of the surface being $2\zeta = \xi^2/\rho + \eta^2/\rho' + \&c.$, the equations of a normal at $(r\cos\theta, r\sin\theta, \frac{1}{2}r^2/R)$ are, neglecting r^2,

$$\rho\,(\xi - r\cos\theta)/r\cos\theta = \rho'\,(\eta - r\sin\theta)/r\sin\theta = -\zeta,$$

when $\eta = 0$, $\zeta = \rho'$, and when $\xi = 0$, $\zeta = \rho$, hence all normals in the neighbourhood of O intersect two focal lines, each of which passes through the centre of curvature of one of the principal normal sections and is perpendicular to the plane of that section. This theorem is due to Sturm.

696. Certain properties of the principal radii of curvature may be conveniently investigated by considering the angle between the two inflexional tangents. In these directions three consecutive points lie in a straight line, and the radius of curvature of a normal section through either of these tangents is therefore infinite. Hence, if θ be the angle which one of these tangents makes with the tangent to a section of principal curvature, we shall have $0 = \cos^2\theta/\rho + \sin^2\theta/\rho'$, ρ, ρ' being the algebraic magnitudes of the radii of principal curvature. Thus, for points at which the radii of principal curvature are equal in magnitude and opposite in sign, we shall have $\tan^2\theta = 1$, and the tangents to the curve of intersection will therefore also be at right angles.

697. As an example of this method we shall take the following:

To prove that at every point where the surface $x\,(x^2 + y^2 + z^2) = 2a\,(x^2 + y^2)$ meets the plane $x = a$, the radii of curvature will be equal in magnitude, and of opposite signs. This, by what has been said, will be true if we can prove that the two straight lines, drawn through any such point, to meet the surface in three consecutive points, are at right angles to each other.

Let $(\xi - a)/\lambda = (\eta - y)/\mu = (\zeta - z)/\nu = r$ be a straight line which meets the surface in three consecutive points; the equation determining r is

$$(a + \lambda r)(z + \nu r)^2 = (a - \lambda r)\{(a + \lambda r)^2 + (y + \mu r)^2\},$$

which must have its three roots equal to zero; the conditions of which are

$$z^2 = a^2 + y^2, \quad y^2\lambda - ay\mu + az\nu = 0, \text{ and } 2z\lambda\nu + a\nu^2 = -a\lambda^2 + a\mu^2 - 2y\lambda\mu,$$

which becomes $(z\lambda + a\nu)^2 = (a\mu - y\lambda)^2 = \nu^2 a^2 z^2 / y^2$;

$$\therefore\ yz\lambda/\nu = a(-y \pm z) \text{ and } yz\mu/\nu = a^2 \pm yz,$$

hence, if $(\lambda_1, \mu_1, \nu_1)(\lambda_2, \mu_2, \nu_2)$ be the directions of the inflexional tangents,

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = \nu_1\nu_2\{a^2(y^2 - z^2) + a^4\}/y^2z^2 = 0.$$

698. In connexion with the curvature of surfaces, the most important lines which can be traced on a surface are *lines of curvature.*

DEF. 1. A *Line of Curvature* is a curve traced upon a surface, such that the tangent to the curve at any point is also a tangent to one of the principal normal sections of the surface at that point.

Since there are two principal normal sections at every point, whose planes are at right angles, there will be two lines of curvature through every ordinary point, crossing one another at right angles.

DEF. 2. A *line of curvature* is a curve traced on a surface, such that the normals to the surface at any two consecutive points of the curve intersect each other.

That the curves given by these definitions are identical, is shewn in Art. 690.

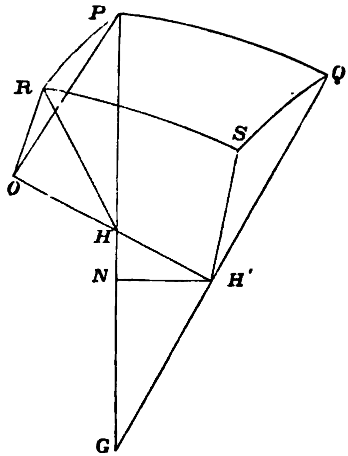
699. *Lines of curvature of a surface of revolution.*

Let PQ be an arc of the generating curve, and PGO, QHO normals to the curve at P and Q, intersecting the axis of revolution in G, H. When the plane of the curve turns round the axis GH, let PQ come into the position $P'Q'$, then the normals to the surface at P and P' intersect in G, also those at P and Q intersect in O; therefore the meridians and the circular sections are lines of curvature.

700. *To find the osculating plane of a line of curvature at any point of a surface.*

Let PQ, PR be small arcs of lines of curvature drawn through P a point in the surface, RS, QS lines of curvature through R, Q respectively; and let PHG, $QH'G$, RH, SH' be normals to the surface at P, Q, R, S, so that PH, QH' are ultimately the radii of curvature of the principal normal sections PR, QS, and PG that of PQ; let these be R', $R' + dR'$, and R, then dR' is the increment of R' due to a change ds along the principal section PQ.

The tangent to PR at P is perpendicular to the plane PH', and therefore to HH', and the tangent at R is, for a similar reason, perpendicular to HH', which is therefore parallel to the binormal to the line PR at P, and determines the osculating plane POR.



Let ϕ be the inclination of the osculating plane of PR to the principal normal section; draw $H'N$ perpendicular to PG,

$$\text{then } \tan\phi = \lim.\frac{HN}{H'N} = \lim.\frac{HN}{PQ}\cdot\frac{PQ}{H'N} = \frac{dR'}{ds}\cdot\frac{R}{R-R'}.$$

COR. In the case of a surface of revolution, since R' is the same for all points in the circular line of curvature supposed to correspond to R, $dR'/ds = 0$, and the osculating plane coincides with the meridian plane.

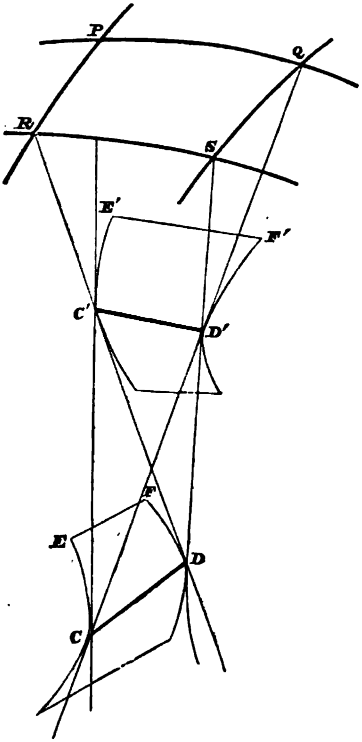
Surface of Centres.

701. DEF. The *surface of centres* is the locus of the centres of principal curvature for every point of a surface.

702. Let L, L' be two lines of curvature passing through the point P of the surface S, and let Q, R be points on L and L' adjacent to P; then QC and RC', normals to S at Q and R, intersect the normal at P in C, C' which are ultimately the centres of curvature of the normal sections touching L and L' respectively, and the planes CPQ, $C'PR$ are at right angles.

Normals drawn at every point of L form a torse whose edge of regression CE is the locus of the centres of curvature of all normal sections of S touching L, these normal planes being tangent planes to the torse.

If an infinite number of lines of curvature of the same system as L be traced on the surface, the corresponding edges such as CE, DF will form a sheet of the surface of centres, and a second sheet will be formed by edges $C'E'$, $D'F'$ corresponding to the lines of the system L'. Call these sheets Σ and Σ'.



Every normal of S such as $PC'C$, $SD'D$ touches each sheet; the two normals PC, QC each touch Σ', and since C does not lie on Σ', PCQ is a tangent plane to Σ', similarly $PC'R$ is a tangent plane to Σ. And, since these two planes are at right angles, the two sheets would appear to cut one another at right angles to an eye situated in a normal and looking along the normal towards both sheets.

703. The two real sheets of the surface of centres for a surface of revolution are the surface generated by the evolute of the generating curve, and the portion of the axis from which normals can be drawn to the generating curve.

Lines of Curvature common to two Surfaces.

704. *When the curve of intersection of two surfaces is a line of curvature on each, the two surfaces cut one another at a constant angle.*

Let $PQRS$ in the figure of page 251 be ultimately the line of curvature common to two surfaces S, S'; and let pa, qa be normals to the surface S, which, since they intersect, must intersect in the polar line aUA, perpendicular to the osculating plane pUq; similarly

qab, rb, normals at q and r, intersect in the polar line bV perpendicular to the consecutive osculating plane, and $brV = bqV = aqU + UqV$.

Let a', b' be corresponding points for the surface S';

$$\therefore\ b'rV = a'qU + UqV; \quad \therefore\ b'rb = a'qa;$$

hence, normals to S and S' at consecutive points q, r are inclined at the same angle, therefore the surfaces cut one another throughout at a constant angle.

In a similar manner it can be shewn that *if two surfaces cut one another at a constant angle, and their curve of intersection be a line of curvature on one surface, it will be a line of curvature on the other also.*

COR. If a line of curvature be a plane curve, its plane will cut the surface at a constant angle.

705. The analytical proof given by Bertrand is very simple. Let P, Q be consecutive points on the curve of intersection of surfaces S, S'; x, y, z and $x+dx$, $y+dy$, $z+dz$ their coordinates; l, m, n and l', m', n' the direction-cosines of the normals at P to S and S'.

If the curve be a line of curvature on S, the normals at P, Q will intersect; $\therefore\ x - l\rho = x + dx - (l+dl)\rho$;

$$\therefore\ dl/dx = dm/dy = dn/dz = \rho^{-1}. \qquad (1)$$

Since PQ is perpendicular to both normals,

$$l\,dx + m\,dy + n\,dz = 0, \text{ and } l'dx + m'dy + n'dz = 0. \qquad (2)$$

i. If the curve be a line of curvature on both surfaces,

$l\,dl' + m\,dm' + n\,dn' = 0$, and $l'dl + m'dm + n'dn = 0$, by (1) and (2);

$$\therefore\ d\,(ll' + mm' + nn') = 0,$$

or the cosine of the angle between the normals is constant.

ii. If the curve be a line of curvature on S, and the surfaces cut one another at a constant angle,

$$l'dl + m'dm + n'dn = 0, \text{ and } d\,(ll' + mm' + nn') = 0;$$

$$\therefore\ l\,dl' + m\,dm' + n\,dn' = 0, \text{ also } l'dl' + m'dm' + n'dn' = 0;$$

therefore, by (2), $dl'/dx = dm'/dy = dn'/dz$, the condition that the curve should be a line of curvature on S'.

706. *When three series of surfaces cut one another orthogonally, the curve of intersection of any two of them is a line of curvature on each.*

Let the origin be a point of intersection of three surfaces, one of each series, and the tangents to their lines of intersection the axes. The equations of the three surfaces may then be written

$$x + ay^2 + 2byz + cz^2 + \ldots = 0, \qquad (1)$$

$$y + a'z^2 + 2b'zx + c'x^2 + \ldots = 0, \qquad (2)$$

$$z + a''x^2 + 2b''xy + c''y^2 + \ldots = 0. \qquad (3)$$

At a consecutive point on the curve of intersection of (2) and (3) we have $y=0$, $z=0$, $x=x'$, and the equations of the tangent planes are, ultimately,

$$2c'x'x + y + 2b'x'z = 0, \quad 2a''x'x + 2b''x'y + z = 0,$$

and since these also are at right angles, $4a''c'x'^2 + 2b''x' + 2b'x' = 0$, or, ultimately, $b' + b'' = 0$; similarly, $b'' + b = 0$, $b + b' = 0$, which can only be satisfied by $b = 0$, $b' = 0$, $b'' = 0$, and therefore the axes are tangents to the lines of curvature on each surface.

Hence, the tangent lines, at any point of intersection of the three surfaces, to their curves of intersection, are tangents to the lines of curvature of the three surfaces through that point, and, consequently, their curves of intersection must coincide with the lines of curvature. This is Dupin's Theorem. A proof is given by Cayley,* which puts in evidence the geometrical ground on which the theorem rests.

Integral and Specific Curvature.

707. Gauss gives the following definition of the Integral Curvature of a finite portion of a surface.

DEF. The *Integral Curvature* of any given portion of a curved surface is the area enclosed on a spherical surface of unit radius by a cone whose vertex is the centre, and whose generating lines are parallel to the normals to the surface at every point of the boundary of the given portion.

Horograph. The curve traced out on the sphere as described above is called the *horograph* of the given portion of the surface.

Average Curvature. The average curvature of any portion of a curved surface is the integral curvature divided by the area of the portion.

Specific Curvature. The specific curvature of a curved surface at any point is the average curvature of an infinitely small area including the point. This is the measure of curvature which was shewn by Gauss to be the reciprocal of the product of the two principal radii of curvature at the point considered.

708. *To shew that the reciprocal of the product of the principal radii of curvature at any point of a surface is a proper measure of the curvature at that point.*

Let an elementary area QRS be described including the point P of a surface, and let a series of lines of curvature divide this area into sub-elementary portions, such as $pqrs$, and let ρ_1, ρ_1' be the principal radii of curvature at p in the directions pq, ps; the horograph for $pqrs$ will be a small rectangle whose sides are pq/ρ_1 and ps/ρ_1', and area $= pqrs/\rho_1\rho_1'$.

But if ρ, ρ' be the principal radii of curvature at P,

$$\rho_1\rho_1' = \rho\rho'(1+\epsilon),$$

* *Quarterly Journal*, vol. XII. p. 185.

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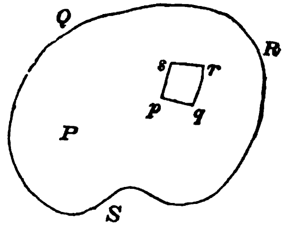
where ε vanishes in the limit; therefore the specific curvature

$$= \lim. \frac{\Sigma(pqrs/\rho_1\rho_1')}{\Sigma(pqrs)} = \frac{1}{\rho\rho'}.$$



This expression is independent of the form of the elementary portion including P, and is analogous to the measure of curvature in plane curves, the solid angle of the cone corresponding to the angle between the normals to a plane curve at the extremities of the small arc on which a point of the curve lies.

709. *To determine the radius of curvature of the normal section of a surface through a given tangent line at a given point in terms of the coordinates.*

Let the equation of the surface be $F(\xi, \eta, \zeta) = 0$; and let (x, y, z) be the given point P, (λ, μ, ν) the direction of the given tangent; also let $(x+dx, y+dy, z+dz)$ be a consecutive point Q taken in the normal section, so that ultimately $dx : dy : dz = \lambda : \mu : \nu$.

Then, if QR be perpendicular to the tangent plane, R the radius of curvature of the normal section will be the limit of $\frac{1}{2}PQ^2/QR$.

The equation of the tangent plane is

$$U(\xi - x) + V(\eta - y) + W(\zeta - z) = 0;$$

$$\therefore QR = \pm (Udx + Vdy + Wdz)/P, \text{ if } P^2 \equiv U^2 + V^2 + W^2;$$

but, Q being a point on the surface,

$$Udx + Vdy + Wdz + \tfrac{1}{2}\{u(dx)^2 + \ldots + 2u'dydz + \ldots\} = 0,$$

neglecting terms of degrees higher than the second;

$$\therefore R = \pm \frac{\{(dx)^2 + (dy)^2 + (dz)^2\} P}{2(Udx + Vdy + Wdz)}$$

$$= \mp \frac{\sqrt{(U^2 + V^2 + W^2)}}{u\lambda^2 + v\mu^2 + w\nu^2 + 2u'\mu\nu + 2v'\nu\lambda + 2w'\lambda\mu}.$$

Since $U\lambda + V\mu + W\nu = 0$ and $\lambda^2 + \mu^2 + \nu^2 = 1$, the problem of finding the directions of the principal sections and the magnitude of the principal radii of curvature is the same as that of finding the direction and magnitude of the principal axes of the section of the conicoid, $ux^2 + \ldots + 2u'yz + \ldots = 1$, made by the plane $Ux + Vy + Wz = 0$.

710. *To determine the principal normal sections, and the radii of principal curvature at any point of a surface, in terms of the coordinates of the point.*

The radius of curvature of a normal section containing the tangent whose direction-cosines are λ, μ, ν is given by

$$u\lambda^2 + v\mu^2 + w\nu^2 + 2u'\mu\nu + 2u'\nu\lambda + 2w'\lambda\mu - \frac{P}{R}(\lambda^2 + \mu^2 + \nu^2) = 0, \quad (1)$$

$$\text{where } U\lambda + V\mu + W\nu = 0, \quad (2)$$

and when R is given, the corresponding tangent lines are the lines of intersection of the cone and plane represented by these equations, λ, μ, ν being considered current coordinates. When R is a maximum or minimum these directions coincide, and the plane is a tangent plane of the cone; hence the directions of the principal sections are given by

$$\{(u-\sigma)\lambda + w'\mu + v'\nu\}/U = \{w'\lambda + (v-\sigma)\mu + u'\nu\}/V$$
$$= \{v'\lambda + u'\mu + (w-\sigma)\nu\}/W, \text{ where } \sigma = P/R,$$

which, by the equation $U\lambda + V\mu + W\nu = 0$, leads to

$$U^2\{(v-\sigma)(w-\sigma) - u'^2\} + \ldots + 2VW\{v'w' - u'(u-\sigma)\} + \ldots = 0. \quad (3)$$

This equation gives the values of the principal radii of curvature, and the values of $\lambda : \mu : \nu$, corresponding to each root, are given by the preceding system of equations.

711. *To find an expression for the specific curvature at any point of a surface.*

The product of the roots of (3) is

$$\frac{U^2(vw - u'^2) + \ldots + 2VW(v'w' - uu') + \ldots}{U^2 + V^2 + W^2}$$
$$= (U^2 + V^2 + W^2) \times \text{specific curvature}.$$

Since the specific curvature vanishes at every point of a developable surface, the numerator equated to zero is the condition that a surface should be developable.

712. We cannot help calling attention to another form of the quadratic giving the principal radii, which was set in an examination paper for Clare and Caius Colleges in 1873.

Since $2VW\mu\nu = U^2\lambda^2 - V^2\mu^2 - W^2\nu^2$, &c., the expression for P/R can be put into the form $A\lambda^2 + B\mu^2 + C\nu^2$, where $A = u + (Uu' - Vv' - Ww')U/VW$, &c.

Construct the conicoid $A\xi^2 + B\eta^2 + C\zeta^2 = P$, having its centre at the point (x, y, z) of the surface, the directions of the axes of the section made by the plane $U\xi + V\eta + W\zeta = 0$ are the directions of the tangents to the principal sections of the surface, and the corresponding values of R will be the squares of the semi-axes of the section; hence, by Art. 237 or 260,

$$U^2/(AR - P) + V^2/(BR - P) + W^2/(CR - P) = 0,$$

a quadratic giving ρ, ρ' the principal radii of curvature.

Also the direction cosines of the tangents to the lines of curvature are as

$$U/(AR-P) : V/(BR-P) : W/(CR-P),$$

where ρ, ρ' are to be written for R.

713. *To determine the conditions satisfied at an umbilic.*

At an umbilic R retains a constant value for all directions (λ, μ, ν) satisfying the two conditions (1) and (2), Art. 710. Hence, at an umbilic the cone (1) must break up into two planes, one of which is the tangent plane (2).

The left-hand member of equation (2) must therefore be a factor of the left-hand member of (1), and the other factor will therefore be $\lambda(u-\sigma)/U+\mu(v-\sigma)/V+\nu(w-\sigma)/W$; multiplying the two, and equating coefficients,

$$(w-\sigma)\ V/W+(v-\sigma)\ W/V=2u',$$
$$(u-\sigma)\ W/U+(w-\sigma)\ U/W=2v',$$
$$(v-\sigma)\ U/V+(u-\sigma)\ V/U=2w',$$

which, on eliminating σ, lead to the two conditions

$$\frac{W^2v+V^2w-2VWu'}{V^2+W^2}=\frac{U^2w+W^2u-2WUv'}{W^2+U^2}$$
$$=\frac{V^2u+U^2v-2UVw'}{U^2+V^2} \qquad (1).$$

These two equations, together with the equation of the surface, will, in general, determine a definite number of points, among which are included all the umbilics. It may happen that a common factor exists, so that the three equations are satisfied by the coordinates of any point lying on a certain curve. Such a curve is called *a line of spherical curvature.*

It should also be observed that U, V, W have been assumed to be *finite* in the foregoing investigation. Should one of them, say U, vanish, we must have in the same manner $V\mu+W\nu$ a factor, and must therefore have $(u-\sigma)\lambda^2+\ldots+2u'\mu\nu+\ldots$

$$\equiv (V\mu+W\nu)\{k\lambda+(v-\sigma)\mu/V+(w-\sigma)\nu/W\} \text{ identically};$$

this gives three equations, and eliminating σ and k,

$$Vv'=Ww',\ 2u'=(v-u)\ W/V+(w-u)\ V/W,$$

which with $U=0$, and the equation of the surface, give *four* relations between the coordinates, unless v' and w' are identically zero, and these will not, in general, be simultaneously true of any point on the surface.

714. The conditions (1) for an umbilic are obtained by the method of Art. 712, from the consideration that the section of the conicoid must be circular, whence, when U, V, W are finite, it follows that $A=B=C$.

715. *To determine the number of umbilics on a surface of the* n^{th} *degree.*

Writing the equations for an umbilic $P/P' = Q/Q' = R/R'$, the degree of P', Q', R' will be $2(n-1)$, and of P, Q, R will be $3n-4$. The degree of the surfaces $QR' - Q'R = 0$, $RP' - R'P = 0$ is therefore $5n-6$, and the degree of their curve of intersection is $(5n-6)^2$. But the curve $R = 0$, $R' = 0$ is part of their intersection, and does not lie on the surface $PQ' - P'Q = 0$. The degree of the curve $P/P' = Q/Q' = R/R'$ is therefore

$$(5n-6)^2 - 2(n-1)(3n-4) \equiv 19n^2 - 46n + 28;$$

but this curve includes three curves similar to

$$U = 0, \quad W^2v + V^2w - 2VWu' = (V^2 + W^2)u,$$

which do not meet the surface in umbilics, and the degree of this curve is $(n-1)(3n-4)$.

Hence the degree of the curve, which meets the surface in umbilics only is

$$19n^2 - 46n + 28 - 3(n-1)(3n-4) \equiv 10n^2 - 25n + 16.$$

The whole number of real and impossible umbilics is therefore

$$n(10n^2 - 25n + 16).$$

Thus in a conicoid the number is 12, four in each of the principal planes; but, if the conicoid be a ruled surface, none will be real, and in the other cases only one system will be real.

There can never be real umbilics on a ruled surface of any degree whatever, since every point of a ruled surface is either parabolic or hyperbolic.

716. *To find the differential equation of the lines of curvature, and the principal radii of curvature at any point.*

Let (ξ, η, ζ) be the point of intersection of normals at consecutive points (x, y, z) and $(x+dx, y+dy, z+dz)$;

$$\therefore (\xi - x)/U = (\eta - y)/V = (\zeta - z)/W = \rho/P = \sigma^{-1},$$

in which equations ξ, η, ζ are unaltered when $x + dx$ is written for x, &c., and $\xi = x + U\sigma^{-1}$, $\eta = y + V\sigma^{-1}$, $\zeta = z + W\sigma^{-1}$;

$$\therefore \; 0 = dx + \sigma^{-1}dU - U\sigma^{-2}d\sigma, \quad 0 = dy + \sigma^{-1}dV - V\sigma^{-2}d\sigma,$$

$$\text{and } 0 = dz + \sigma^{-1}dW - W\sigma^{-2}d\sigma;$$

$$\text{therefore} \quad \begin{vmatrix} dx, & dU, & U \\ dy, & dV, & V \\ dz, & dW, & W \end{vmatrix} = 0,$$

which is one of the differential equations of the lines of curvature, the other being $Udx + Vdy + Wdz = 0$.

Expanding dU, dV, dW, and eliminating dx, dy, dz, and $d\sigma$,

$$\begin{vmatrix} u-\sigma, & w', & v', & U \\ w', & v-\sigma, & u', & V \\ v', & u', & w-\sigma, & W \\ U, & V, & W, & 0 \end{vmatrix} = 0.$$

The coefficient of U^2 is $-(v-\sigma)(w-\sigma) + u'^2$, and that of VW is $u'(u-\sigma) - v'w'$, whence we obtain the quadratic given in Art. 710.

717. The foregoing equations for determining the principal curvatures undergo a considerable simplification, if the equation of the surface be of the form $\phi_1(x)+\phi_2(y)+\phi_3(z)=0$.

We shall then have u', v', w' all zero; the equation giving the length of the radius of curvature of any normal section, the direction of whose tangent line is (λ, μ, ν), will be $PR^{-1}=u\lambda^2+v\mu^2+w\nu^2$; the quadratic equation for the principal radii of curvature will be

$$\frac{U^2}{Ru-P}+\frac{V^2}{Rv-P}+\frac{W^2}{Rw-P}=0;$$

and the differential equation of the lines of curvature will be

$$U(v-w)\,dy\,dz+V(w-u)\,dz\,dx+W(u-v)\,dx\,dy=0. \qquad (1)$$

The conditions for an umbilic in this case reduce to $u=v=w$ when U, V, W are finite, but since this is the exceptional case mentioned in Art. 713, in which u', v', and w' vanish identically, there are other umbilics which are given by $U=0$ and $(v-u)W^2+(w-u)V^2=0$, and similar equations when $V=0$ and $W=0$. The whole number of umbilics is therefore, as before,

$$n\{(n-2)^2+3(n-1)(3n-4)\}\equiv n(10n^2-25n+16).$$

718. *To obtain the differential equation of the lines of curvature, and to find the centres and radii of principal curvature when the equation of the surface gives one of the coordinates explicitly in terms of the other two.*

Let the equation of the surface be $\zeta=f(\xi,\eta)$, and let P, Q be consecutive points on a line of curvature whose coordinates are x, y, z, and $x+dx$, $y+dy$, $z+dz$, then the normals at P, Q intersect; and if (ξ, η, ζ) be their point of intersection,

$$\xi-x+p(\zeta-z)=0, \text{ and } \eta-y+q(\zeta-z)=0, \qquad (1)$$

but ξ, η, ζ remain the same when $x+dx$, $y+dy$, $z+dz$ are written for x, y, z; therefore

$$dp(\zeta-z)=dx+pdz, \text{ and } dq(\zeta-z)=dy+qdz; \qquad (2)$$

$$\therefore \frac{rdx+sdy}{sdx+tdy}=\frac{dx+p(pdx+qdy)}{dy+q(pdx+qdy)};$$

$$\therefore \{(1+q^2)s-pqt\}(dy)^2+\{(1+q^2)r-(1+p^2)t\}dx\,dy$$
$$-\{(1+p^2)s-pqr\}(dx)^2=0, \qquad (3)$$

which is the differential equation of the projection of the lines of curvature on the plane of xy.

Let ρ be the radius of curvature of the principal section through PQ, hence by (1) $\rho^2=(1+p^2+q^2)(z-\zeta)^2$, therefore, writing in (2) σ for $z-\zeta$ or $\rho(1+p^2+q^2)^{-\frac{1}{2}}$,

$$(rdx+sdy)\sigma+dx+p(pdx+qdy)=0;$$

$$\therefore (r\sigma+1+p^2)dx+(s\sigma+pq)dy=0,$$

and similarly $(t\sigma+1+q^2)dy+(s\sigma+pq)dx=0$;

$$\therefore (r\sigma+1+p^2)(t\sigma+1+q^2)-(s\sigma+pq)^2=0,$$

or $(rt-s^2)\sigma^2+\{(1+q^2)r-2pqs+(1+p^2)t\}\sigma+1+p^2+q^2=0.$

COR. The reciprocal of the specific curvature is

$$\rho\rho' = \sigma\sigma'(1+p^2+q^2) = (1+p^2+q^2)^2/(rt-s^2),$$

which is infinite for a developable surface.

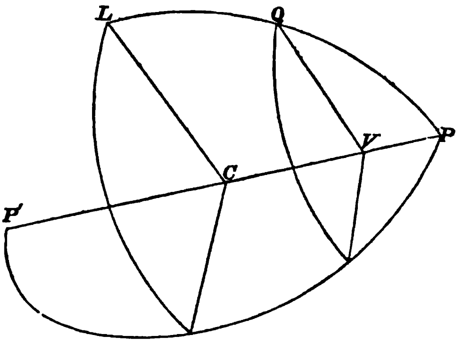
719. *To find the umbilics of the surface $z = f(x, y)$.*

Since the normals at points passing in any direction from an umbilic intersect the normal at the umbilic, neglecting small quantities of the third order in dx and dy, the equation (3) must be true independently of the value of $dy : dx$, and this condition is satisfied by $(1+p^2)/r = pq/s = (1+q^2)/t$, which, with the equation of the surface, determine the umbilics.

Curvature of Conicoids.

720. *To find the radii of principal curvature at any point of a central conicoid.*

Let P be any point on the conicoid, supposed in the figure to be an ellipsoid, PCP' the diameter through P, CL the radius parallel to the tangent at P to any normal section whose radius



of curvature is required, PQL the central section having the same tangent. Let a plane be drawn through a point Q near P parallel to the tangent plane at P, meeting CP in V, and let p, ϖ be the perpendiculars on the tangent plane from C and Q, so that $\varpi : p :: VP : CP$. The radius of curvature of the normal section is the limit of $PQ^2/2\varpi$ or $QV^2/2\varpi$, and

$$QV^2 : CL^2 :: PV.VP' : CP^2 :: \varpi . VP' : p.CP = 2\varpi : p \text{ ultimately,}$$

hence the radius of curvature $= CL^2/p$.

If α, β be the semi-axes of the central section parallel to the tangent plane, α^2/p and β^2/p will be the principal radii of curvature, which we shall call ρ and ρ'.

721. *To find the coordinates of the centre of curvature.*

Let the equation of the conicoid be $x^2/a + y^2/b + z^2/c = 1$, (1) and let (f, g, h) be the point P, then ξ, η, ζ, the coordinates of the

centre of curvature corresponding to ρ, satisfy the equations

$$\frac{\xi-f}{f/a}=\frac{\eta-g}{g/b}=\frac{\zeta-h}{h/c}=-p\rho=-\alpha^2;$$

$$\therefore\ \xi=f(1-\alpha^2/a),\quad \eta=g(1-\alpha^2/b),\quad \zeta=h(1-\alpha^2/c),$$

and by Art. 296, if the equations of confocal conicoids through P be $x^2/a'+y^2/b'+z^2/c'=1$ and $x^2/a''+y^2/b''+z^2/c''=1$, α^2 and β^2 are respectively $a-a''$ and $a-a'$; therefore the centres of curvature are $(fa''/a,\ gb''/b,\ hc''/c)$ and $(fa'/a,\ gb'/b,\ hc'/c)$.

722. *If three confocal conicoids A, B, C intersect in P, the centres of principal curvature of A at P are the poles with respect to B and C of the tangent plane to A at P.*

Let the three conicoids A, B, and C be

$$x^2/a+y^2/b+z^2/c=1,\ x^2/a'+y^2/b'+z^2/c'=1,\ x^2/a''+y^2/b''+z^2/c''=1,$$

intersecting in (f, g, h).

The coordinates of the centre of curvature of the normal section containing the tangent to the intersection of A and B are fa''/a, gb''/b, hc''/c, and its polar, with respect to C, is $fx/a+gy/b+hz/c=1$, the tangent plane to A at P.

Similarly for the other centre of principal curvature. This proposition is due to Salmon.

723. *The curve of intersection of two confocal conicoids is a line of curvature on each.*

Let PT be a tangent at P to the curve of intersection of two confocals S and S', PN, PN' normals at P to S and S'; and suppose a central section of S made by a plane parallel to the tangent plane $N'PT$, and therefore to the indicatrix to S at P. Now it is shewn, Art. 296, that PN' is parallel to one axis of this section, therefore PT is parallel to the other axis; hence, the tangent to the curve of intersection of S and S' at any point is parallel to an axis of the indicatrix of either surface at that point, and the curve is a line of curvature.

724. *At any point in a line of curvature of a conicoid, the rectangle contained by the diameter parallel to the tangent at that point and the perpendicular from the centre on the tangent plane at the point is constant.*

Let the line of curvature on the conicoid S be the curve of intersection with S', and let PT be a tangent to it at any point P, and PN, PN' normals to S and S' at P; then, if α, β be the semi-axes of the central section parallel to $N'PT$, the tangent plane to S, which are parallel respectively to PT and PN', it is shewn, Cor., Art. 296, that β is constant, and, if p be the perpendicular from the centre on the tangent plane, $p\alpha\beta$ is constant, therefore $p\alpha$ is constant.

725. The following proof is independent of the properties of confocal surfaces.

Let P, Q be consecutive points on a line of curvature, O the corresponding centre of curvature, ρ the radius of curvature, p the perpendicular on the tangent plane, α, β the semi-axes of the central section parallel to the tangent plane, α being parallel to PQ, and let $CP = r$, then by the triangle OCP

$$OC^2 = \rho^2 + r^2 - 2\rho p,$$

and since, for a change from P to Q, both O and C are unchanged in position and ρ is unaltered, $rdr = \rho dp$. But, by Art. 277,

$$\alpha^2 + \beta^2 + r^2 = a^2 + b^2 + c^2, \ \alpha\beta p = abc;$$

$$\therefore \alpha d\alpha + \beta d\beta + rdr = 0, \text{ and } d\alpha/\alpha + d\beta/\beta + dp/p = 0;$$

multiplying the last equation by α^2 or $p\rho$, and subtracting the preceding, we obtain $(\alpha^2 - \beta^2)\,d\beta = 0$; therefore β is constant, unless $\alpha = \beta$, which is true only at an umbilic, therefore $p\alpha$ is also constant.

726. *To shew that the curves of intersection of a given conicoid with all confocal conicoids which intersect it satisfy the differential equations of a line of curvature.*

Let the equation of the surface be $x^2/a + y^2/b + z^2/c = 1$, (1) then for the curve of intersection with the confocal

$$x^2/(a+k) + y^2/(b^2+k) + z^2/(c+k) = 1, \qquad (2)$$

$$x\,dx/a + y\,dy/b + z\,dz/c = 0, \qquad (3)$$

$$\text{and } \ x\,dx/(a+k) + y\,dy/(b+k) + z\,dz/(c+k) = 0;$$

$$\therefore \ a(a+k)(b-c)/x\,dx = b\,(b+k)(c-a)/y\,dy = c(c+k)(a-b)/z\,dz;$$

but subtracting (2) from (1) $x^2/a(a+k) + y^2/b(b+k) + z^2/c(c+k) = 0$;

$$\therefore \ x(b-c)/dx + y\,(c-a)/dy + z\,(a-b)/dz = 0,$$

and this is one differential equation of a line of curvature, see (1) Art. 717, the other being (3).

The two equations (1) and (2) involving an arbitrary constant k are therefore the complete integrals of the differential equations of the lines of curvature.

Having given any one point (x', y', z'), we shall have the quadratic equation $x'^2/a(a+k) + y'^2/b(b+k) + z'^2/c(c+k) = 0$ for determining k, and to each value we shall have a corresponding line of curvature passing through the point (x', y', z').

727. *To find the lines of curvature of a central conicoid from the differential equation of their projections on a principal plane.*

The differential equation of the projections of the lines of curvature on the plane of xy may be obtained either by eliminating dz from the equations $x\,dx/a + y\,dy/b + z\,dz/c = 0$

$$\text{and } \ x\,(b-c)\,dy\,dz + y\,(c-a)\,dz\,dx + z\,(a-b)\,dx\,dy = 0,$$

or, by (3) Art. 718, the equation is

$$xy\,(dy)^2\,(b-c)/b+\{(a-c)\,x^2/a+(c-b)\,y^2/b+b-a\}\,dx\,dy$$
$$+\,xy\,(dx)^2\,(c-a)/a=0. \qquad (1)$$

Multiply by $4xy/ab$, and assume $x^2/a=u$, $y^2/b=v$; therefore

$$(b-c)\,u\,(dv)^2+\{(a-c)\,u+(c-b)\,v+b-a\}\,du\,dv+(c-a)\,v\,(du)^2=0,$$

$$\text{or } \{(b-c)\,dv-(c-a)\,du\}\,(udv-vdu)+(b-a)\,du\,dv=0. \qquad (2)$$

If we assume $v=\alpha+\alpha_1 u+\ldots+\alpha_r u^r+\ldots$,

$$udv-vdu=(-\alpha+\alpha_2 u^2+2\alpha_3 u^3+\ldots)\,du,$$

hence the equation (2) cannot be identically satisfied unless α_2, $\alpha_3\ldots$ are all zero, and substituting $\alpha+\alpha_1 u$ for v,

$$\{(b-c)\,\alpha_1-(c-a)\}\,(-\alpha)+(b-a)\,\alpha_1=0. \qquad (3)$$

The solution $v=\alpha+\alpha_1 u$ is therefore the complete solution, since it involves one arbitrary constant in the second degree.

The projections of the lines of curvature are therefore conics of the form $x^2/a'+y^2/b'=1$, where $b'=b\alpha$, $a'\alpha_1+a\alpha=0$, so that, dividing (3) by α_1,

$$a'\,(a-c)/a+b'\,(c-b)/b+b-a=0. \qquad (4)$$

It can be shewn from this relation between the axes, or directly from the singular solution of the differential equation (1), that the system of conics is enveloped by the four straight lines whose projections are $x\sqrt{\{(a-c)/a\}}\pm y\sqrt{\{(c-b)/b\}}=\pm\sqrt{(a-b)}$; also, that each of these four straight lines is the projection of a generating line containing three umbilics real or imaginary.

The projections of the intersection of the two confocals (1) and (2) of Art. 726 are $x^2\,(a-c)/a\,(a+k)+y^2\,(b-c)/b\,(b+k)=1$, the axes of which satisfy the conditions (4), and the solutions agree.

Another solution is given in Boole's *Diff. Eq.*, p. 135, Ex. 3.

728. *Lines of curvature of the paraboloids.*

Let $2z=x^2/a+y^2/b$ (1) be the equation of a paraboloid; the differential equation of the projections of the lines of curvature on the plane of xy is, by (3) Art. 718,

$$axy\,(dy)^2+\{(a-b)\,ab+bx^2-ay^2\}\,dx\,dy-bxy\,(dx)^2=0,$$

the solution of which may be obtained as in Art. 727, viz. $x^2/a'+y^2/b'=1$ (2), where a', b' are connected by the equation

$$a'/a-b'/b+a-b=0.$$

The equation of a paraboloid, confocal with (1), is

$$2z+k=x^2/(a+k)+y^2/(b+k),$$

and of the projection of the curve of intersection

$$x^2/a\,(a+k)+y^2/b\,(b+k)+1=0,$$

which is one of the system of conics (2).

729. The differential equation of the lines of curvature of a hyperbolic paraboloid, whose equation is $az = xy$, is

$$(a^2 + x^2)(dy)^2 - (a^2 + y^2)(dx)^2 = 0,$$

and the lines of curvature are the intersections of the paraboloid and hyperbolic cylinders, whose equations are

$$y^2 - 2Cxy + x^2 = a^2(C^2 - 1),$$

the positive and negative values of C determining the two systems of lines.

Lines of Curvature through an Umbilic.

730. *To shew that there are three directions passing from an umbilic in which the normals at the consecutive points intersect.*

If the axis of z be a normal at an umbilic, the equation of the surface is of the form $\zeta = a(\xi^2 + \eta^2) + u_3(1 + \epsilon)$, where u_3 is of the third degree in ξ and η, and ϵ vanishes in the limit; the equations of a normal at (x, y, z) are

$$\xi - x + (2ax + du_3/dx)(\zeta - z) = 0, \quad \eta - y + (2ay + du_3/dy)(\zeta - z) = 0;$$

but if this normal meet that at the umbilic, the equations are satisfied by $\xi = 0$; $\eta = 0$; $\therefore\ x\,du_3/dy - y\,du_3/dx = 0$, which gives the three directions in which the point (x, y, z) must be taken.

731. *To find the three directions for which normals to a conicoid intersect the normal at an umbilic.*

Let $a\xi^2 + b\eta^2 + c\zeta^2 = 1$ (1) be the conicoid, $(\alpha, 0, \gamma)$ the umbilic, $(\alpha + \lambda r, \mu r, \gamma + \nu r)$ a point adjacent to it in the direction (λ, μ, ν), the equations of the two normals are

$$\frac{\xi - \alpha - \lambda r}{a(\alpha + \lambda r)} = \frac{\eta - \mu r}{b\mu r} = \frac{\zeta - \gamma - \nu r}{c(\gamma + \nu r)}, \text{ and } \frac{\xi - \alpha}{a\alpha} = \frac{\zeta - \gamma}{c\gamma}, \ \eta = 0;$$

one condition that they may intersect is $\mu = 0$, and therefore one direction is that of the principal section containing the umbilic; for the other conditions

$$\xi - \alpha = \lambda r - (\alpha + \lambda r)\, a/b \text{ and } \zeta - \gamma = \nu r - (\gamma + \nu r)\, c/b;$$

$$\therefore\ c\gamma(b - a)\lambda = a\alpha(b - c)\nu,$$

$$\text{but } a^2\alpha^2/(b - a) = c^2\gamma^2/(c - b), \ \therefore\ a\alpha\lambda + c\gamma\nu = 0, \tag{2}$$

$$\text{and, by (1), } a(\alpha + \lambda r)^2 + b\mu^2 r^2 + c(\gamma + \nu r)^2 = 1; \tag{3}$$

$$\therefore\ a\lambda^2 + b\mu^2 + c\nu^2 = 0; \tag{4}$$

(2) and (4) give the two other directions for which the normals intersect; and, since (3) is satisfied for all values of r, they are the directions of the imaginary generatrices through the umbilic. Since the argument is independent of the magnitude of r, it is true that all the normals at points along one of the umbilical generatrices intersect, and they have therefore this character of lines of curvature; but Cayley has remarked in a note on a paper upon "the direction of lines of curvature in the neighbourhood of an umbilic,"* that they are the envelopes of the lines of curvature, and belong to the singular solution of the differential equation of these lines, as stated in Art. 727.

We may observe also that since $(b - a)\lambda^2 = (c - b)\nu^2$,

$$\therefore\ a\lambda^2 + b\mu^2 + c\nu^2 = b(\lambda^2 + \mu^2 + \nu^2); \ \therefore\ \lambda^2 + \mu^2 + \nu^2 = 0 \text{ by (4)},$$

which shews that these generatrices pass through the imaginary circle at infinity.

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* Frost, *Quart. Journ. of Math.*, vol. x. p. 78, and Cayley, *ibid.* p. 111.

732. In the note referred to above, Cayley remarks that since, at an umbilic, dy/dx is determined by a cubic equation, there are generally three directions of the line of curvature, which may arise from three distinct curves, or from a curve with a triple point at the umbilic; and, referring to a paper by Serret,* he states that the lines of curvature on the surface $xyz = 1$ are its intersection with the series of surfaces $h = (x^2 + \omega y^2 + \omega^2 z^2)^{\frac{3}{2}} + (x^2 + \omega^2 y^2 + \omega z^2)^{\frac{3}{2}}$, (1), where ω is an imaginary cube root of unity; now at the umbilic (1, 1, 1), corresponding to which $h = 0$,

$$(x^2 + \omega y^2 + \omega^2 z^2)^3 = (x^2 + \omega^2 y^2 + \omega z^2)^3; \quad \therefore \; x^2 + \omega y^2 + \omega^2 z^2 = x^2 + \omega^2 y^2 + \omega z^2,$$

or $= \omega (x^2 + \omega^2 y^2 + \omega z^2)$, or $= \omega^2 (x^2 + \omega^2 y^2 + \omega z^2)$; $\therefore \; y^2 = z^2$, or $x^2 = y^2$, or $z^2 = x^2$; hence, through the umbilic (1, 1, 1) three distinct lines of curvature pass, viz. the curves $y = z$, $xy^2 = 1$; $x = y$, $zx^2 = 1$; and $z = x$, $yz^2 = 1$.

733. To shew that (1) of the last article is one of the equations of the lines of curvature of $xyz = 1$, we have

$$xdydz\,(y^2 - z^2) + ydzdx\,(z^2 - x^2) + zdxdy\,(x^2 - y^2) = 0.$$

Multiply by xyz, and let $x^2 = p$, $y^2 = q$, $z^2 = r$;

$$\therefore \; p\,(q - r)\,dqdr + q\,(r - p)\,drdp + r\,(p - q)\,dpdq = 0. \qquad (1)$$

Again, if $h = (p + \omega q + \omega^2 r)^{\frac{3}{2}} + (p + \omega^2 q + \omega r)^{\frac{3}{2}}$,

$$(dp + \omega dq + \omega^2 dr)^2\,(p + \omega q + \omega^2 r) - (dp + \omega^2 dq + \omega dr)^2\,(p + \omega^2 q + \omega r) = 0.$$

The coefficient of $(dp)^2 + 2dqdr = (\omega - \omega^2)\,(q - r)$,

.... $(dq)^2 + 2drdp = (\omega - \omega^2)\,(r - p)$,

.... $(dr)^2 + 2dpdq = (\omega - \omega^2)\,(p - q)$,

and $dp/p + dq/q + dr/r = 0$, $\therefore \; -(dp)^2 = dpdq\,p/q + dpdr\,p/r$;

$$\therefore \; (-dpdq\,p/q - dpdr\,p/r + 2dqdr)\,(q - r) + \ldots = 0,$$

in which the coefficient of $dqdr = 2\,(q - r) - (r - p)\,q/r - (p - q)\,r/q$

$$= q - r + p\,(q/r - r/q) = p\,(q - r)\,(p^{-1} + q^{-1} + r^{-1});$$

$$\therefore \; p\,(q - r)\,dqdr + \ldots = 0 \text{ the same as (1)};$$

hence, the curve of intersection is a line of curvature.

XLVIII.

(1) A surface is generated by the revolution of a parabola about its directrix; shew that one principal radius of curvature at any point is double the other.

(2) If at any point of a surface R, R' be radii of curvature of normal sections at right angles to each other, and ρ, ρ' be principal radii of curvature, the sections corresponding to R and ρ being inclined at an angle α, prove that

$$R^{-1}\cos^2\alpha - R'^{-1}\sin^2\alpha = \rho^{-1}\cos 2\alpha.$$

(3) The principal radii of curvature at the points of the surface $a^2x^2 = z^2(x^2 + y^2)$, where $x = y = z$, are given by the equation $2\rho^2 + 2\sqrt{3}a\rho - 9a^2 = 0$.

(4) If θ be the inclination of any tangent to that of the principal section of least curvature, and ψ the inclination of a section through this tangent to the corresponding normal section such that the curvature is equal to that of the other principal section, ρ, ρ' being the radii of curvature of the two principal sections, prove that $2\rho\sin^2\frac{1}{2}\psi = (\rho - \rho')\cos^2\theta$.

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* *Liouv. Journ.*, t. 12 (1847), pp. 241—254.

(5) If ρ, ρ' be the principal radii of curvature at a point of a surface, prove that with the notation of Art. 710

$$\frac{1}{\rho}+\frac{1}{\rho'}=\frac{d}{dx}\left(\frac{U}{P}\right)+\frac{d}{dy}\left(\frac{V}{P}\right)+\frac{d}{dz}\left(\frac{W}{P}\right).$$

(6) Shew that the projection, on the plane of xy, of the indicatrix at any point of the surface $z=(e^y+e^{-y})\cos x$ is a rectangular hyperbola.

(7) Prove that the radius of curvature of the surface $x^m+y^m+z^m=a^m$ at an umbilic is $3^{\frac{m-2}{2m}}a/(m-1)$.

(8) Prove that the specific curvature at every point of the elliptic paraboloid $2z=x^2/a+y^2/b$, where it is cut by the cylinder $x^2/a^2+y^2/b^2=1$, is $(4ab)^{-1}$.

(9) The integral curvatures of the portions of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ cut off by the cone $x^2/a^4+y^2/b^4=z^2/c^4$ are in the ratio of $\sqrt{2}-1$ to $\sqrt{2}+1$.

(10) A series of central sections of an ellipsoid are taken such that one axis of the section is constant, and planes parallel to these touch the ellipsoid, prove that the points of contact lie on a line of curvature.

XLIX.

(1) Prove that the principal curvatures are equal and opposite at points in the surface $x^2(y-z)+ayz=0$ where it is met by the cone $(x^2+6yz)yz=(y-z)^4$.

(2) If ρ, ρ' be the principal radii of curvature at any point of an ellipsoid on its line of intersection with a given concentric sphere, prove that the expression $\rho\rho'/(\rho+\rho')^4$ will be invariable.

(3) At any point of the curve of contact of a cylinder circumscribed to a surface, the product of the radius of curvature of the right section of the cylinder and the radius of curvature of the normal section of the surface drawn through the generator of the cylinder, is equal to the product of the principal radii of curvature of the surface at the point.

(4) Shew that a sphere whose centre is the origin, and the reciprocal of whose radius is $a+b+c$ touches the surface whose equation is $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}+(cz)^{\frac{2}{3}}=1$ at an umbilic.

(5) Prove that the specific curvature at any point of an ellipsoid is proportional to p^4, p being the perpendicular from the centre on the tangent plane.

(6) Shew that the integral curvature of the whole surface

$$(x^2+y^2)/a^2-z^2/c^2=1 \text{ is } 4\pi\{1-c/\sqrt{(a^2+c^2)}\}.$$

(7) The planes drawn through the centre of an ellipsoid, parallel to the tangent planes at points along a line of curvature, envelope a cone which intersects the ellipsoid in a sphero-conic.

(8) On an umbilical conicoid, the projections of the lines of curvature on the planes of circular section, by lines parallel to an axis, form a series of confocal conics, the foci of which are the projections of the umbilics.

(9) In the helicoid, whose equation is $y = x\tan(z/a)$, the lines of curvature are the intersections of the helicoid with the surfaces represented by the equation $2\sqrt{(x^2+y^2)} = a\sinh\{(z+c)/a\}$, for different values of c.

Also, prove that the principal radii of curvature are, at every point in the intersection of the helicoid with a coaxial cylinder, constant and equal in magnitude, but of opposite signs.

(10) A helix is drawn on a cylinder, and the surface which has this curve as its edge of regression is cut by a coaxial cylinder. Prove that the principal radii of curvature of the surface at all points of the curve of section are equal.

L.

(1) A series of conicoids, $ax^2 + by^2 + cz^2 = 1$, is drawn through a fixed point (α, β, γ); shew that the locus of the centres of principal curvature at the fixed point is the surface $\alpha yz(x-\alpha)^2 + \beta zx(y-\beta)^2 + \gamma xy(z-\gamma)^2 = 0$.

(2) If a surface have contact of the second order at (x, y, z) with the conicoid

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + 2a''x + 2b''y + 2c''z + f = 0,$$

then $r^{-1}(a + cp^2 + 2b'p) = s^{-1}(cqp + a'p + b'q + c') = t^{-1}(b + cq^2 + 2a'q)$.

(3) Deduce the conditions for an umbilicus from the equation giving the radii of curvature, by making the roots of the equation equal.

(4) Prove that the specific curvature at any point of the paraboloid $y^2/b + z^2/c = x$ varies as $(p/z)^4$, p being the perpendicular from the centre on the tangent plane.

(5) If a plane curve be given by the equations

$$x/a = \cos\theta + \log\tan\tfrac{1}{2}\theta, \quad y/a = \sin\theta,$$

the surface produced by the revolution of this curve about the axis of x will have its specific curvature constant.

(6) Shew that the integral curvature of the portion of a surface of revolution cut off by two planes perpendicular to the axis is $2\pi(\cos\beta - \cos\alpha)$, where α, β are the angles which the normals to the surface at any points on the curves of intersection of the planes and surface make with the axis.

(7) A cone of revolution circumscribes an ellipsoid, prove that the plane of contact divides the ellipsoid into two portions whose integral curvatures are $2\pi(1 \pm \sin\alpha)$, where 2α is the vertical angle of the cone.

(8) If one series of lines of curvature on a surface be plane curves, lying in parallel planes, the other series will also be plane curves.

(9) Prove that the three surfaces $yz = ax$, $\sqrt{(x^2+y^2)} + \sqrt{(x^2+z^2)} = b$, $\sqrt{(x^2+y^2)} - \sqrt{(x^2+z^2)} = c$, intersect each other always at right angles; and hence prove that, on a hyperbolic paraboloid whose principal sections are equal parabolas, the sum or the difference of the distances of any point on a line of curvature from the two generators through the vertex is constant.

(10) Prove that, if the normals drawn at the points where a straight line meets a conicoid intersect, those drawn at the points where the same line meets any confocal conicoid will also intersect. Hence shew that four lines can be drawn from any point to touch the lines of curvature on a given conicoid.

LI.

(1) The normal at each point of a principal section of an ellipsoid is intersected by the normal at a consecutive point not on the principal section; shew that the locus of the point of intersection is an ellipse having four (real or imaginary) contacts with the evolute of the principal section.

(2) The points of the surface $xyz = a(yz + zx + xy)$, at which the principal curvatures are equal and opposite, lie on the cone

$$x^4(y+z) + y^4(z+x) + z^4(x+y) = 0.$$

(3) If the product of the principal radii of curvature of a surface of revolution be constant and equal to a^2, prove that $\rho^2a^{-2} + b^2y^{-2} = 1$, where ρ is the radius of curvature of the generating curve, y the distance from the axis, and b some constant. Prove also that, if the generating curve cut the axis at right angles, the surface will be a sphere.

(4) A surface is generated by the motion of a straight line which always intersects the axis of x, prove that the radii of curvature at any point on the axis of x are $\cot\frac{1}{2}\theta\, dx/d\psi$ and $-\tan\frac{1}{2}\theta\, dx/d\psi$, x being the distance of the point from the origin, θ the angle which the corresponding generator makes with the axis of x, and ψ that which its projection upon the plane of yz makes with the axis of y.

(5) Prove that the specific curvature at a point on a right conoid, whose equation in cylindrical coordinates r, θ, z is $z = f(\theta)$, is $-\{f'(\theta)\}^2/[r^2+\{f'(\theta)\}^2]^2$.

(6) Shew that the specific curvature at any point on the surface $xyz = abc$, varies as the fourth power of the perpendicular from the origin on the tangent plane at the point, and that at an umbilic it is $\frac{1}{3}(abc)^{-\frac{2}{3}}$.

(7) If any cylinder circumscribe an ellipsoid, it will divide the ellipsoid into portions whose integral curvatures are equal. Hence, if three cylinders circumscribe an ellipsoid, the integral curvature of the portion of the ellipsoid cut off is $\pi - POQ - QOR - ROP$, where O is the centre, and OP, OQ, OR are the directions of the axes of the cylinders.

(8) Find the umbilic of the surface $ax^3 + by^3 + cz^3 = k^3$, and shew that, at the umbilic, $ax = by = cz$, the directions of the three lines of curvature are given by the equations $adx = bdy$, $bdy = cdz$, and $cdz = adx$ respectively.

(9) Tangent planes are drawn to a series of confocal conicoids from a fixed point on one of the axes, the locus of the points of contact is a surface; prove that three such surfaces corresponding to three points one on each axis cut one another orthogonally; also that part of the curves of intersection are circles.

(10) Shew that the line which separates the synclastic from the anticlastic parts of a surface is not generally a line of curvature, and that along it the inflexional tangents coincide.

LII.

(1) The only surface of revolution, such that the curvatures of the principal sections at every point are equal and opposite, is that produced by the revolution of a catenary about its directrix.

(2) A plane curve is wrapped upon a developable surface. If ρ be the radius of curvature of the plane curve at any point, ρ' the corresponding radius of curvature of the curve upon the surface, R the corresponding principal radius of curvature of the surface, and θ the angle at which the curve intersects the generator of the surface, $R^{-2}\sin^4\theta = \rho'^{-2} - \rho^{-2}$.

(3) A surface is generated by the motion of a variable circle which always intersects the axis of x, and is parallel to the plane of yz. If r be the radius of the circle at a point on the axis of x, and θ the inclination of the diameter through that point to the axis of z, prove that the principal radii of curvature at the point are given by the equation $\rho^2 r + p^2(\rho - r) = 0$, where p is the value of $dx/d\theta$ at the point.

(4) A surface is generated by a straight line which always intersects a given circle, and the straight line through the centre of the circle normal to its plane, prove that the principal radii of curvature of the surface, at any point on the circle, are given by the equation $\rho^2(d\theta/d\psi)^2 + a\rho\cos\theta - a^2 = 0$, a being the radius of the circle, θ the angle which the generator at the point makes with the fixed line, and ψ the angle which the radius of the circle through the point makes with a fixed radius.

(5) Two surfaces touch each other at the point P; if the principal curvatures of the first surface at P be denoted by $a \pm b$, those of the second by $a' \pm b'$; and if ω be the angle between the principal planes to which $a+b$, $a'+b'$ belong, θ the angle between the two branches at P of the curve of intersection of the surfaces, shew that $(b^2 - 2bb'\cos 2\omega + b'^2)\cos^2\theta = (a-a')^2$.

(6) In a surface, generated as in (4), if $\psi = \log\tan\frac{1}{2}\theta$, the measure of curvature will be the same at corresponding points on the fixed line, and on the circle.

(7) Through any point of the hyperboloid $x^2 + y^2 - z^2 = a^2$ two generators are drawn; shew that the integral curvature of the surface bounded by these generators and the plane of xy is $-\sin^{-1}\{h^2/(a^2+h^2)\}$, where h is the distance of the point from that plane.

(8) Find the differential equations of the first order of surfaces possessing the property that the projections, on a fixed plane, of their lines of curvature cross each other everywhere at right angles. Prove that they are satisfied by surfaces of revolution whose axes are perpendicular to the fixed plane.

(9) Prove that the surface of centres of the helicoid, whose equation is $z = m\theta$ in cylindrical coordinates, will be found by eliminating ω between

$$r^2 = -m^2\sec 2\omega, \quad z = \tfrac{1}{2}m\tan 2\omega + m(\theta - \omega).$$

(10) Through a given generator of a hyperboloid a variable plane is drawn; this will touch the hyperboloid at one point A, and contain a normal at another point B; prove that the sum of the square roots of the specific curvatures at A and at B is constant for all positions of the plane.

CHAPTER XXIII.

GEODESIC LINES.

734. Geodesic lines are among the most interesting lines which can be drawn on a surface, and they have long occupied the attention of mathematicians. The principal theorems connected with these lines contained in this Chapter are due to Cayley, Chasles, Gauss, Hart, Jacobi, Joachimsthal, Liouville, Roberts (M. and W), and Salmon.

When a string passes through small rings at two points on a smooth sphere, and is then stretched tight, it is easily seen that it must lie on a great circle of the sphere, the reason being that the resultant of the tensions at the extremity of any element of the string will then be balanced by the reaction of the surface. Moreover, when it occupies the smaller arc of the circle it will be in a position of stable equilibrium, since if slightly disturbed it would return to the original position; whereas, if it occupied the larger arc, it would, on being displaced, slide along the surface of the sphere until it assumed the position of the smaller arc; the equilibrium would in this latter case be unstable. If the two rings were at opposite ends of a diameter the equilibrium would be neutral.

Any great circle is a geodesic line on a sphere, and the idea involved in this illustration may be generalised by either of the following definitions, which are so closely connected that it is indifferent which is considered to be the better description of the geodesic lines on a surface, since, when one is accepted, the property involved in the other is almost obvious.

735. DEF. 1. *A geodesic line* on a surface is a line whose osculating plane at any point contains the normal to the surface at that point.

DEF. 2. *A geodesic line* on a surface is a line along which, if an inelastic string were placed, it would be in equilibrium when strained by forces applied in the direction of the tangents at any two points of the line; the surface being supposed smooth.

This definition includes the cases both of stable and unstable equilibrium mentioned above, which may occur on any surface.

From either definition it follows that a proposition similar to Euclid I, prop. 11, Cor. about straight lines holds also with respect to geodesic lines on a surface.

736. The connecting link between the two definitions is obtained by considering that an element of a strained string on a smooth

surface is in equilibrium when the resultant of the equal tensions at the extremities of the element is balanced by the reaction of the surface, thus the two tangents must be in the same plane with the normal to the surface, and the first definition follows from the second; and *vice versâ* if we reverse the argument.

Unless specially mentioned we shall suppose that a geodesic is described by the first definition.

737. A geodesic line has also been defined as the line of maximum or minimum length which can be drawn between two points on a surface, but this is not exact enough for a definition; for take the case of a great circle of a sphere $APBQA$, in which APB is the smaller arc, then if BQ be less than half the circumference, BQ would be a minimum line between B and Q, therefore some line BRQ would be greater than BQ, and $BRQA$ would be greater than BQA, so that it is hard to say in what sense the larger arc is a maximum.

The following definition which, as well as the first, is independent of mechanical considerations, sufficiently expresses the properties of a geodesic.

DEF. 3. *A geodesic line* is a curve such that its arc between any two neighbouring points is shorter than the arc of any other curve on the surface joining the two points.

The connection between this and the first definition is easily established, as follows, by the use of Meunier's theorem.

738. *The line of minimum length between two points on a surface, measured along the surface, is a geodesic line.*

The curvature of any curve on a surface is the same at any point as that of the section of the surface made by the osculating plane at that point, since the two curves will have three coincident points. Also, of all sections having a common tangent line, the normal section is that whose curvature is the least, by Meunier's theorem, Art. 686; hence any infinitesimal arc of the proposed line being manifestly the shortest possible between its extremities, the osculating plane must contain the normal to the surface, and so the line must be a geodesic, Def. 1.

739. The particular case, that the line of minimum length joining two points on a torse is a geodesic line, can be shewn by considering that, if the torse be developed into a plane, the minimum line must be the straight line joining the two points. In the figure on page 202 let $ABC...K$ be the polygon which in the limit becomes the minimum line joining A and K; since on development this becomes a straight line, two consecutive sides EF, FG must be inclined at equal angles to the line Ff. Hence a straight line, drawn through F perpendicular to the line Ff in the plane bisecting the angle between the planes EFf, GFf, will manifestly lie in the plane EFG, and bisect the angle EFG. This line will be in the limit the normal to the surface, and the plane EFG will be the osculating plane of the curve $AB...K$ at the point F; therefore, by Def. 1, the curve is a geodesic.

A similar proof could be given of the general proposition with respect to any surface, but we prefer the proof by Meunier's theorem given in the last article.

740. *Differential equations of geodesic lines on a surface.*

By the first definition the principal normal of a geodesic line at

any point of a surface coincides with the normal to the surface at that point, hence, if $F(x, y, z) = 0$ be the equation of the surface, using the notation of Arts. 461 and 634, $x''/U = y''/V = z''/W$ (1).

One integral of these equations is, of course, $F(x, y, z) = 0$, and if another can be found it will involve two constants, which can be determined so as to make the line pass through any two points on the surface, or to satisfy any two conditions consistent with its nature.

The form of the equations connecting the constants of integration with the coordinates of two proposed points may be such that many values can be given to the constants, to each of which will correspond a geodesic through the given points; examples of this will be given when the surfaces are right circular cylinders and cones.

When the equation of the surface is given in the form $z = f(x, y)$ the fundamental equations (1) of a geodesic assume the forms $x'' + pz'' = 0$, $y'' + qz'' = 0$.

741. *If a geodesic be either a plane curve or a line of curvature it will be both; but a plane line of curvature is not necessarily a geodesic.*

i. Along a plane geodesic consecutive normals to the surface intersect, which is the fundamental property of a line of curvature.

The case of a generator on a scroll is an exception, for although it is a geodesic and plane it is not a line of curvature.

ii. In a geodesic line of curvature take A, B, C, D any four consecutive points; since these are in a line of curvature the normals at B, C intersect in some point O, and since they are in a geodesic line ABC, the osculating plane at B contains BO, therefore A lies in the plane BCO; similarly D lies in the same plane. Thus, the whole curve lies in one plane.

iii. A plane line of curvature is not necessarily a geodesic, as in the case of a circular section of a surface of revolution, Art. 699.

742. The student should prove i. and ii. analytically by considering that, if (l, m, n) be the direction of the normal at a point (x, y, z), and $l' = dl/ds$, &c. for a geodesic $x''/l = y''/m = z''/n$, Art. 740, for a line of curvature $x'/l' = y'/m' = z'/n'$, (1) Art. 705, and for a plane curve

$$\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix} = 0, \text{ Art. 627.}$$

743. *If a series of lines of curvature be geodesics, they will all be repetitions of the same plane curve.*

Let one of these lines of curvature, which have been shewn to be plane curves, lie in the plane xOy, and let Oy be the intersection of this plane with the plane ξOy containing a consecutive line of the same system. Let a line of curvature of the other system cut these two curves in P, Q, then PR, the projection of PQ on the plane xOy, is a normal to the surface being the

principal normal of the plane line of curvature in xOy, and thus the curve PQ cuts the plane xOy at right angles, and PR is ultimately of the order QR^2.

If x, y and ξ, η be the coordinates of P and Q in their respective planes, $\xi - x$ and $\eta - y$ will be of the order of PR, and therefore of the square of the angle $xO\xi$ between the planes; hence the plane lines of curvature are plane curves equal in all respects, and by giving the angle between the planes successive increments, so small that the squares may be neglected, the result follows.

744. *If tangents to a geodesic make constant angles with a fixed line, the normals to the surface along it will be always perpendicular to that line.*

Let (l, m, n) be the direction of the given line, (x, y, z) a point of the geodesic, then $lx' + my' + nz'$ is constant,

$$\therefore\ lx'' + my'' + nz'' = 0, \text{ and } lU + mV + nW = 0.$$

745. Salmon has shewn that the edge of regression of the torse generated by the normals to the surface at every point of a line of curvature is a geodesic line on the sheet of the surface of centres in which it lies. For, in the figure page 287, PCQ is the osculating plane at C to the edge CE, and $PC'R$ is a tangent plane at C to the sheet on which CE lies; and, since the planes PCQ and $PC'R$ are at right angles, the normal to that sheet at C lies in the osculating plane of CE, which is the condition that CE should be a geodesic on the sheet on which it lies.

746. *Geodesic Polar Coordinates.* If on a surface a fixed point S and a fixed geodesic line SX be chosen, the position of any point P on the surface will be fully defined when the length of the geodesic joining S and P and the angle at S between the geodesics SP and SX are given; these are *geodesic polar coordinates* of P.

For the study of curves referred to these coordinates, the following theorem, which we give with Gauss' geometrical proof, is important.

747. *If geodesic lines of equal length be drawn on a surface from the same point in every direction the curve which is the locus of their extremities will cut all the lines at right angles.*

Let SP, SQ be two geodesics of equal length inclined at an infinitely small angle at S, and let PQ, if possible, make angles different from a right angle with SP, SQ; by the law of continuity, if one be greater the other will be less than a right angle. Let SPQ be the acute angle, and draw QR perpendicular to PQ meeting SP in R, then, treating the small triangle as plane, RQ is less than RP, hence $SR + RQ < SR + RP$ or SP, that is, SRQ is a shorter path from S to Q than SQ, which is equal to SP, and, by Art. 738, this is contrary to the supposition that SQ is a geodesic line.

The curve is called a *geodesic circle.*

748. It is easy to see that this theorem enables us to prove by the method of infinitesimals many propositions relating to curves on surfaces almost in the same words as are used in proving the corresponding propositions in plane geometry; thus, *if a curve be such that either the sum or difference of the geodesic distances of any point*

in it from two fixed points in the surface is constant, the tangent at any point will bisect the angle between the geodesics which join it to the two fixed points: the converse can also be shewn.

749. If P, P' be two consecutive points on a curve referred to geodesic polar coordinates and PM be drawn perpendicular to SP', PM will be a small arc of a geodesic circle and will be equal to the angle PSP' multiplied by some function of the position of P.

Let ρ, ω be the geodesic coordinates of P; then

$$(ds)^2 = (d\rho)^2 + (Pd\omega)^2,$$

where P indicates the function mentioned above.

The proposition of Art. 747 can be shewn analytically as follows: The small arc of a geodesic circle is perpendicular to the normal to the surface, therefore, by Art. 740, $\frac{dx}{Pd\omega}\frac{d^2x}{d\rho^2} + \frac{dy}{Pd\omega}\frac{d^2y}{d\rho^2} + \frac{dz}{Pd\omega}\frac{d^2z}{d\rho^2} = 0$; also, differentiating the equation $(dx/d\rho)^2 + (dy/d\rho)^2 + (dz/d\rho)^2 = 1$, we have

$$\frac{d^2x}{d\omega\,d\rho}\frac{dx}{d\rho} + \frac{d^2y}{d\omega\,d\rho}\frac{dy}{d\rho} + \frac{d^2z}{d\omega\,d\rho}\frac{dz}{d\rho} = 0,$$

and, by these equations, $\frac{dx}{d\omega}\frac{dx}{d\rho} + \frac{dy}{d\omega}\frac{dy}{d\rho} + \frac{dz}{d\omega}\frac{dz}{d\rho}$ is shewn to be constant for all values of ρ, and is zero, when ρ is indefinitely diminished, in which case the geodesic circle is ultimately a plane circle; hence, all geodesic circles cut their radii at right angles.

750. In the case of a surface of revolution which intersects the axis in S, P is a function of ρ only. Draw PM perpendicular to the axis of revolution, and let $SM = x$, $MP = y$, $SP = s$, the normal and radius of curvature at P are y/x' and $-x'/y'$, therefore, by Art. 699, the product of the principal radii of curvature of the surface at P is $-y/y''$, and since $yd\omega$ is the elementary arc of the geodesic circle, $y = P$ and $s = \rho$, therefore $\frac{d^2P}{d\rho^2} + \frac{P}{RR'} = 0$, where R, R' are the principal radii of curvature. It will be shewn in Art. 757 that this equation holds for any surface.

751. *To find the measure of tortuosity at any point of a geodesic line.*

Take a point in the geodesic as origin, and refer the surface to the tangent plane and the planes of principal curvature, so that its equation assumes the form $2z = x^2/\rho_1 + y^2/\rho_2 + \ldots$; ρ_1, ρ_2 being the principal radii of curvature.

Let the geodesic make an angle θ with the axis of x at the origin O, and let ds be a small element of the geodesic whose extremities are O and P, so that ultimately $x = ds\cos\theta$, $y = ds\sin\theta$. Then, if $d\chi$ be the angle between the normals to the surface at O and P, $\cos d\chi = (1 + x^2/\rho_1^2 + y^2/\rho_2^2)^{-\frac{1}{2}}$, $\therefore \sin^2 d\chi = x^2/\rho_1^2 + y^2/\rho_2^2$ ultimately.

But, since the normals to the surface at all points of the geodesic are the principal normals of the geodesic, by Art. 647, and with the same notation, $(d\varepsilon)^2 + (d\tau)^2 = (d\chi)^2$, $\therefore \rho^{-2} + \sigma^{-2} = \cos^2\theta/\rho_1^2 + \sin^2\theta/\rho_2^2$,

and $\rho^{-1} = \cos^2\theta/\rho_1 + \sin^2\theta/\rho_2$, $\therefore \sigma^{-2} = \cos^2\theta\sin^2\theta(\rho_1^{-1} - \rho_2^{-1})^2$,

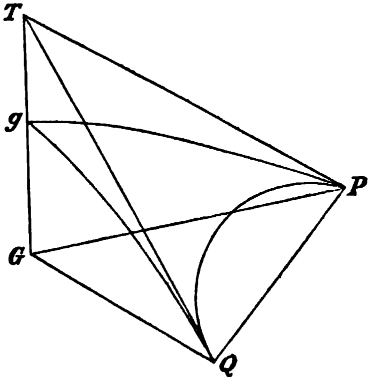
$$\text{or } \sigma^{-1} = \cos\theta\sin\theta\,(\rho_1^{-1} \sim \rho_2^{-1}).$$

Cor. A geodesic has no tortuosity at points where it touches a line of curvature, or at any umbilic when it passes through one.

752. Geodesic lines on a surface correspond to straight lines on a plane, and the geodesic which touches a curve traced on the surface is called a *geodesic tangent* to the curve at the point of contact. If geodesic tangents be drawn at the extremities of a small arc ds of a curve traced on a surface, and du be the angle at which they intersect, the ultimate value of du, when ds is indefinitely diminished, corresponds to the angle of contingence in a plane curve, and the limit of ds/du is the *geodesic radius of curvature.*

753. *To find the geodesic radius of curvature of a curve traced on a surface.*

Let PQ be the chord of a small arc of the curve, PT, QT tangents to the curve, which will also be tangents to the geodesic



tangents. Draw a plane through PQ parallel to the tangent plane at g, the intersection of the geodesic tangents Pg, Qg, and let PG, QG be the projections on this plane of PT and QT, therefore PTG, QTG are ultimately the osculating planes of the geodesic tangents, and du is the angle between PG produced and GQ, which is the projection of $d\epsilon$, the angle of contingence of the curve; hence, if ϕ be the angle between the osculating plane and the plane of the normal section containing the tangent line to the curve, which is ultimately the complement of the angle between the planes PTQ, PGQ, $PG.QG \sin du = PT.QT \sin d\epsilon \sin\phi$, therefore, since $PG=PT$ and $QG = QT$ ultimately,

$$\therefore du = d\epsilon \sin\phi, \text{ and } ds/du = \operatorname{cosec}\phi \, d\epsilon/ds = \rho \operatorname{cosec}\phi,$$

where ρ is the radius of curvature of the curve at P.

754. *Change of direction of motion on a surface.*

It is remarked in Thomson and Tait's *Natural Philosophy* that as, when a ship is sailing in a meridian or on the equator, her direction is said to be unchanged, so we ought to say that her direction is not

changed if she move on *any* great circle; and, since on a sphere, an arc of a great circle is a geodesic line, we see the connexion with the case of a plane polygon, when it is said that the *integral change* of direction of motion of a point, moving on a surface along the perimeter of a polygon whose sides are geodesic lines, is the sum of the exterior angles of the polygon.

755. *The integral change of direction for a closed curve on a surface is equal to that of the horograph.*

If tangent planes be drawn at every point of a closed curve traced on a surface they will envelope a torse, and if this torse be slit down one of its generating lines and developed, the trace of the original closed curve will be a curve cutting at the same angle the bounding lines which were coincident before the slitting process, and consecutive geodesic tangents to the closed curve will become rectilinear tangents to the curve on the developed torse.

Let $\beta_1, \beta_2 \ldots$ be the angles of successive facets of the enveloping tangent planes which ultimately form the torse, $\alpha, \alpha_1, \alpha_2 \ldots$ the angles which the tangents make with the generating lines, $u_1, u_2 \ldots$ the angles between the tangents, which are the same as the angles between the consecutive geodesic tangents on the surface.

Then $\alpha_1 + u_1 = \alpha + \beta_1$, $\alpha_2 + u_2 = \alpha_1 + \beta_2$, &c. and $\alpha_n = \alpha$, $\therefore \Sigma(u) = \Sigma(\beta)$.

Hence the integral change of direction for the closed curve is the sum of all the facets of the torse, which is clearly the same for all torses with parallel facets; and, therefore, for the torse which touches the unit sphere along the horograph, so that the integral change of direction for a closed curve on a surface is equal to that of the horograph.

This proof was suggested by Moulton, and leads to Gauss' proposition respecting a geodesic triangle on a surface.

756. *The excess of four right angles above the change of direction of a point moving round any closed curve on a surface is equal to the area of the horograph of the enclosed portion of the surface.*

The area of a closed polygon whose sides are arcs of great circles of a sphere of unit radius is the excess of 2π over the sum of the exterior angles; this is readily shewn by dividing the polygon into a number of triangles whose common vertex is a point within the polygon. If the number of sides be increased and their magnitude diminished indefinitely we shall have proved that the excess of 2π over the change of direction of a point moving round any closed curve on the sphere is equal to the area of the enclosed portion of the sphere, and by the last article the proposed theorem is true.

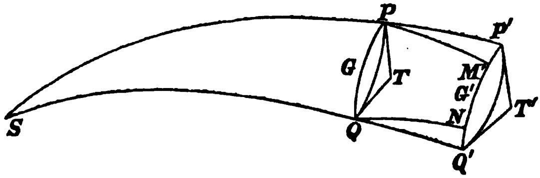
COR. 1. *If a closed geodesic be drawn on any closed surface without singular points, the integral curvature of the two parts into which the surface is divided will be equal.*

For there is no change of direction in motion round the curve, and therefore the area of the horograph must be 2π, or half the surface of the unit sphere.

Cor. 2. One of Gauss' theorems is a particular case of the proposition, viz. *that the excess over π of the sum of the angles of a geodesic triangle is the area of that portion of the unit sphere enclosed by the horograph.*

757. *To prove that* $\dfrac{d^2P}{d\rho^2}+\dfrac{P}{RR'}=0$ *when* $Pd\omega$ *is the length of an element of a geodesic circle.*

Let SPP', SQQ' be geodesic lines inclined at a small angle $d\omega$, PQ and $P'Q'$ arcs of geodesic circles whose radii are ρ and $\rho+d\rho$. PT, QT, and $P'T'$, $Q'T'$ geodesic tangents to these arcs, du the angle at which PT, TQ intersect.



By Art. 756, the excess of 2π over the change of direction in passing round $QSPT$ is equal to the arcs of the horograph corresponding to it, which is therefore $2\pi-(\pi-d\omega+\frac{1}{2}\pi+du+\frac{1}{2}\pi)$. But, ultimately, the difference of the areas of the horographs of $SPTQ$ and $SP'T'Q'$ is $PQQ'P'/RR'$, where R, R' are the principal radii of curvature at P; hence

$$Pd\omega d\rho/RR'=-\frac{d\,(du)}{d\rho}\,d\rho,\ \therefore P/RR'=-\frac{d}{d\rho}\left(\frac{du}{d\omega}\right). \quad (1)$$

Let PGQ, $P'G'Q'$ be geodesic lines, and draw PM, QN perpendicular to PGQ, meeting $P'G'Q'$ in M, N, then

$$du=\angle TPG+\angle TQG=\angle P'PM+\angle Q'QN=(P'Q'-PQ)/d\rho=d(Pd\omega)/d\rho;$$

$$\therefore \text{ by (1)},\quad \frac{d^2P}{d\rho^2}+\frac{P}{RR'}=0. \quad (2)$$

758. Another form of the measure of geodesic curvature arises from the equation (2).

For, by (1) and (2) $\dfrac{d}{d\rho}\left(\dfrac{du}{d\omega}\right)=\dfrac{d^2P}{d\rho^2}$, integrating and observing that the constant vanishes, since, when ρ is indefinitely diminished $P=\rho$ and $du/ds=\rho^{-1}$, we obtain $\dfrac{du}{ds}=\dfrac{dP}{Pd\rho}$.

In the case of a sphere of unit radius, $P = \sin\rho$, and for a small circle whose geodesic radius is ρ, the measure of geodesic curvature is $\cot\rho$, which is easily identified with the result of Art. 753.

759. *Geodesics joining two points on a circular cylinder.*

Let the equation of the cylinder be $x^2 + y^2 = a^2$, and let a, θ, z be the cylindrical coordinates of any point in a geodesic line joining two given points A and B, for which $\theta = 0$, $z = 0$ and $\theta = \alpha$, $z = c$ respectively.

By the equations of a geodesic $z'' = 0$ and $xy'' - yx'' = 0$, $\therefore$ z' and $a^2\theta'$ are constant, so that $ad\theta/dz$ is constant, that is, any geodesic cuts the generating lines at a constant angle β, and $z = a\theta\cot\beta$, where β is given by the equation $c\tan\beta = a\,(2n\pi + \alpha)$, n being any positive or negative integer or zero.

These equations have been obtained from the fundamental equations of geodesics, but it is obvious that we can wind a string round a cylinder in either direction as many times as we please, so as to start from A and pass through B, retaining its form when under tension, and the equations given above are easily deduced.

760. *Geodesics joining two given points on a right circular cone.*

Let the equation of the cone be $r^2 \equiv x^2 + y^2 = z^2\tan^2\alpha$, and let the cylindrical coordinates of the given points A, B be $a\sin\alpha$, 0, $a\cos\alpha$ and $b\sin\alpha$, $2n\pi + \beta$, $b\cos\alpha$, n being any positive or negative integer or zero.

The equations of the geodesic give

$$xy'' - yx'' = 0, \quad \therefore\ xy' - yx' = r^2\theta' = C$$

a constant quantity, and

$$1 = r'^2\operatorname{cosec}^2\alpha + r^2\theta'^2, \quad \therefore\ \operatorname{cosec}^2\alpha\,(dr/d\theta)^2 + r^2 = r^4/C^2,$$

the general solution of which is $r = C\sec(\theta\sin\alpha + D)$ and the constants are determined by the equations

$$a\sin\alpha = C\sec D \text{ and } b\sin\alpha = C\sec\{(2n\pi + \beta)\sin\alpha + D\},$$

whence the equation of one of the geodesics joining AB is given by

$$a^{-1}\sin\{(2n\pi + \beta - \theta)\sin\alpha\} + b^{-1}\sin(\theta\sin\alpha) = r^{-1}\sin\alpha\sin\{(2n\pi + \beta)\sin\alpha\},$$

If the cone, vertex V, be developed into a plane, and r', θ' be the polar coordinates of the point corresponding to r, θ, VA being the initial radius vector, $r'\sin\alpha = r$, $\theta' = \theta\sin\alpha$. The point B will occupy different positions corresponding to the number of sheets unwrapped, so that

$$AVB = \beta\sin\alpha = \beta',\ BVB_1 = B_1VB_2 = \ldots = 2\pi\sin\alpha = \delta,$$

and in the opposite direction $BVB'_1 = B'_1VB'_2 = \ldots = \delta$; the equations of the geodesics developed on the plane are thus given by

$$a^{-1}\sin(n\delta + \beta' - \theta') + b^{-1}\sin\theta' = r'^{-1}\sin(n\delta + \beta'),$$

shewing that the geodesics become straight lines joining A with B_1, B_2,... B_1', B_2'...

The number of geodesics is not infinite as in the case of the cylinder but is limited by the consideration that $\beta' + n\delta$ must be between $+\pi$ and $-\pi$ inclusive; the number therefore exceeds by unity the sum of the greatest integers in $(\pi - \beta')/\delta$ and $(\pi + \beta')/\delta$.

A geodesic on a right circular cone cannot have its two ends on the same generating line unless the vertical angle be less than 60°, for in that case $2\pi \sin\alpha$ must be less than π.

761. *Throughout a geodesic on a surface of revolution the distance of any point from the axis varies as the cosecant of the angle between the geodesic and the meridian.*

Let the equation of the surface be $z = f(x^2 + y^2)$, the axis of revolution being taken for that of z, $\therefore xq - yp = 0$, and for the geodesic $py'' - qx'' = 0$, $\therefore xy'' - yx'' = 0$, or $xy' - yx' = C$ a constant; thus if r, θ be cylindrical coordinates, $r^2 d\theta = C ds$, hence $r = C/r\theta'$, which represents the property stated.

762. *At every point of the same geodesic on a central conicoid pD is constant, where p is the perpendicular from the centre on the tangent plane, and D is the central radius parallel to the tangent to the geodesic.*

Let the equation of the conicoid be $ax^2 + by^2 + cz^2 = 1$, and (l, m, n) the direction of the tangent at any point of the geodesic,

$$\therefore\ l'/ax = m'/by = n'/cz = k \text{ suppose,} \qquad (1)$$

$$\text{also } axl + bym + czn = 0, \qquad (2)$$

$$a^2x^2 + b^2y^2 + c^2z^2 = p^{-2}, \qquad (3)$$

$$\text{and } al^2 + bm^2 + cn^2 = D^{-2}. \qquad (4)$$

Differentiating (2) and observing that $x' = l$, &c.

$$axl' + bym' + czn' + al^2 + bm^2 + cn^2 = 0,$$

$\therefore$ by (1), (3) and (4) $kp^{-2} + D^{-2} = 0$.

Again, differentiating (3) and (4),

$$a^2xl + b^2ym + c^2zn = \tfrac{1}{2}dp^{-2}/ds, \text{ and } k(a^2xl + b^2ym + c^2zn) = \tfrac{1}{2}dD^{-2}/ds;$$

$$\therefore\ D^{-2}dp^{-2} + p^{-2}dD^{-2} = d(p^{-2}D^{-2}) = 0, \text{ thus } pD \text{ is constant.}$$

This first integral of the equations is due to Joachimsthal.*

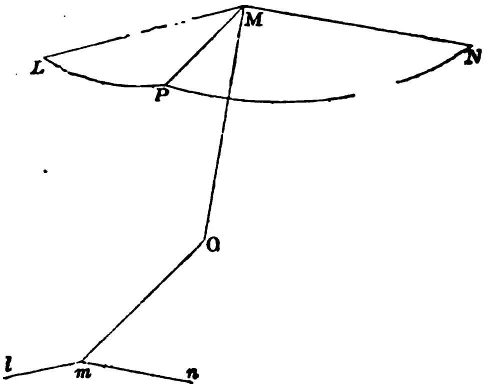
763. The important property given in the last article has been proved geometrically by Joyce† as follows.

Let LM, MN be tangents to a curve traced on a central conicoid at the points L and N, MP the intersection of the tangent planes to the surface at L and N, $LM + MN$ is ultimately equal to the arc between L and N, whose osculating plane is LMN. If this curve be a geodesic line, LMN will be shorter than for any curve whose osculating plane $LM'N$ meets PM in any other point, hence LM, MN make equal angles with PM and PM produced. Through the

* *Crelle*, XXVI.

† *Quar. Jour.* V. p. 265.

centre O of the conicoid draw Om a central radius parallel to the conjugate line MP, and draw ml, mn chords of the conicoid parallel to ML, MN. Then ml, mn will be ultimately tangents to the central



sections made by planes parallel to the tangent planes at L, N. Also, since Oml and Omn are supplementary angles, the perpendiculars on these tangents are equal; that is, if a plane be drawn through the centre parallel to the tangent plane at any point of a geodesic, and a tangent line be drawn to this section parallel to the tangent line to the curve, the perpendicular q on this tangent line will be of constant length. The area of the section will be πqD, and $8qpD$ will be the volume of a parallelepiped enveloping the conicoid and having its faces parallel to conjugate planes, and will therefore be constant; hence, since q is constant, pD will also be constant.

When the geodesic is perpendicular to the conjugate tangent, it touches at that point one of the lines of curvature, being in the direction of an axis of the indicatrix, Art. 690, and pD is obviously the same for both curves.

This proof holds equally for the lines of curvature, since along a line of curvature lm and mn are always perpendicular to Om.

764. The property that pD is constant at every point of a geodesic on a central conicoid has been replaced by another form expressed in terms of the primary axes of the confocals which intersect in any point of the geodesic.*

Let a point P of a geodesic on a central conicoid be determined by the confocals, the squares of whose primary semi-axes are $a^2-k_1^2$, $a^2-k_2^2$, then, Art. 296, k_1 and k_2 are the semi-axes of the central

* *Liouville*, IX, 401.

section by a plane parallel to the tangent plane to the conicoid at P. Let θ be the inclination of the geodesic to the line of curvature corresponding to k_2, D the radius of the central section will be given by $k_2^{-2}\cos^2\theta + k_1^{-2}\sin^2\theta = D^{-2}$, also $k_1k_2p = abc$, and $k_1k_2 = qD$, where q is the perpendicular from the centre on a tangent line parallel to the direction of the geodesic at P,

$$\therefore\ k_1^2\cos^2\theta + k_2^2\sin^2\theta = q^2,$$

which, assuming Joachimsthal's theorem, is a constant; or, if a_1, a_2 be the primary semi-axes of the confocals through P,

$$a_1^2\cos^2\theta + a_2^2\sin^2\theta = a^2 - q^2.$$

COR. Since at an umbilic $q = b$ it follows that $a_1^2\cos^2\theta + a_2^2\sin^2\theta$ has the same value for every umbilical geodesic.

765. Chasles has deduced the second form of the first integral independently of Joachimsthal's theorem by making use of the proposition of Art. 310. The osculating plane of a geodesic at P is taken for the plane U, therefore, if a be the primary semi-axis of the confocal touching the osculating plane, which contains the normal to the conicoid on which the geodesic lies, since $\theta_1 = \frac{1}{2}\pi$, $a_2^2\cos^2\theta_2 + a_3^2\cos^2\theta_3 = a^2$.

Hence, by Art. 310, since a consecutive point on the geodesic is also a point in U, for which $\theta_1 = \frac{1}{2}\pi$, $a_2^2\cos^2\theta_2 + a_3^2\cos^2\theta_3$ is unaltered, and remains the same throughout the geodesic.

766. *The constant pD has the same value for all geodesic tangents to the same line of curvature.*

For pD is constant throughout a line of curvature, and, at the point of contact with any geodesic, both p and D are the same for the two curves.

767. *Two geodesic tangents drawn to a line of curvature on a central conicoid make equal angles with the lines of curvature which pass through their point of intersection.**

For pD will be the same for both, and therefore at the point of intersection, since p is the same, D will also be the same, and then the axes of the central section will bisect the angle between the two directions of D.

768. The same proof as for plane confocals shews that, if two geodesic tangents be drawn to a line of curvature from a point on another line of curvature of the same system, the sum of the tangents will exceed the intercepted arc by a constant quantity.

769. *The locus of a point on an ellipsoid, the sum or difference of whose geodesic distances from two adjacent umbilics is constant, is a line of curvature.*

For, let U, V be adjacent umbilics, P any point on the ellipsoid for which the sum or difference of the geodesic distances PU, PV

* M. Roberts, *Liouville* XI. p. 1.

is constant, it is easily shewn by the theorem of Art. 747 that if Q be a point in the locus very near to P, PQ bisects the external or the internal angle between PU and PV, according as we consider the sum or difference constant. But in the geodesics PU, PV the value of pD is the same, being ac, the value at each umbilic, and p is the same for both geodesics at P, therefore the central radii parallel to the tangents to PU, PV are equal, and therefore equally inclined to the axes of the central section parallel to the tangent plane at P, which axes are parallel to the lines of curvature through P.

Thus the two lines of curvature through P bisect the angles between PU and PV, and the proposition is proved. Hence a mode of constructing lines of curvature is obtained similar to that by which an ellipse or hyperbola may be generated by means of a string fixed at the foci.

770. *All geodesics joining two opposite umbilics are of equal length.*

Let U, U' be opposite umbilics, V one of the umbilics between them; let a line of curvature through any point P of a geodesic UPU' meet the principal plane containing the umbilics in R, between V and U', then $PU+PV=RU+RV$, and $PU'-PV=RU'-RV$; $\therefore$ $PU+PU'=RU+RU'$, that is, UPU' is of constant length.

771. *Tangents to a geodesic on a central conicoid all touch the same confocal.*

Let θ be the angle made by the tangent at any point P of a geodesic on a central conicoid with a line of curvature through that point, then this tangent will touch only one confocal of the central conicoid, Art. 304; let α be the primary semi-axis of this confocal.

If a cone whose vertex is P envelope the conicoid (α), and l, m, n be the direction-cosines of one of its sides referred to the normals to the three confocals (a), (a_1), (a_2) passing through P, $l^2/(a^2-\alpha^2)+m^2/(a_1^2-\alpha^2)+n^2/(a_2^2-\alpha^2)=0$, Arts. 301 or 309. And the tangent at P to the geodesic is a side of this cone for which $l=0$, $m=\sin\theta$, $n=\cos\theta$, $\therefore a_1^2\cos^2\theta+a_2^2\sin^2\theta=\alpha^2$. Therefore, by Art. 764, α^2 is constant for all tangents to the same geodesic, being equal to a^2-q^2.

COR. 1. The plane of two consecutive tangents to the geodesic, since it contains two consecutive tangent lines to the confocal will be a tangent plane to that confocal. Hence *the osculating planes of a given geodesic on a central conicoid all touch the same confocal.*

COR. 2. The conicoid on which the geodesic lies and the confocal which is touched by all the tangents to the geodesic intersect in a line of curvature on each, and the geodesic touches this line, since, where they meet, the tangent to the geodesic is a common tangent to both surfaces.

772. The equation of the conicoid which is touched by the tangents to all the geodesics for which q or abc/pD has the same value is $x^2/(a^2-q^2)+y^2/(b^2-q^2)+z^2/(c^2-q^2)=1$, (1) and determines, by its intersection with the conicoid on which the geodesics lie, the line of curvature which is the envelope of all such geodesics.

If the geodesics be on an ellipsoid whose axes are AOA', BOB', COC', and U, V, U', V' be the umbilics in AC, CA', $A'C'$, $C'A$,

i. $b^2>q^2>c^2$, the equation (1) gives a hyperboloid of one sheet, and the line of curvature consists of two similar portions on opposite sides of the plane $ABA'B'$, both of which are touched by the geodesics, for which q is the constant parameter. When q is nearly equal to b, these two portions, lying in a very flat hyperboloid, are narrow curves nearly coincident with UCV and $U'C'V'$.

ii. $a^2>q^2>b^2$, (1) gives a hyperboloid of two sheets and the line of curvature consists of two portions on opposite sides of the plane $CBC'B'$. When q^2 is nearly equal to b^2 these become narrow curves nearly coincident with UAV' and $U'A'V$.

iii. $q=b$, (1) gives the focal hyperbola which is the limit of the edges of either of the flat hyperboloids mentioned above, the tangents to the geodesics are tangents to these flat hyperboloids, and therefore ultimately pass through the focal hyperbola.

In cases i. and ii. the system of geodesics for which q is constant, fill up the space between the two portions of the corresponding line of curvature. In case iii. they pass through the umbilics, the principal section containing the four umbilics being both a line of curvature and a geodesic line.

COR. *If tangents to an ellipsoid from any point in the focal hyperbola be produced geodesically, they will all pass through the opposite umbilics.*

773. *If A and B be two confocals of different species, and a string have its two extremities at two fixed points of B, then a pencil, whose point is on A, stretching the string so that it rests on both surfaces with rectilinear portions between, will trace on A a line of curvature.*

Let the string be fixed at two points O, O' on B, and let SQ, SQ' be the two curved portions of the string on A. Since the tangents to these geodesics both touch the same confocal B, the value of pD is the same for both, and therefore that of D at the point S; hence SQ and SQ' make equal angles with a line of curvature on A through S, and it follows, as in Art. 769, that S remains on that line of curvature as the pencil moves.

COR. The focal ellipse can be constructed by a pencil guided by a string fixed at two points on the focal hyperbola.

774. *The locus of the intersection of geodesic tangents to two lines of curvature on an ellipsoid, which cut at a constant angle, is the curve of intersection of the ellipsoid with any one of three distinct*

quartics, each of which is a surface of revolution round a corresponding principal axis of the ellipsoid.

Let $a^2-k_1^2$, $a^2-k_2^2$ be the squares of the primary semi-axes of the confocal hyperboloids of one and two sheets which pass through the point of intersection P, whose distance from the centre is r; then k_2, k_1 are the major and minor semi-axes of the central section of the ellipsoid parallel to the tangent plane at P. Let D, D' be the central radii parallel to the two geodesic tangents at P, inclined to the semi-axis k_2 at angles $\theta \mp \alpha$, whose difference is constant. Then, if q, q' be the perpendiculars from the centre on the tangents to the central section which are parallel to D and D', q and q' will be constant throughout the two lines of curvature and their respective geodesic tangents. Hence $k_1k_2 = qD = q'D'$, and

$$k_1^2 + k_2^2 = a^2 + b^2 + c^2 - r^2 \equiv u \text{ suppose};$$

$$\therefore\ k_1^2 \cos^2(\theta-\alpha) + k_2^2 \sin^2(\theta-\alpha) = q^2,$$

$$\text{whence}\quad (k_2^2 - k_1^2)\cos(2\theta - 2\alpha) = u - 2q^2.$$

$$\text{Similarly}\quad (k_2^2 - k_1^2)\cos(2\theta + 2\alpha) = u - 2q'^2;$$

$$\therefore\ (k_2^2 - k_1^2)^2 = (u - q^2 - q'^2)^2 \sec^2 2\alpha + (q'^2 - q^2)^2 \operatorname{cosec}^2 2\alpha$$

$$= (k_2^2 + k_1^2)^2 - 4k_2^2k_1^2 = u^2 - 4a^2b^2c^2/p^2,$$

and since it can be shewn that

$$a^2b^2c^2/p^2 = a^2(b^2 + c^2 - r^2) + (a^2 - b^2)(a^2 - c^2)\,x^2/a^2,$$

the equation is of the form $F(r^2, x^2) = 0$, therefore the locus lies on a quartic surface of revolution round the principal axis $2a$, and similarly for the other principal axes.

COR. 1. If the geodesics touch the same line of curvature the equation reduces to $\cos^2 2\alpha\ a^2b^2c^2/p^2 = (q^2 - u\sin^2\alpha)(u\cos^2\alpha - q^2)$.

COR. 2. If the geodesics cut at right angles the locus is a sphero-conic, the square of the radius of the sphere being $a^2+b^2+c^2-q^2-q'^2$.

This may be shewn independently of the general proposition, since, D and D' being perpendicular radii, $D^{-2} + D'^{-2} = k_2^{-2} + k_1^{-2}$; $\therefore\ q^2 + q'^2 = a^2 + b^2 + c^2 - r^2$.

Or, by adding the equations of the two geodesics, $k_1^2\cos^2\theta + k_2^2\sin^2\theta = q^2$, $k_1^2\sin^2\theta + k_2^2\cos^2\theta = q'^2$, we obtain the same result.

COR. 3. *The foot of the geodesic perpendicular from an umbilic on any geodesic tangent to a line of curvature lies on a sphero-conic.*

For any geodesic through an umbilic touches the limit of the lines of curvature for which $q = b$.

775. *To find the conditions that it may be possible to draw geodesic tangents to two lines of curvature on an ellipsoid which shall intersect at right angles.*

The surfaces $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, and $x^2 + y^2 + z^2 = a^2 + b^2 + c^2 - q^2 - q'^2$ intersect only when the hyperbolic cylinder

$$(a^2 - b^2)\,x^2/a^2 - (b^2 - c^2)\,z^2/c^2 = a^2 + c^2 - q^2 - q'^2$$

intersects the ellipsoid, which can happen only

when $q^2 + q'^2 - c^2$ lies between a^2 and b^2,

or $q^2 + q'^2 - a^2$ lies between b^2 and c^2.

776. *A torse circumscribing a conicoid along a geodesic line has its edge of regression on another conicoid.*

Each point of the edge of regression is the intersection of three consecutive tangent planes of the torse, which touch the enveloped conicoid A in three points on the geodesic, hence each point of the edge is the vertex of a cone enveloping A, and having an osculating plane of the geodesic for its plane of contact; and since, by Cor. 1, Art. 771, these planes all touch the same conicoid B, it follows that every point of the edge of regression is the pole with respect to A of a tangent plane to B and therefore lies on a fixed conicoid C.

For umbilical geodesics on an ellipsoid this fixed conicoid, being the polar with respect to the ellipsoid of $x^2/(a^2-b^2)-z^2/(b^2-c^2)=1$, reduces to the hyperbola $(a^2-b^2)x^2/a^4-(b^2-c^2)z^2/c^4=1$.

777. *Geodesic lines of paraboloids.*

Let (l, m, n) be the direction of the tangent to a geodesic line at a point (x, y, z) of a paraboloid whose equation is $y^2/b+z^2/c=2x$, (1) and let D be the length of the chord drawn through the vertex parallel to the tangent, p the perpendicular from the vertex on the tangent plane at (x, y, z).

By the equations of the geodesic

$$m'=-l'y/b, \quad n'=-l'z/c, \tag{2}$$

$$\text{also } x^2p^{-2}=y^2/b^2+z^2/c^2+1, \tag{3}$$

$$2lD^{-1}=m^2/b+n^2/c. \tag{4}$$

Differentiating (1) twice with respect to s, we have

$$my/b+nz/c=l, \quad \text{and } m^2/b+n^2/c+ym'/b+zn'/c=l';$$

$$\therefore \text{ by (2), (3) and (4)} \quad 2lD^{-1}=l'(1+y^2/b^2+z^2/c^2)=l'x^2p^{-2}.$$

Differentiating (3) and (4)

$$d(x^2p^{-2})/ds=2ym/b^2+2zn/c^2,$$

$$d(lD^{-1})/ds=mm'/b+nn'/c=-l'(ym/b^2+zn/c^2);$$

$$\therefore x^2p^{-2}d(lD^{-1})+lD^{-1}d(x^2p^{-2})=0;$$

$$\therefore lx^2/p^2D \text{ is constant,}$$

hence $(m^2/b+n^2/c)(1+y^2/b^2+z^2/c^2)=$ constant is one form of a first integral of the equations.

778. A second form of the first integral is derived from the confocals of the paraboloid.

The confocals which form the lines of curvature on the paraboloid are given by the equations $y^2/(b-k)+z^2/(c-k)=2x-k$, Art. 312, where k is either of the roots k', k'' of the equation

$$(c-k)y^2/b+(b-k)z^2/c+(b-k)(c-k)=0,$$

$$\text{or } k^2-(b+c+2x)k+(y^2/b^2+z^2/c^2+1)bc=0,$$

Referring to Arts. 242–244, we find that the semi-axes of a section parallel to the tangent plane at (x, y, z), whose equation is $\eta y/b+\zeta z/c-\xi=x$, are the two values of r given by

$$(r^2/\rho)^2-(b+c+2x)\,r^2/\rho+(y^2/b^2+z^2/c^2+1)\,bc=0,$$

hence, comparing this equation with the quadratic in k, the squares of the semi-axes of the section are $\rho k'$ and $\rho k''$.

If θ be the angle which the geodesic makes with the line of curvature corresponding to k', and D' be the corresponding radius of the section, $D'^2\cos^2\theta/\rho k'+D'^2\sin^2\theta/\rho k''=1$, but, by Art. 244,

$$(m^2/b+n^2/c)\,D'^2=\rho,$$

$$\therefore\ k''\cos^2\theta+k'\sin^2\theta=k'k''\rho/D'^2=bc\,(1+y^2/b^2+z^2/c^2)\,(m^2/b+n^2/c),$$

which is constant.

At an umbilic, $y=0,\ z^2=c\,(b-c),\ l^2/(b-c)=n^2/c,$

$$\therefore\ k''\cos^2\theta+k'\sin^2\theta=b.$$

779. The results of the preceding articles can be obtained from the corresponding properties of geodesics on central conicoids by transferring the origin to the extremity of the axis of x and making the axes infinite, so as to reduce the equation $x^2/a'^2+y^2/b'^2+z^2/c'^2=1$ to the form $y^2/b+z^2/c=2x$.

Thus, let $b'^2=a'b$ and $c'^2=a'c$, and the equation becomes

$$x^2/a'-2x+y^2/b+z^2/c=0,$$

which is of the required form when a' is infinite.

The condition $pD=$ constant makes

$$(l^2/a'^2+m^2/b'^2+n^2/c'^2)\,\{(x-a')^2/a'^4+y^2/b'^4+z^2/c'^4\}\text{ constant,}$$

$$\therefore\ (l^2/a'+m^2/b+n^2/c)\,\{(x-a')^2/a'^2+y^2/b^2+z^2/c^2)\},$$

or, making a' infinite, $(m^2/b+n^2/c)\,(1+y^2/b^2+z^2/c^2)$ is constant.

Again, if we write $x-a'$ for x and $a'k$ for k'^2 in the equation $x^2/(a'^2-k'^2)+\ldots=1$ of a confocal of the central conicoid, and make a' infinite, we obtain the confocal to the paraboloid, and also the constancy of $k''\cos^2\theta+k'\sin^2\theta$.

780. *Tangents to a geodesic on a paraboloid all touch the same confocal.*

Following the steps of the proof for central conicoids in Art. 771, let the equation of the confocal touched by a tangent at P to the geodesic be $y^2/(b-\alpha)+z^2/(c-\alpha)=2x-\alpha$; then, by Art. 318, the relation between l, m, n the direction cosines of a side of the cone enveloping this paraboloid referred to the normals to the three confocals passing through P, is $(l^2/(-\alpha)+m^2/(k'-\alpha)+n^2/(k''-\alpha)=0$, and for the tangent to the geodesic which is one of the sides $l=0$, $m=\cos\theta$, $n=\sin\theta$; $\therefore\ (k''-\alpha)\cos^2\theta+(k'-\alpha)\sin^2\theta=0$, and

$$k''\cos^2\theta+k'\sin^2\theta=\alpha,$$

which, by Art. 778, is constant for all the tangents to the sam geodesic.

781. *To find the locus of the intersection of geodesic tangents t two lines of curvature of a paraboloid at right angles to each other*

The geodesic tangents make angles θ, $\frac{1}{2}\pi-\theta$ with the lines c curvature, the parameters of which are α and α';

$$\therefore\ k''\cos^2\theta+k'\sin^2\theta=\alpha \text{ and } k''\sin^2\theta+k'\cos^2\theta=\alpha';$$

$$\therefore\ k''+k'=b+c+2x=\alpha+\alpha', \text{ Art. } 778;$$

the locus is therefore a section of the paraboloid by a plane per pendicular to its axis.

782. *A point of an ellipsoid and a line of curvature on it are projected on th plane of a circular section by lines parallel to the greatest or least axis; prov that the angle between geodesic tangents drawn from the point to the line c curvature is equal to the angle between the tangents from the projection of th point to that of the line of curvature.*

Let (x, y, z) be any point on the ellipsoid $x^2/a+y^2/b+z^2/c=1$, X, Y th coordinates of its projection by a line parallel to the axis of x on a plane c circular section whose inclination to the plane of xy is ω, the axis of Y coinciding with that of y, and the axis of X lying in the plane of the circula section, so that $y=Y$ and $z=X\sin\omega$; $\therefore\ x^2/a=1-Y^2/b-X^2\sin^2\omega/c$.

If the line of curvature be the intersection with the confocal $x^2/(a+\lambda)+\ldots=$ by substituting the above values of x, y, z in the equation

$$x^2/a(a+\lambda)+y^2/b(b+\lambda)+z^2/c(c+\lambda)=0,$$

and observing that $b(a-c)\sin^2\omega=c(a-b)$, the equation of the projection c the line of curvature will be reduced to

$$X^2/(c+\lambda)+Y^2/(b+\lambda)+b/(a-b)=0, \tag{1}$$

a conic, the focus of which is the projection of an umbilic.

Let k_1, k_2 be the elliptic coordinates, defined as in Art. 287, of the point fron which the geodesic tangents are drawn, then 2θ the angle between the tangent is given by

$$k_1\cos^2\theta+k_2\sin^2\theta=\lambda. \tag{2}$$

The projection of the point being (X', Y'), k_1, k_2 are the roots of th equation

$$X'^2/(c+k)+Y'^2/(b+k)+b/(a-b)=0, \tag{3}$$

and if 2ϕ be the angle between the tangents from (X', Y') to the conic

$$X^2/(c+\lambda)+Y^2/(b+\lambda)+b/(a-b)=0,$$

writing d for $b/(a-b)$,

$$\{X'^2+Y'^2+d(b+c+2\lambda)\}^2\tan^2 2\phi=-4d\{X'^2(b+\lambda)+Y'^2(c+\lambda)+d(b+\lambda)(c+\lambda)\}$$

But, by (3),

$$X'^2(b+k)+Y'^2(c+k)+d(b+k)(c+k)\equiv d(k-k_1)(k-k_2);$$

$$\therefore\ X'^2+Y'^2+d(b+c)=-d(k_1+k_2),$$

$$\text{and } X'^2(b+\lambda)+Y'^2(c+\lambda)+d(b+\lambda)(c+\lambda)=d(\lambda-k_1)(\lambda-k_2);$$

$$\therefore\ \tan^2 2\phi=-4(\lambda-k_1)(\lambda-k_2)/(2\lambda-k_1-k_2)^2,$$

$$\text{and } \sec^2 2\phi=(k_1-k_2)^2/(2\lambda-k_1-k_2)^2;$$

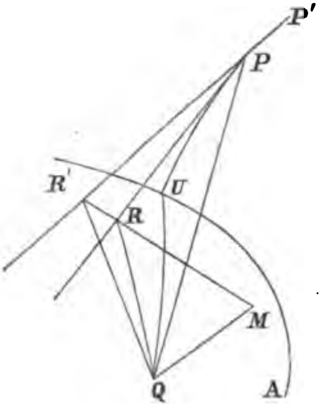
$$\therefore\ 2\lambda-k_1-k_2=\pm(k_1-k_2)(\cos^2\phi-\sin^2\phi),$$

$$\text{whence } k_1\cos^2\phi+k_2\sin^2\phi=\lambda, \text{ or } k_1\sin^2\phi+k_2\cos^2\phi=\lambda,$$

therefore, by (2), $\phi=\theta$ or $\frac{1}{2}\pi-\theta$, which proves the theorem.

783. *Umbilical geodesics on an ellipsoid.*

Let the tangent QP to an umbilical geodesic UQ intersect the focal hyperbola in P, and let PR be a tangent to the hyperbola at P; QP is a side of the circular cone enveloping the ellipsoid, the axis of which is PR in the plane of the umbilics, Art. 322, and the normal at Q intersects PR, so that QPR is the osculating plane of the geodesic at Q. Draw QM, QR perpendicular to the



plane of the umbilics and to PR, then the angle QRM is the angle of inclination of the osculating plane to the plane of the umbilics, measured in the figure towards the extremity of the primary axis nearest to U. The osculating plane turns round QP to its next position QPR', $P'PR'$ being a consecutive tangent to the focal hyperbola, and its inclination is then $QR'M$ ultimately, R' being the intersection of MR and $P'R'$.

Let θ, $\theta + d\theta$ be these inclinations, and α, $\alpha + d\alpha$ the semi-vertical angles of the corresponding cones, let ϕ, $\phi + d\phi$ be the inclinations of PR, $P'R'$ to the primary axis of the ellipsoid. Then $-d\theta = \angle RQR' = RR' \sin\theta / QR$ and $-d\phi = \angle RPR' = RR'/PR$, and $QR = PR \tan\alpha$, $\therefore d\theta/\sin\theta = d\phi \cot\alpha$.

Let $x^2/a + \ldots = 1$ and $x^2/(a+k) + \ldots = 1$ be the equations of the ellipsoid on which the geodesic lies and of the confocal ellipsoid through P, and let p, p' be the perpendiculars on the normal and tangent to the focal hyperbola at P which is an umbilic of the confocal ellipsoid, the squares of whose coordinates are

$$(a+k)(a-b)/(a-c),\ 0,\ \text{and}\ (c+k)(b-c)/(a-c);$$

$$\therefore p^2 = (a+k)(c+k)/(b+k) \text{ and } p'^2 = (a-b)(b-c)/(b+k),$$

$$\text{also } p = dp'/d\phi; \quad \therefore d\theta/\sin\theta = dp' \cot\alpha/p,$$

and, by Art. 324, $\tan^2\alpha = b/k$,

$$\therefore \frac{d\theta}{\sin\theta} = -\frac{\sqrt{(a-b)}\sqrt{(b-c)}\sqrt{k}\,dk}{2(b+k)\sqrt{b}\sqrt{(a+k)}\sqrt{(c+k)}},$$

where θ and α diminish when k increases, and if ω be the value of θ at the umbilic U,

$$\text{hence } \log\frac{\tan\frac{1}{2}\omega}{\tan\frac{1}{2}\theta} = \int_0^k \frac{\sqrt{(a-b)}\,\sqrt{(b-c)}\,\sqrt{k}\,dk}{2\,(b+k)\,\sqrt{b}\,\sqrt{(a+k)}\,\sqrt{(c+k)}}. \quad (1)$$

Since $k = b\cot^2\alpha$, the integral (1) is easily reduced to the form

$$\int_\alpha^{\frac{1}{2}\pi} \frac{\sqrt{(a-b)}\,\sqrt{(b-c)}\,d\alpha}{\sqrt{(b+a\tan^2\alpha)}\,\sqrt{(b+c\tan^2\alpha)}}, \quad (2)$$

or, if ρ be the primary semi-axis of the confocal ellipsoid, to the form

$$\frac{\sqrt{(a-b)}\,\sqrt{(b-c)}}{b}\int_{\sqrt{a}}^{\rho} \frac{d\rho\,\sqrt{(\rho^2-a)}}{\sqrt{(\rho^2-a+b)\,(\rho^2-a+c)}},$$

784. Let $\log m$ be the value of the definite integral (1) taken from 0 to ∞, or of (2) from 0 to $\frac{1}{2}\pi$, and let θ_1 be the value of θ at the point where the geodesic runs parallel to the asymptote of the focal hyperbola, then $\tan\frac{1}{2}\omega = m\tan\frac{1}{2}\theta_1$ (3); if we work backwards from the opposite umbilic U', at which $\theta = \omega'$, to this point we must write in the formula (3) $\pi-\theta_1$ and $\pi-\omega'$ for θ_1 and ω, therefore $\cot\frac{1}{2}\omega' = m\cot\frac{1}{2}\theta_1$ and so $\tan\frac{1}{2}\omega = m^2\tan\frac{1}{2}\omega'$; also, if ω'' be the angle at which the geodesic passes through U a second time $\tan\frac{1}{2}\omega' = m^2\tan\frac{1}{2}\omega''$. An umbilical geodesic will therefore pass and repass the opposite umbilics, making with the plane of the umbilics a series of angles such that the tangents of their halves are in geometrical progression.

785. The points where the geodesics are parallel to an asymptote of the focal hyperbola are in the curve of contact of the circular cylinder which is the limit of the circular cone when its vertex passes to infinity; the radius of this cylinder is b, and the geodesic crosses the curve at a point whose distance from the plane of umbilics is $b\sin\theta_1$. If β be the value of ω for the particular geodesic which passes through an extremity of the mean axis, for which $\theta_1 = \frac{1}{2}\pi$, then $m = \tan\frac{1}{2}\beta$, and θ_1 is determined by the equation $b\sin\theta_1 = b\sin\beta\sin\omega/(1-\cos\beta\cos\omega)$.

For the geodesic which leaves the umbilic at right angles to the plane of umbilics $\tan\frac{1}{2}\theta_1 = \cot\frac{1}{2}\beta$, $\therefore\ \theta_1 = \pi-\beta$.

786. *To find the point at which a particular umbilical geodesic meets the curve of contact of the cone whose vertex is a given point on the focal hyperbola.*

Using the same notation as in Art. 783, let the geodesic be particularized by the angle ω at which it leaves the umbilic. By Art. 302, the equation of the plane of contact referred to the normals to the confocals through the given point P is $px/k + p'z/b = 1$, or since $x = r\cos\alpha$, $z = r\sin\alpha\cos\theta$, where r is the length of the tangent from P to the geodesic, $r^{-1} = p\cos\alpha/k + p'\sin\alpha\cos\theta/b$, (1) whence r is known, since θ, p, p' and α are all given in terms of k.

Again, if y be the distance of the point from the plane of umbilics, $y = r \sin\alpha \sin\theta$, therefore by (1), since $\tan^2\alpha = b/k$, $b \sin\theta / y = p \tan\alpha + p' \cos\theta$,

$$\text{or } \tan^2 \tfrac{1}{2}\theta \,(p \tan\alpha - p') - 2 \tan \tfrac{1}{2}\theta \, b/y + p \tan\alpha + p' = 0. \quad (2)$$

It follows from (2) that if ω, ω' correspond to two geodesics which cut the plane of contact at points which are the same distance from the plane of umbilics $\tan \frac{1}{2}\omega \tan \frac{1}{2}\omega'$ is constant.

The cotangent of the angle which the geodesic (ω) makes with the curve of contact is $dr/r \sin\alpha d\theta$, which, by (1), is equal to $rp' \sin\theta / b = yp' / b \sin\alpha$.

787. If P be a point in the focal hyperbola which meets the ellipsoid, on which the geodesic lies, in U, and the confocal ellipsoid which passes through P be intersected by a tangent plane at U, it has been shown in Art. 354, and could be shown directly from the equations, that U will be a focus of the section; also, if PQ, a side of the circular cone enveloping the ellipsoid, be a tangent at Q to the umbilical geodesic UQ, then Q', the point in the confocal ellipsoid which corresponds to Q, will be a point in the section of which U is a focus, Art. 334, and since P corresponds to U, by Art. 332, $UQ' = PQ$, which is Ivory's theorem, and since every side of the cone is equal to the focal distance of the corresponding point, it follows that the equation (1) of Art. 786 is the polar equation of the section, which may be written in the form

$$\frac{1}{r} = \int \frac{(a+k)(c+k)\,dk}{k\,(b+k)^2} + \cos\theta \int \frac{(a-b)(b-c)\,dk}{b\,(b+k)^2}.$$

Hence the inclination of the osculating plane of an umbilical geodesic to the plane of umbilics is equal to that of UQ' to the axis of the ellipse.

788. *To find the element $Pd\omega$ of the arc of a geodesic circle whose radii are umbilical geodesics inclined at an angle $d\omega$.*

Let PQ, PQ' be tangents to two neighbouring geodesics from P in the focal hyperbola, then since k is constant, $\log\tan \frac{1}{2}\omega - \log \tan \frac{1}{2}\theta$ is constant for Q and Q', $\therefore d\omega / \sin\omega = d\theta / \sin\theta$, and if QM be perpendicular on PQ', $Pd\omega = QM = r \sin\alpha d\theta$ ultimately;

$$\therefore P = r \sin\alpha \sin\theta / \sin\omega = y / \sin\omega.$$

Cor. 1. If U, V be neighbouring umbilics, and $UQ = \rho$, $VQ = \rho'$ be the geodesic distances of a point Q, let (P', ω') for VQ correspond to (P, ω) for UQ, then $P \sin\omega = P' \sin\omega'$.

Cor. 2. If QQ' be a small arc of a line of curvature, the perpendiculars from Q on UQ' and VQ' will be ultimately equal; $\therefore Pd\omega \pm P'd\omega' = 0$, $\pm$ as the line of curvature bisects the angle UQV externally or internally; $\therefore d\omega / \sin\omega \pm d\omega' / \sin\omega' = 0$, so that throughout a line of curvature $\tan \frac{1}{2}\omega \tan \frac{1}{2}\omega'$ or $\tan \frac{1}{2}\omega \cot \frac{1}{2}\omega'$ is constant.

789. *To find the length of an element of a curve traced on an ellipsoid, the elliptic coordinates of whose extremities are μ, ν and $\mu + d\mu$, $\nu + d\nu$.*

The four lines of curvature corresponding to constant values of these coordinates form an infinitesimal rectangle of which the element is a diagonal, and if (x, y, z) be the point (μ, ν) using the notation of Art. 287,

$$-\beta\gamma x^2 = a(a+\mu)(a+\nu)\,; \;\therefore\; 2dx/x = d\mu/(a+\mu) + d\nu/(a+\nu),$$

$$-4\beta\gamma (dx)^2 = a(a+\mu)(a+\nu)\{d\mu/(a+\mu) + d\nu/(a+\nu)\}^2\,;$$

similarly for $(dy)^2$ and $(dz)^2$,

$$\therefore\; 4ds^2 = M(d\mu)^2 + 2R\,d\mu\,d\nu + N(d\nu)^2,$$

where $M = \dfrac{a(a+\nu)}{(a-b)(a-c)(a+\mu)} + \dfrac{b(b+\nu)}{(b-a)(b-c)(b+\mu)} + \dfrac{c(c+\nu)}{(c-a)(c-b)(c+\mu)}$,

which are the partial fractions of $\dfrac{\mu(\mu-\nu)}{(a+\mu)(b+\mu)(c+\mu)}$, and similarly for N;

$$\text{also } R = -(a\alpha + b\beta + c\gamma)/\alpha\beta\gamma = 0.$$

$$\text{Hence } (ds)^2 = \frac{\mu(\mu-\nu)(d\mu)^2}{4(a+\mu)(b+\mu)(c+\mu)} + \frac{\nu(\nu-\mu)(d\nu)^2}{4(a+\nu)(b+\nu)(c+\mu)}.$$

The two terms in the expression for $(ds)^2$ are the squares of elements ds_1 and ds_2 of the lines of curvature $\nu = \text{constant}$ and $\mu = \text{constant}$ respectively. Hence, if θ be the inclination of the curve to the line $\mu = \text{constant}$,

$$\tan^2\theta = \frac{\mu(d\mu)^2}{-\nu(d\nu)^2} \cdot \frac{(a+\nu)(b+\nu)(c+\nu)}{(a+\mu)(b+\mu)(c+\mu)}.$$

790. A first integral of a geodesic on an ellipsoid is

$$\mu\cos^2\theta + \nu\sin^2\theta + k = 0\,; \;\therefore\; \tan^2\theta = -(\mu+k)/(\nu+k),$$

hence, by the last article,

$$d\mu\sqrt{\frac{\mu}{(a+\mu)(b+\mu)(c+\mu)(k+\mu)}} \pm d\nu\sqrt{\frac{\nu}{(a+\nu)(b+\nu)(c+\nu)(k+\nu)}} = 0\,;$$

and since the variables are separated, this gives rise to the complete integral of the differential equations of a geodesic on an ellipsoid.

791. *To find the length of any portion of an umbilical geodesic.*

Let ds_2, ds_1 be the elements of the lines of curvature $\mu = \text{constant}$ and $\nu = \text{constant}$, corresponding to the element $d\rho$ of the geodesic inclined at an angle θ to the former;

$$\therefore\; d\rho = ds_2\cos\theta + ds_1\sin\theta, \text{ and } \mu\cos^2\theta + \nu\sin^2\theta + b = 0,$$

$$\therefore\; \cos^2\theta = (b+\nu)/(\nu-\mu) \text{ and } \sin^2\theta = (b+\mu)/(\mu-\nu)\,;$$

$$\therefore\; d\rho = \tfrac{1}{2}d\nu\sqrt{\frac{\nu}{(a+\nu)(c+\nu)}} + \tfrac{1}{2}d\mu\sqrt{\frac{\mu}{(a+\mu)(c+\mu)}}.$$

LIII.

(1) On a right circular cone, whose vertical angle is $\cos^{-1}\frac{1}{3}$, two points are joined by a geodesic which completely surrounds the cone; prove that the two tangents to this geodesic at the double point are at right angles to each other.

(2) If two surfaces touch each other along a curve, and if the curve be a geodesic line on one surface, prove that it will also be a geodesic line on the other surface.

(3) The umbilical geodesics through the extremities of the mean axes of a system of confocal ellipsoids all touch one or other of two planes.

(4) The geodesic distance of the extremity of the mean axis of an ellipsoid from an umbilic is equal to the perimeter of a quadrant of the ellipse passing through the four umbilics.

(5) If ρ^{-1}, σ^{-1}, be the measures of curvature and tortuosity at any point of a geodesic, and ρ_1, ρ_2 the principal radii of curvature at the corresponding point of the surface, prove that $\sigma^{-2} + (\rho^{-1} - \rho_1^{-1})(\rho^{-1} - \rho_2^{-1}) = 0$.

(6) A geodesic is drawn on a torse, and cuts any generating line of the surface at an angle ψ, and at a distance t from the edge of regression measured along the generator, ρ is the radius of curvature of the edge of regression at the point where the generator touches it; prove that $dt/d\psi + t\cot\psi = \rho$.

(7) The angle between the geodesics passing through a point (x, y, z) on the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, and through the umbilics, is given by the equation $b^2(x^2 + y^2 + z^2 - a^2 + b^2 - c^2)^2 \tan^2\theta = 4(b^2 - c^2)(a^2 - b^2)y^2$.

(8) The ratio of the radii of torsion and curvature in a geodesic line on a torse at any point is equal to the tangent of the inclination of the curve to the corresponding generating line.

LIV.

(1) If a geodesic on a torse be a plane curve, it must be one of the generators, or else the torse must be a cylinder.

(2) If a geodesic be drawn on an ellipsoid from an umbilic to an extremity of the mean axis, prove that its radius of torsion at the latter point will be $a^2c^2/\{b\sqrt{(a^2 - b^2)}\sqrt{(b^2 - c^2)}\}$.

(3) If an ellipsoid be such that its central circular sections pass each through a pair of umbilics, prove that the geodesics drawn from two adjacent umbilics to any point in either section will be at right angles.

(4) When an umbilical geodesic which passes through an extremity of the mean axis is produced until it meets a second time the curve of contact of the cylindrical envelope whose axis is parallel to an asymptote of the focal hyperbola, prove that at that point the inclination of the osculating plane to the plane of the umbilics is $2\cot^{-1}m^2$, with the notation of Art. 784.

(5) AB, AC are geodesic lines on a surface, ab, ac corresponding lines on the horograph, shew that, if torses touch the surface along AB and AC, the angle between the two generating lines which pass through A is equal to the angle between the tangents at a to ab and ac.

(6) If P be a point on a given line of curvature of a conicoid, PQ, PQ' geodesic tangents to another given line of curvature of the same system. PU, PV geodesics to two adjacent umbilics, prove that the ratio $\cos\frac{1}{2}QPQ' : \cos\frac{1}{2}UPV$ will be constant.

(7) D is the semi-diameter of an ellipsoid parallel to the tangent at a point P of a geodesic traced upon it, σ is the radius of torsion at P, A is the area of the central section parallel to the tangent plane at P, A' the projection upon the osculating plane of the geodesic of the central section conjugate to D; prove that $\sigma = AD/A'$.

(8) Prove that, if ψ be the inclination of a geodesic line to a generating line of the helicoid whose equations are $x = r\cos\theta$, $y = r\sin\theta$, $z = a\theta$, and ϕ be the inclination of the tangent plane at the corresponding point to the axis of z, $\sin\psi\sec\phi$ will be constant.

(9) If the geodesic on a surface lie on a sphere, the radius of curvature of the geodesic at any point will be equal to the perpendicular from the centre of the sphere on the tangent plane to the surface.

(10) The sides of a geodesic triangle traced on a surface of revolution make with the meridians which pass through the angular points six angles; prove that the product of the sines of the three angles not adjacent is equal to the product of the three others.

LV.

(1) The angle between the geodesics passing through the point (x, y, z) of the paraboloid $y^2/b + z^2/c = 2x$, and through the umbilici, is given by the equation $b(b-c-2x)^2\tan^2\theta = 4(b-c)y^2$, b being greater than c.

(2) If λ be the parameter of a geodesic on the conicoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, and σ be the radius of torsion of the geodesic at a point (x, y, z), prove that $a^4b^4c^4\sigma^{-2} = \lambda^2p^6(a^2+b^2+c^2-x^2-y^2-z^2-\lambda^2) - a^2b^2c^2p^4$, p being the central distance of the tangent plane.

(3) If a line of curvature on an ellipsoid be the intersection with a hyperboloid, one of whose principal sections is a rectangular hyperbola, no geodesic tangents, which are at right angles, can be drawn to it, except from the extremities of one of the axes.

(4) Prove that the projection upon the plane xy of a geodesic on the surface $ar^2 \equiv z(x^2+y^2) = a^3$ has the equation $c^2r^2/p^2 = (r^6+4a^4c^2)/(r^6+4a^6)$.

(5) P is any point on a geodesic AP drawn on a conoidal surface, s is the length of the geodesic AP, σ, N, O are the projections of s, P, and the axis of the conoid on a plane perpendicular to the axis. Prove that, if q be the projection of ON on the tangent to the geodesic at P, $dq/d\sigma = d\sigma/ds$.

(6) If a geodesic bisect the angle between the lines of curvature on a central conicoid at every point along it, prove that the sum of the principal curvatures will vary as the cube of the perpendicular on the tangent plane at any point of the geodesic.

(7) If ABC be a geodesic triangle on a conoidal helicoid, and (α_1, α_2), (β_1, β_2), (γ_1, γ_2) be the angles at which the sides cut the generating lines through A, B, C, prove that $\sin\alpha_1\sin\beta_1\sin\gamma_1 = \sin\alpha_2\sin\beta_2\sin\gamma_2$.

(8) The intersection of a tangent to a given geodesic on a central conicoid with a tangent plane to which it is perpendicular lies on a sphere.

LVI.

(1) A geodesic is traced on any conical surface, and any involute of the geodesic is taken, prove i, that it lies on a sphere whose centre is at the vertex of the cone, ii, that each tangent to the geodesic makes a constant angle with the corresponding radius of the sphere, iii, that the radius of curvature of the geodesic is perpendicular to the radius of the sphere.

(2) At every point of a geodesic on a conicoid for which $pd = \lambda^2$, and perpendicular to it, lines are drawn touching the conicoid.

Prove that the surface generated by these lines is a scroll, whose equation is given by eliminating x, y, z from the equations $\xi x/a^2 + \eta y/b^2 + \zeta z/c^2 = 1$, $\{(\xi - x)^2 + \ldots\} a^2b^2c^2/\lambda^4 = a^2(\xi - x)^2 + \ldots - \{(\xi - x)x + \ldots\}^2$, and those of the geodesic.

If two planes be drawn perpendicular to the tangent to the geodesic at P, one through the centre and the other touching the conicoid, p', p'' being the distances of P from the first and of the centre from the second, $p''^2 - p'^2 = (a^2b^2c^2)/\lambda^4$.

(3) The angle of geodesic contingence of a curve traced on a surface of revolution $= d(r \sin\theta)/r \cos\theta$, where θ is the angle made with a meridian, and r the radius of the corresponding parallel.

(4) The curve $s = c \tan^2\psi$ revolves about a straight line in its plane perpendicular to its axis, at a distance $2c$ from the cusp, not cutting the curve. Shew that the polar equation of the projection of the geodesics on a plane perpendicular to the axis of revolution is $r = a \cosh(2c\theta/a)$.

(5) A surface is formed by the revolution of a catenary about its directrix; prove that the projection on a plane perpendicular to the axis of revolution of a geodesic upon the surface is given by the equation $\rho \operatorname{sn}(\phi/k, k) = c$.

(6) Find the differential equation of the projection of a geodesic on a spheroid upon a plane of circular section, and shew that it is the same curve which the base of a right elliptic cone would trace if the cone rolled on a plane.

(7) The meridian curve of a surface is the roulette of the focus of an ellipse rolling on the axis of revolution, α, β are the greatest and least values of the focal distance; prove that the projection of a geodesic on a plane perpendicular to the axis is given by the polar equation $r^{-2} = \alpha^{-2}\operatorname{cn}^2(\mu\theta) + \beta^{-2}\operatorname{sn}^2(\mu\theta)$. Shew that if the geodesic cut the meridian at its maximum distance from the axis at an angle γ, then $\mu = \beta \cot\gamma/(\alpha + \beta)$ and $k^2 = (\alpha^2/\beta^2 - 1)\tan^2\gamma$.

(8) The polar equation of the projection of any geodesic of a helicoid on its principal plane is of one of the forms

$$r \operatorname{tn}\{(\alpha - \theta)/k\} = m, \quad \text{or} \quad r \operatorname{cn}(\alpha - \theta) = mk'/k, \text{ mod. } k,$$

m being the pitch of the helicoid, and α, k arbitrary constants.

CHAPTER XXIV.

CURVILINEAR COORDINATES. CURVES TRACED ON SURFACES. DEFORMATION OF INEXTENSIBLE SURFACES.

792. In this chapter we give an account of the method of dealing with the geometry of surfaces by curvilinear coordinates employed by Gauss, and we call attention to work done by Liouville, Cayley, Bour, and Maxwell in the general treatment of the subject, but especially in establishing propositions concerning deformation of inextensible surfaces, and the applicability of such surfaces to one another.

793. The coordinates x, y, z of a point are regarded as functions of two parameters p and q; by eliminating p and q from the three equations, we should obtain the equation of a surface S on which the point must move as the parameters change their values. When a constant value is given to q, we can, by eliminating p, obtain two equations which will determine a curve lying on the surface S; thus the surface can be striated by a series of curves corresponding to a series of constant values of q.

In the same manner S can be striated by a second series of curves corresponding to constant values of p. These curves can be conveniently described as the curves (q) and (p) respectively.

Every point on S will be the intersection of two curves corresponding to certain values of p and q, which are therefore *curvilinear coordinates* of the point considered.

794. *To express the length of an element of a curve traced on a surface in terms of curvilinear coordinates.*

Let PQ be a small arc of any curve passing through the point P, the values of the parameters being p, q at P, and $p + dp$, $q + dq$ at Q, and let PM, NQ be the curves for which q and $q + dq$ respectively are constant, PN, MQ those for which p and $p + dp$ are constant.

Suppose x, y, z the coordinates of P to be expressed as functions of the two parameters p and q, and let the partial differential coefficients of x with respect to p and q be denoted, those of the first order by a and a', and those of the second order by α, α', α'', and let a similar notation be used for y and z. If δx, δy, δz be the projections of PQ on the coordinate axes, we shall have,

neglecting terms of the third order,

$$\delta x = dx + \tfrac{1}{2}d^2x = a\,dp + a'dq + \tfrac{1}{2}\{\alpha\,(dp)^2 + 2\alpha' dp\,dq + \alpha''\,(dq)^2\},$$

$$\delta y = dy + \tfrac{1}{2}d^2y = b\,dp + b'dq + \tfrac{1}{2}\{\beta\,(dp)^2 + 2\beta' dp\,dq + \beta''\,(dq)^2\},$$

and $$\delta z = dz + \tfrac{1}{2}d^2z = c\,dp + c'dq + \tfrac{1}{2}\{\gamma\,(dp)^2 + 2\gamma' dp\,dq + \gamma''\,(dq)^2\};$$

eliminating dp and dq, we have ultimately $A\,dx + B\,dy + C\,dz = 0$, where $A = bc' - b'c$, &c., and this is the differential equation of the surface. For the length of the element of the curve, if $PQ = ds$,

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = E\,(dp)^2 + 2F\,dp\,dq + G\,(dq)^2,$$

where $E = a^2 + b^2 + c^2$, $G = a'^2 + b'^2 + c'^2$, $F = aa' + bb' + cc'$, also $EG - F^2 = A^2 + B^2 + C^2 = V^2$ suppose.

It should be observed that, since for PM $dq = 0$, $PM = dp\sqrt{E}$, and that $a/\sqrt{E}$, $b/\sqrt{E}$, c/E are the direction cosines of the tangent to PM at P, also that A/V, B/V, C/V are those of the normal to the surface at P.

795. *To express the direction of a curve at any point by curvilinear coordinates.*

Let ds_1, ds_2 denote the elements PN and PM of the curves (p) and (q) which are the sides of the infinitesimal parallelogram of which PQ $(= ds)$ is the diagonal, then if θ, ω be the angles QPN, NPM, since $ds_2 = dp\sqrt{E}$ and $ds_1 = dq\sqrt{G}$,

also $$(ds_2)^2 + 2ds_1 ds_2 \cos\omega + (ds_1)^2 = E\,(dp)^2 + 2F\,dp\,dq + G\,(dq)^2;$$

$\therefore \cos\omega = F/\sqrt{(EG)}$ and, by the triangle QPM,

$$ds\cos\theta = ds_1 + ds_2\cos\omega = dq\sqrt{G} + dp\,F/\sqrt{G},$$

$$ds\sin\theta = \sqrt{\{(ds)^2 - (dq\sqrt{G} + dp\,F/\sqrt{G})^2\}} = dp\,V/\sqrt{G}.$$

If the two systems of curves be orthotomic, $dq\sqrt{G}\tan\theta = dp\sqrt{E}$.

COR. For geodesic polar coordinates, with the notation of Art. 749, $p = \rho$ and $q = \omega$; therefore, since $(ds)^2 = (d\rho)^2 + (P\,d\omega)^2$, $E = 1$, $F = 0$, $G = P^2$.

796. *To express the radii and directions of principal curvature by curvilinear coordinates.*

Let R be the radius of curvature of the normal section corresponding to the direction $dp : dq$ of the element PQ; QT, the perpendicular from Q on the tangent plane at P,

$$= (A\,\delta x + B\,\delta y + C\,\delta z)/V = \tfrac{1}{2}\{E'\,(dp)^2 + 2F' dp\,dq + G'\,(dq)^2\}/V,$$

where E', F' and G' are written for $A\alpha + B\beta + C\gamma$, $A\alpha' + B\beta' + C\gamma'$, and $A\alpha'' + B\beta'' + C\gamma''$, and $R = PQ^2/2\,QT$ ultimately;

$$\therefore \frac{R}{V} = \frac{E\,(dp)^2 + 2F\,dp\,dq + G\,(dq)^2}{E'\,(dp)^2 + 2F' dp\,dq + G'\,(dq)^2}. \quad (1)$$

Making R a maximum or minimum by the variations of $dp : dq$ we obtain for the radii and directions of principal curvature

$$\frac{R}{V} = \frac{Edp + Fdq}{E'dp + F'dq} = \frac{Fdp + Gdq}{F'dp + G'dq}, \quad (2)$$

$$\text{whence } -\frac{dq}{dp} = \frac{RE' - VE}{RF' - VF} = \frac{RF' - VF}{RG' - VG},$$

$$R^2(E'G' - F'^2) - RV(EG' + E'G - 2FF') + V^2(EG - F^2) = 0, \quad (3)$$

$$\text{and } \begin{vmatrix} (dq)^2, & -dp\,dq, & (dp)^2 \\ E, & F, & G \\ E', & F', & G' \end{vmatrix} = 0, \quad (4)$$

(3) and (4) give the radii and directions of principal curvature independently of each other. By (2), when the magnitude of one of the principal radii of curvature is given, we can determine the direction of the corresponding line of curvature, and vice versâ.

(4) is the differential equation of the lines of curvature of the surface, and by (3) the specific curvature or Gauss' measure of curvature is $(E'G' - F'^2)/(EG - F^2)^2$.

797. We shall in what follows use the notation employed by Cayley in the memoirs referred to, viz., if U be a function of p and q, the first and second differential coefficients with respect to p and q will be written U_1, U_2, U_{11}, U_{12}, U_{22}.

798. *To express the specific curvature in terms of E, F and G and their partial differential coefficients.*

The following functions can be obviously so expressed, viz.

$$\begin{aligned} u &\equiv a\alpha + b\beta + c\gamma, & u' &\equiv a'\alpha + b'\beta + c'\gamma, \\ v &\equiv a\alpha' + b\beta' + c\gamma', & v' &\equiv a'\alpha' + b'\beta' + c'\gamma', \\ w &\equiv a\alpha'' + b\beta'' + c\gamma'', & w' &\equiv a'\alpha'' + b'\beta'' + c'\gamma''. \end{aligned}$$

We have thus $\frac{1}{2}E_1 = u$, $\frac{1}{2}E_2 = v$, $\frac{1}{2}G_1 = v'$, $\frac{1}{2}G_2 = w'$, $F_1 = v + u'$, or $F_1 - \frac{1}{2}E_2 = u'$ and $F_2 = w + v'$, or $F_2 - \frac{1}{2}G_1 = w$.

Also $E' = A\alpha + B\beta + C\gamma$, $u = a\alpha + b\beta + c\gamma$, $u' = a'\alpha + b'\beta + c'\gamma$; therefore since

$$Cb' - Bc' = (ab' - a'b)b' - (ca' - c'a)c' = a(b'^2 + c'^2) - a'(bb' + cc') = aG - a'F$$

$$\text{and similarly } Bc - Cb = a'E - aF$$

eliminating successively β, γ; γ, α; and α, β;

$$\begin{aligned} AE' + (aG - a'F)\,u + (a'E - aF)\,u' &= V^2\alpha, \\ BE' + (bG - b'F)\,u + (b'E - bF)\,u' &= V^2\beta, \\ CE' + (cG - c'F)\,u + (c'E - cF)\,u' &= V^2\gamma, \\ E'G' + (wG - w'F)\,u + (w'E - wF)\,u' &= V^2(\alpha\alpha'' + \beta\beta'' + \gamma\gamma''); \end{aligned}$$

in a similar manner, since $F'=A\alpha'+B\beta'+C\gamma'$, we obtain

$$F'^2+(vG-v'F)v+(v'E-vF)v'=V^2(\alpha'^2+\beta'^2+\gamma'^2).$$

$$\text{Now } u'_2=a\alpha''+a'a_2+\beta\beta''+b'\beta_2+\gamma\gamma''+c'\gamma_2,$$

$$\text{and } v'_1=\alpha'^2+a'\alpha'_1+\beta'^2+b'\beta'_1+\gamma'^2+c'\gamma'_1;$$

therefore, since $a_2=a_{12}=a_{21}=a'_1$, &c.,

$$u'_2-v'_1=a\alpha''+\beta\beta''+\gamma\gamma''-\alpha'^2-\beta'^2-\gamma'^2,$$

hence

$$E'G'-F'^2=V^2(u'_2-v'_1)+E(v'^2-u'w')+G(v^2-uw)+F(uw'+u'w-2vv'),$$

the first term of which is $-\frac{1}{2}(EG-F^2)(E_{22}-2F_{12}+G_{11})$; the third differential coefficients of the coordinates, which seem to enter the expression, disappear, as can be shewn directly, or indirectly by considering that the term $E'G'-F'^2$ from which it was derived involves second differential coefficients only; since u, u', &c., can be expressed as above in terms of the differential coefficients of E, F and G, the specific curvature $(E'G'-F'^2)/(EG-F^2)^2$ can be so expressed.

COR. In the case of geodesic polar coordinates, $E=1$, $F=0$, $G=P^2$, u, v, u' vanish and $v'=\frac{1}{2}G_1$, $\therefore$ the specific curvature

$$=-\tfrac{1}{2}G_{11}/G+\tfrac{1}{4}G_1^2/G^2=-P_{11}/P=1/R'R'',$$

as in Art. 757.

799. If the curves (p) be geodesic lines, and the q system cut these orthogonally, since, by Art. 794, A, B, C are proportional to the direction-cosines of the normal to the surface, which is in the osculating plane of the geodesic,

$$\begin{vmatrix} A, & B, & C \\ a', & b', & c' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}=0,$$

$$\text{also } Bc'-Cb'=-aG, \text{ \&c.}; \quad \therefore a\alpha''+b\beta''+c\gamma''=0,$$

hence $-\frac{1}{2}G_1=0$, or G is a function of q only.

This is apparent geometrically, for the two curves (q), $(q+dq)$ by Art. 747, cut off from the curves (p), $(p+dp)$ equal lengths, so that $\sqrt{G}dq$ is independent of p.

Hence, by assuming q' a new value of the second variable such that $\sqrt{G}dq=dq'$, the expression $E(dp)^2+G(dq)^2$ can always be put in the form $E(dp)^2+(dq')^2$.

800. Liouville* has shewn that the expression $E(dp)^2+\ldots$ for $(ds)^2$ can be put in the form $\lambda\{(d\alpha)^2+(d\beta)^2\}$, for, being never negative, it is expressible in the form of the product of two imaginary expressions

$$dp\sqrt{E}+dq\{F\pm i\sqrt{(EG-F^2)}\}/\sqrt{E},$$

* *Liou. Jour.* tome 12.

and, if $\mu+\nu i$ be an integrating factor of the above expression with the uppe[r] sign, $\mu-\nu i$ will be one for that with the lower sign, and multiplying b[y] these factors, the expressions become $d(\alpha+\beta i)$ and $d(\alpha-\beta i)$,

$$\therefore\ (\mu^2+\nu^2)(ds)^2=(d\alpha)^2+(d\beta)^2,$$

and since the number of integrating factors is infinite the expression for (ds) can be expressed in the form $\lambda\{(d\alpha)^2+(d\beta)^2\}$ in an infinite number of ways.

The specific curvature referred to these coordinates is

$$-\frac{1}{2\lambda}\left\{\frac{d^2(\log\lambda)}{d\alpha^2}+\frac{d^2(\log\lambda)}{d\beta^2}\right\}.$$

801. *Lines of curvature with curvilinear coordinates.*

If $p=$ constant, $q=$ constant be the equations of two system[s] of lines of curvature the equation (4) of Art. 796 must reduce t[o] $dp\,dq=0$, $\therefore F=0$, $F'=0$, the first of which, viz. $aa'+bb'+cc'=0$ represents the fact that the two sets of lines are at right angles and the second enables us to obtain a differential equation in term[s] of E and G, which is satisfied by the coordinates x, y and z.

$$\text{For }\ \tfrac{1}{2}E_2=a\alpha'+b\beta'+c\gamma',$$
$$\tfrac{1}{2}G_1=a'\alpha'+b'\beta'+c'\gamma',$$
$$\text{and }\ F'=0=A\alpha'+B\beta'+C\gamma';$$

hence, eliminating β' and γ', observing that $b'C-c'B=aG$, &c.,

$$\tfrac{1}{2}aGE_2+\tfrac{1}{2}a'EG_1=(A^2+B^2+C^2)\,\alpha'=EG\alpha';$$

$$\therefore\ \frac{d^2x}{dp\,dq}-\frac{1}{2}\frac{dE}{Edq}\frac{dx}{dp}-\frac{1}{2}\frac{dG}{Gdp}\frac{dy}{dq}=0,\quad (1)$$

and similarly for y and z.

The equations (2) of Art. 796 shew that p constant gives th[e] line of curvature for which $R=VG/G'$, and q constant that fo[r] which $R=VE/E'$.

802. If the surface be the conicoid $x^2/a+y^2/b+z^2/c=1$,

$$(a-b)(a-c)x^2=a(a+p)(a+q),$$
$$E=(dx/dp)^2+\ldots=p(p-q)/\{(a+p)(b+p)(c+p)\},$$
$$d\log E/dq=-(p-q)^{-1}=-d\log G/dp,$$
$$\text{and }\ \tfrac{1}{2}\frac{dx}{dp}:\tfrac{1}{2}\frac{dx}{dq}:\frac{d^2x}{dp\,dq}::a+q:a+p:1,$$

so that the equation (1) of the last article is satisfied.

803. *To find the direction of the inflexional tangents at an[y] point of a surface, and the differential equation of the inflexion o[r] chief curves.*

If $dp:dq$ give the direction of an inflexional tangent, for which the radius of curvature is infinite, by (1) Art. 796,

$$E'(dp)^2+2F'dp\,dq+G'(dq)^2=0,$$

which is the differential equation of the inflexion curves, called by Cayley *chief curves.*

If $E'=0$ and $G'=0$ the equation becomes $dpdq=0$, so that the urface is striated by chief curves.

In this case, by (4) Art. 796, the differential equation of the ines of curvature is $G(dq)^2-E(dp)^2=0$, (1), representing that he lines of curvature bisect the angles between the striating urves. Also the principal radii of curvature are

$$\{\pm\sqrt{(EG)}+F\}\sqrt{(EG-F^2)}/F'.$$

In the case of the hyperboloid of one sheet and the hyperbolic araboloid these chief curves are the generating lines.

804. If a paraboloid be given by $x=lp+mq$, $y=l'p+m'q$, $z=pq$, the eneral equation of the lines of curvature obtained by integrating (1) Art. 803 is

$$p^2/\alpha^2+q^2/\beta^2-2Dpq/\alpha\beta=D^2-1,$$

here D is an arbitrary constant, and $l^2+m^2=\alpha^2$, $l'^2+m'^2=\beta^2$, and if $D=\operatorname{cosec}2\gamma$,

$$(p/\alpha-\tan\gamma\,.\,q/\beta)(p/\alpha-\cot\gamma\,.\,q/\beta)=\cot^2 2\gamma.$$

805. *Geodesic lines.* Suppose p, q to be functions of a new ariable t, and write p', p'', &c. for $\dfrac{dp}{dt}$, $\dfrac{d^2p}{dt^2}$,

$$x'=ap'+a'q',$$
$$x''=ap''+a'q''+\alpha p'^2+2\alpha'p'q'+\alpha''q'^2;$$

nd the direction-cosines of the normal being as $A:B:C$, Art. 794, he condition that it lies in the osculating plane of the geodesic is

$$\begin{vmatrix} A, & B, & C \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix}=0;$$

or, if ξ, ξ' be written for the terms involving p'', q'' and p', q' respectively,

$$\begin{vmatrix} A, & B, & C \\ x', & y', & z' \\ \xi, & \eta, & \zeta \end{vmatrix}+\begin{vmatrix} A, & B, & C \\ x', & y', & z' \\ \xi', & \eta', & \zeta' \end{vmatrix}=0.$$

Now $y'\zeta-z'\eta=(bc'-b'c)(p'q''-p''q')$;

ence the first determinant is $V^2(p'q''-p''q')$;

lso $Bz'-Cy'=p'(Bc-Cb)+q'(Bc'-Cb')=p'(Ea'-Fa)-q'(Ga-Fa')$;

ence the differential equation of a geodesic in curvilinear coorlinates is

$$(EG-F^2)(p'q''-q'p'')$$
$$+(\alpha p'^2+2\alpha'p'q'+\alpha''q'^2)\{p'(Ea'-Fa)+q'(Fa'-Ga)\}+\ldots=0.$$

When the curves $p=0$, $q=0$ intersect at right angles $F=0$, nd the geodesic equation is, having regard to the values of $a'\alpha+\ldots$ &c., given in Art. 798,

$$2EG(p'q''-q'p'')+Ep'(-E_2p'^2+2G_1p'q'+G_2q'^2)$$
$$-Gq'(E_1p'^2+2E_2p'q'-G_1q'^2)=0.$$

If, moreover, $q=$ constant be a geodesic line, the condition will be $E_2=0$, whence E is a function of p only.

806. *Radius of relative geodesic curvature of two curves traced on a surface.* Let PQ, PQ' be small elements of two curves having common rectilinear and geodesic tangents PT and PM, and let a plane through T perpendicular to the tangent PT intersect the surface in the curve MQ', so that $PQ=PQ'=PT$ ultimately. The rate of increase of the geodesic angle of contact QPM of PQ per unit of length is MQ/PQ^2, and, by analogy with plane curves, $PQ^2/2MQ$ is called the radius of geodesic curvature. The measure of the relative geodesic curvature of PQ and PQ' is $2QQ'/PQ^2$.

Cayley* has fully worked out the formulæ connected with geodesic curvature. The following articles sufficiently represent his method of working.

807. *Expression for the radius of relative geodesic curvature in curvilinear coordinates.*

Let p, q be given functions of s an arc of PQ measured from some fixed point in it, p', q' their differential coefficients with respect to s, also let P'', Q'' be written for p'', q'' when p, q belong to the curve PQ'. Therefore, with the notation of Art. 794, since $dp=p'ds+\frac{1}{2}p''(ds)^2$, $dq=q'ds+\frac{1}{2}q''(ds)^2$,

$$\delta x=(ap'+a'q')\,ds+\tfrac{1}{2}(ap''+a'q'')(ds)^2+\tfrac{1}{2}(\alpha p'^2+2\alpha'p'q'+\alpha''q'^2)(ds)^2$$

is the projection of PQ on the axis of x, and, if $PQ'=PQ$, that of PQ' differs from δx by writing P'', Q'' for p'', q''.

$$\text{Hence,}\quad QQ'^2=\tfrac{1}{4}\{a(p''-P'')+a'(q''-Q'')\}^2(ds)^4+\dots$$
$$=\tfrac{1}{4}(E,F,G\between p''-P'',\,q''-Q'')^2(ds)^4,$$

and the relative curvature $=2QQ'/(ds)^2=\sqrt{\{(E,F,G\between p''-P'',\,q''-Q'')^2\}}$.

For orthotomic coordinates the expression becomes

$$\sqrt{\{E(p''-P'')^2+G(q''-Q'')^2\}}=\rho^{-1},\quad\text{and}\quad 1=Ep'^2+Gq'^2\ \text{for both curves,}$$
$$\therefore\quad 0=2Ep'p''+2Gq'q''+(E_1p'+E_2q')\,p'^2+(G_1p'+G_2q')\,q'^2,$$
$$\text{and}\quad 0=2Ep'P''+2Gq'Q''+\dots\dots\dots\dots;$$
$$\therefore\quad 0=2Ep'(p''-P'')+2Gq'(q''-Q''),$$
$$\therefore\quad (p''-P'')\sqrt{E}/q'\sqrt{G}=(q''-Q'')\sqrt{G}/-p'\sqrt{E}=\rho^{-1}/(Ep'^2+Gq'^2);$$
$$\therefore\quad \rho^{-1}=\sqrt{(EG)}\{p''q'-q''p'-(P''q'-Q''p')\}.$$

The general expression for the relative curvature, when the curves (p) and (q) are not orthotomic, viz. $V\{p''q'-q''p'-(P''q'-Q''p')\}$ may be found in the same way, but the particular case is sufficient to shew how the problem can be dealt with.

808. *To find the radius of geodesic curvature in terms of orthotomic curvilinear coordinates.*

The differential equation of the geodesic line is

$$2EG(P''q'-Q''p')=Ep'(-E_2p'^2+2G_1p'q'+G_2q'^2)-Gq'(E_1p'^2+2E_2p'q'-G_1q'^2),\quad (1)$$

whence the radius required can be expressed in terms of p, q, and their first and second differential coefficients belonging to the curve itself.

For the special curve $q=$ constant, $q'=0$ and $Ep'^2=1$; let $\rho=\rho_2$ for this curve, then $G(P''q'-Q''p')=-\frac{1}{2}E_2p'^3=-\frac{1}{2}E_2/E\sqrt{E}$,

$$\rho_2^{-1}=-E_2/2E\sqrt{G}.$$

Similarly, if $\rho=\rho_1$ for the curve $p=$ constant,

$$\rho_1^{-1}=G_1/2G\sqrt{E},$$

ρ_1 being considered to be measured in the same direction as ρ_2 when the concavities of the curves (p) and (q) are in the same direction.

* *Lond. Math. Soc.*, vol. XII. p. 110.

809. If θ be the inclination of the curve to the line $q = \text{constant}$,

$$\cos\theta = p'\sqrt{E}, \quad \sin\theta = q'\sqrt{G};$$

$$\therefore\ p'' = -\theta' \sin\theta/\sqrt{E} - \tfrac{1}{2}\cos\theta\,(E_1p' + E_2q')/E\sqrt{E},$$

$$\text{hence } \sqrt{(EG)}\,p'' = -\theta' Gq' - \tfrac{1}{2}p'\sqrt{G}\,(E_1p' + E_2q')/\sqrt{E},$$

$$\text{similarly } \sqrt{(EG)}\,q'' = \ \theta' Ep' - \tfrac{1}{2}q'\sqrt{E}\,(G_1p' + G_2q')/\sqrt{G},$$

$$\therefore\ \sqrt{(EG)}\,(p''q' - q''p') = -\theta' - \tfrac{1}{2}p'q'\,\{(E_1p' + E_2q')\sqrt{G}/\sqrt{E} - (G_1p' + G_2q')\sqrt{E}/\sqrt{G}\},$$

and by (1), Art. 808, since $Ep'^2 + Gq'^2 = 1$,

$$2EG\,(P''q' - Q''p') = -E_2p' + G_1q' + p'q'\,\{E\,(G_1p' + G_2q') - G\,(E_1p' + E_2q')\};$$

$$\therefore\ \rho^{-1} = -\tfrac{1}{2}\,(E_2p' - G_1p')\sqrt{(EG)} + \theta' = \rho_2^{-1}\cos\theta + \rho_1^{-1}\sin\theta + d\theta/ds.$$

810. The relations between the coefficients E, E', &c. being so much simplified when the striating curves are orthotomic, it is worth while to go through the work for this case of curvilinear coordinates in a form better adapted for shewing the geometrical meaning of the operations. Considering the infinitesimal rectangle $QNPM$, let $PM = P\delta p$, $PN = Q\delta q$ be the elements of the curves (q), (p) respectively, and let λ, μ, ν and λ', μ', ν' be the direction-cosines of the tangents to these elements at P, l, m, n those of the normal to the surface at P, and let δx, δy, δz be the excess of the coordinates of Q over those of P,

$$\delta x = P\lambda\delta p + Q\lambda'\delta q$$
$$+ \tfrac{1}{2}\,[(P\lambda)_1\,\delta p^2 + 2\,\{(P\lambda)_2 \text{ or } (Q\lambda')_1\}\,\delta p\,\delta q + (Q\lambda')_2\,(\delta q)^2] + \ldots;$$

and R, the radius of curvature of the normal section containing the tangent at P to PQ, is the limit of

$$\frac{\tfrac{1}{2}(\delta s)^2}{l\delta x + m\delta y + n\delta z} = \frac{P^2(\delta p)^2 + Q^2(\delta q)^2}{KP^2(\delta p)^2 - 2TPQ\delta p\delta q + HQ^2(\delta q)^2},$$

$$\text{where } KP^2 = l\,(P\lambda)_1 + \ldots = P\,(l\lambda_1 + m\mu_1 + n\nu_1),$$
$$-TPQ = P\,(l\lambda_2 + m\mu_2 + n\nu_2) = Q\,(l\lambda'_1 + m\mu'_1 + n\nu'_1),$$
$$HQ^2 = Q\,(l\lambda'_2 + m\mu'_2 + n\nu'_2).$$

It is obvious that K, H are the curvatures at P of the curves (q), (p) respectively; T will be interpreted in the following article. R', R'', the maximum and minimum values of R, are given by the condition of equality of the roots of

$$(RK - 1)\,(Pdp)^2 - 2RTPdp\,.\,Qdq + (RH - 1)\,(Qdq)^2 = 0;$$
$$\therefore\ (K - R^{-1})\,(H - R^{-1}) - T^2 = 0,$$

and the specific curvature at $P = HK - T^2$.

It follows from the above equation that

$$(K - R^{-1})\,Pdp = TQdq, \qquad (H - R^{-1})\,Qdq = TPdp;$$

$$\text{so that } R^{-1} = (KPdp - TQdq)/Pdp = (HQdq - TPdp)/Qdq;$$

$$\therefore\ T\,(Qdq)^2 - (K - H)\,Qdq\,.\,Pdp - T\,(Pdp)^2 = 0$$

the differential equation of the lines of curvature.

811. *Torsion of a geodesic.* Let R, R' be the principal rad[ii] of curvature of the surface at the point (p, q) of a geodesic incline[d] at an angle θ to (q), so that $\cos\theta = Pdp/ds = Pp'$ and let θ_1 be th[e] inclination to (q) of the normal section corresponding to R. B[y] Art. 751 the torsion of the geodesic is

$$\sigma^{-1} = \sin(\theta - \theta_1)\cos(\theta - \theta_1)(R^{-1} - R'^{-1}).$$

Now, by Art. 810,

$$\cos\theta_1/\sqrt{(H - R^{-1})} = \sin\theta_1/\sqrt{(K - R^{-1})} = 1/\sqrt{(R'^{-1} - R^{-1})};$$

$$\therefore \sqrt{(R'^{-1} - R^{-1})}\sin(\theta - \theta_1) = Qq'\sqrt{(H - R^{-1})} - Pp'\sqrt{(K - R^{-1})},$$

and $\sqrt{(R'^{-1} - R^{-1})}\cos(\theta - \theta_1) = Pp'\sqrt{(H - R^{-1})} + Qq'\sqrt{(K - R^{-1})};$

$$\therefore \sigma^{-1} = \{(Pp')^2 - (Qq')^2\}\,T + Pp'.Qq'\,(K - H),$$

$$= T\cos 2\theta + \tfrac{1}{2}(K - H)\sin 2\theta;$$

$\therefore$ T is the torsion of the geodesic tangent at P to either o[f] curves (q) or (p).

If the geodesic tangent to either of the curves or the curves thems[elves] were rectilinear, as might happen in the case of a scroll, it might appear $T = 0$, but what is called the torsion in that case is rate of increase o[f] angle between consecutive principal normals which are the normals t[o] surface along the line; T will not vanish except where the normals ar[e] parallel to the same plane, as in a torse.

812. *Specific curvature in terms of P and Q.*

$$\text{Since } KP = l\lambda_1 + m\mu_1 + n\nu_1, \quad (1)$$

$$0 = \lambda\lambda_1 + \mu\mu_1 + \nu\nu_1,$$

$$\text{let } L = \lambda'\lambda_1 + \mu'\mu_1 + \nu'\nu_1 = -\lambda\lambda'_1 - \mu\mu'_1 - \nu\nu'_1; \quad (2)$$

and, by Art. 146, $\mu\nu' - \nu\mu' = l$, $\mu'n - \nu'm = \lambda$, $m\nu - n\mu = \lambda'$;

$$\therefore\ KPl + L\lambda' = \lambda_1,$$

$$\text{similarly } KPm + L\mu' = \mu_1, \quad (3)$$

$$\text{and } KPn + L\nu' = \nu_1.$$

In the same way, if $M = \lambda\lambda'_2 + \mu\mu'_2 + \nu\nu'_2$, $HQl + M\lambda = \lambda'_2$.

Multiply the left side of equations (3) by lHQ, mHQ, $nH[Q]$ and the right side by their equivalents, and add,

$$KHPQ = \lambda_1\lambda'_2 + \mu_1\mu'_2 + \nu_1\nu'_2, \quad (4) \quad \text{since } \lambda\lambda_1 + \ldots = 0.$$

$$\text{Again } -TQ = l\lambda_2 + m\mu_2 + n\nu_2,$$

$$0 = \lambda\lambda_2 + \mu\mu_2 + \nu\nu_2,$$

$$-M = \lambda'\lambda_2 + \mu'\mu_2 + \nu'\nu_2;$$

$$\therefore -TQl - M\lambda' = \lambda_2, \text{ \&c.};$$

similarly $-TPl - L\lambda = \lambda'_1$; $\therefore$ $T^2PQ = \lambda_2\lambda'_1 + \mu_2\mu'_1 + \nu_2\nu'_1$. (5)

$$\text{Now } L_2 = \lambda'_2\lambda_1 + \lambda'\lambda_{12} + \mu'_2\mu_1 + \mu'\mu_{12} + \nu'_2\nu_1 + \nu'\nu_{12},$$
$$\text{and } -M_1 = \lambda'_1\lambda_2 + \lambda'\lambda_{21} + \mu'_1\mu_2 + \mu'\mu_{21} + \nu'_1\nu_2 + \nu'\nu_{21};$$
$$\therefore \text{, by (4) and (5), } (KH - T^2)\,PQ = L_2 + M_1;$$
$$\text{and since } (P\lambda)_2 = (Q\lambda')_1,\ P_2\lambda + P\lambda_2 = Q_1\lambda' + Q\lambda'_1, \text{ \&c.};$$
$$\therefore P_2 = Q(\lambda\lambda'_1 + \mu\mu'_1 + \nu\nu'_1) = -QL, \qquad (6)$$
$$Q_1 = P(\lambda'\lambda_1 + \mu'\mu_2 + \nu'\nu_2) = -PM;$$
k, the specific curvature at P, $= -\{(P_2/Q)_2 + (Q_1/P_1)_1\}/PQ$. (7)

, COR. For geodesic polar coordinates $p = \omega$, $q = \rho$, $Q = 1$;

$$\therefore \frac{d^2P}{d\rho^2} + kP = 0, \text{ as in Art. 757};$$

.d, by Art. 808, or geometrically by Art. 814, the geodesic curva-

f th of ρ is $-\frac{1}{P}\frac{dP}{d\rho}$, as in Art. 758.

selve13. Geometrical interpretations of some of these analytical expressions

: thas follows.

of thince $\lambda_1/P = d\lambda/Pdp = d\lambda/ds_2 = d^2x/(ds_2)^2$, if ρ' be the radius of curvature

o the curve (q), ϕ the inclination of its osculating plane to the normal section

re ah is the osculating plane of the geodesic tangent to (q), $\rho'\lambda_1/P$, $\rho'\mu_1/P$,

P are direction-cosines of the principal normal, therefore, by (1), Art. 812,

$$\cos\phi = (l\lambda_1 + m\mu_1 + n\nu_1)\,\rho'/P = \rho' K, \text{ Meunier's theorem,}$$
$$\text{and by (2), } \sin\phi = (\lambda'\lambda_1 + \mu'\mu_1 + \nu'\nu_1)\,\rho'/P = \rho' L/P;$$

(1) ›y Art. 753, L/P is the geodesic curvature of the curve (q), which also

ırs from (3) since $K^2 + (L/P)^2 = \rho'^{-2}$.

;14. *To express the specific curvature at any point of a surface*

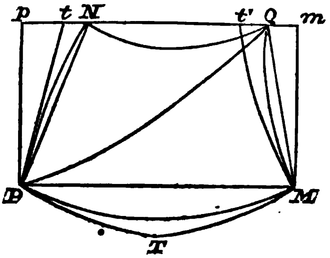
(2) *red to orthotomic striating curves, in terms of the geodesic*

atures of the curves intersecting in the point.

Let PM, NQ be elements of the curves (q) and $(q + dq)$, PN,

of the curves (p) and $(p + dp)$, and let geodesic chords of the

(3)



ıHQ

· curves be drawn; draw PT, TM geodesic t

rc PM at P and M, and let geodesic tangents P

' meet the chord NQ in t and t', and draw θ be the inclination

ılar to NQ. 'Q and PTq to NQ,

(5)

Then, ultimately, $(NQ-PM)/Pp=-(Np+Qm)/Pp$
$=-(\angle tPp+\angle tPN+\angle t'Mm-\angle t'MQ)=-(\angle MPT+\angle PMT)$,
since $\angle t'MQ-\angle tPN$ vanishes compared with either of the angles hence the angle at T, which is the angle of geodesic contingence of the curve (q), $=-d(Pdp)/Qdq$, and dividing by PM or Pdp, the geodesic curvature of $(q)=-P_2/PQ$, similarly that of $(p)=-Q_1/PQ$. If ρ_2, ρ_1 be the two radii of geodesic curvature, by (7) of Art. 812,

$$\frac{1}{R'R''}=\left\{\frac{d}{dq}\left(\frac{P}{\rho_2}\right)+\frac{d}{dp}\left(\frac{Q}{\rho_1}\right)\right\}/PQ$$

$$=\frac{d}{Qdq}\left(\frac{1}{\rho_2}\right)+\frac{P_2}{PQ}\frac{1}{\rho_2}+\frac{d}{Pdp}\left(\frac{1}{\rho_1}\right)+\frac{Q_1}{PQ}\frac{1}{\rho_1};$$

$$\therefore\ \frac{1}{R'R''}=\frac{d}{ds_1}\left(\frac{1}{\rho_2}\right)+\frac{d}{ds_2}\left(\frac{1}{\rho_1}\right)-\frac{1}{\rho_2^2}-\frac{1}{\rho_1^2}.$$

815. *Differential equations connecting H, K and T.*

By Art. 812, $(KP)_2+(TQ)_1=l_2\lambda_1-l_1\lambda_2+m_2\mu_1-m_1\mu_2+n_2\nu_1-n_1\nu_2$.

Now $l=\mu\nu'-\mu'\nu$, $m=\nu\lambda'-\nu'\lambda$, and $n=\lambda\mu'-\lambda'\mu$.

$$l_2=\mu_2\nu'-\nu_2\mu'+\mu\nu'_2-\nu\mu'_2,\quad TQ\lambda=l(\mu\mu_2+\nu\nu_2)-\lambda(m\mu_2+n\nu_2),$$

$$l\mu-m\lambda=\nu',\quad l\nu-n\lambda=-\mu';\quad \therefore\ \mu_2\nu'-\nu_2\mu'=TQ\lambda;$$

$$HQ\lambda'=-l(\mu'\mu'_2+\nu'\nu'_2)+\lambda'(m\mu'_2+n\nu'_2);$$

$$-l\mu'+m\lambda'=\nu,\quad -l\nu'+n\lambda'=-\mu;$$

$$\therefore\ \nu\mu'_2-\mu\nu'_2=HQ\lambda';\quad \therefore\ l_2=Q(T\lambda-H\lambda').$$

Similarly $TP\lambda'=l(\mu'\mu'_1+\nu'\nu'_1)-\lambda'(m\mu'_1+n\nu'_1)=-\nu\mu'_1+\mu\nu'_1$,

$$KP\lambda=-l(\mu\mu_1+\nu\nu_1)+\lambda(m\mu_1+n\nu_1)=-\mu_1\nu'+\nu_1\mu';$$

$$\therefore\ l_1=P(-K\lambda+T\lambda');$$

$\therefore\ (KP)_2+(TQ)_1=-HQL+TPM=HP_2-TQ_1$, (6) Art. 812;

$$\left.\begin{aligned}\therefore\ T_1Q+2TQ_1+K_2P+(K-H)P_2&=0.\\ \text{Similarly}\ T_2P+2TP_2+H_1Q+(H-K)Q_1&=0.\end{aligned}\right\}\quad(7)$$

816. If geodesic polar coordinates be used, and q be the ·desic radius vector, $Q=1$, and the equations (7) become

$$\text{ˉ}+PK_2+(K-H)P_2=0,\ \text{or}\ T_1+(PK)_2-HP_2=0,$$

$$\text{d}\ PT_2+2TP_2+H_1=0,\qquad \text{or}\ (TP^2)_2+PH_1=0;$$

ᵗhe equation derived from the specific curvature, viz. $T^2)=0$, are the three fundamental equations employed memoir on deformation of surfaces.

similarly — 1. *Jour. L'Ec. Polyt.* tome XXII.

817. *To find the differential equation of a geodesic considered as a line of minimum length between two points on the surface.*

Let PQ be a small element of the geodesic joining two points A, B on the surface, and let $AP'Q'B$ be a curve slightly differing from the geodesic; let p, q and $p+dp$, $q+dq$ be the coordinates of P and Q, $p+\delta p$, q and $p+\delta p+d(p+\delta p)$, $q+dq$ those of P' and Q', so that p and $p+\delta p$ are different functions of q.

If $S=\int ds=\int\sqrt{\{(Pdp)^2+(Qdq)^2\}}$ be a minimum, then, for all values of δp, $\delta S=0$,

$$\text{and } \delta S=\int[\{PP_1(dp)^2+QQ_1(dq)^2\}\delta p/ds+P^2dp\,\delta dp/ds],$$

$$=\int\delta p[\{PP_1(dp)^2+QQ_1(dq)^2\}/ds-d(P^2dp/ds)],$$

since $\delta p.\ P^2dp/ds$ vanishes at A and B;

$$\therefore\ PP_1(dp)^2+QQ_1(dq)^2-ds\,d(P^2dp/ds)=0;$$

and if θ be the angle at which the geodesic crosses the curve $q=$ constant, $Pdp=ds\cos\theta$, $Qdq=ds\sin\theta$;

$$\therefore\ P_1dp\cos\theta+Q_1dq\sin\theta=d(P\cos\theta)=(P_1dp+P_2dq)\cos\theta-P\sin\theta\,d\theta;$$

$$\therefore\ d\theta=dpP_2/Q-dq\,Q_1/P,$$

and by eliminating θ we obtain the equation of Art. 805, writing P^2 and Q^2 for E and G respectively.

818. ***Integration of the equation of a geodesic on a central conicoid, referred to lines of curvature.***

By Art. 789, $P^2=-u(q-p)$, $Q^2=v(q-p)$, where u and v are respectively functions of p and q alone, also, by Art. 817,

$$d\theta=dpP_2/Q-dq\,Q_1/P=dq\cot\theta\,.\,P_2/P-dp\tan\theta\,.\,Q_1/Q;$$

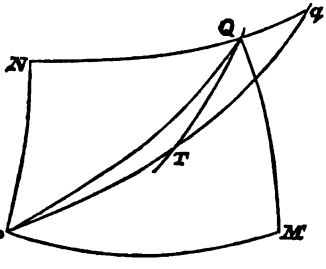
$$\therefore\ 2d\theta\sin\theta\cos\theta=(dq\cos^2\theta+dp\sin^2\theta)/(q-p),$$

$$\therefore\ dq\cos^2\theta+q\,d\cos^2\theta+dp\sin^2\theta+p\,d\sin^2\theta=0,$$

$$\therefore\ q\cos^2\theta+p\sin^2\theta=\text{constant},$$

which is a first integral of the equation.

819. The equation of Art. 817 enables us to obtain Liouville's equation, Art. 809. Let PTq, TQ be geodesic tangents to the



curve PQ, PT intersecting NQ in q; and let θ be the inclination of PT at P to PM, $\theta+d\theta$, $\theta+\delta\theta$ those of TQ and PTq to NQ,

then, by the triangle TQq, $d\theta - \delta\theta$ is the geodesic angle of contingence of PQ, and $(d\theta - \delta\theta)/ds$ the geodesic curvature ρ^{-1}.

But, since $\delta\theta$ is the $d\theta$ of Art. 817, $\delta\theta = dp.P_2/Q - dq.Q_1/P$;

$$\therefore \rho^{-1} = d\theta/ds + \rho_2^{-1}\cos\theta + \rho_1^{-1}\sin\theta, \text{ as in Art. 809,}$$

ρ_1, ρ_2 being measured in the direction towards which θ is increasing.

Deformation of Surfaces.

820. The method of curvilinear coordinates is of special importance in the investigation of surfaces which are applicable to each other when they are considered to be flexible and inextensible.

A familiar example of the deformation of a flexible and inextensible surface is that of a torse which may be developed into a plane and then be transformed into any other torse, and it is obvious that the generating lines of one torse may be curved lines on the other.

It should be noticed that the edge of regression, when the torse becomes a plane, is developed into a plane curve, and although every part of the torse lies on the plane some parts of the plane do not become parts of the torse when reversion of the process of development takes place, viz. those parts from which no tangent can be drawn to the plane curve. And, *a fortiori*, the whole of a torse cannot always be deformed so as to form the whole of any other torse.

821. *Conditions that a given surface may be bent into the form of another surface.*

These can be deduced from the consideration that any line drawn arbitrarily upon the surface will be unaltered in length, although its form will generally change.

Let (x, y, z) be a point on any surface S and let (ξ, η, ζ) be the changed position of the point when the surface S has been deformed into the surface Σ. If x, y, z be expressed as functions of p and q so that p, q are curvilinear coordinates of (x, y, z) on the surface S, then ξ, η, ζ, depending only on x, y, z and the constants which determine any change of position of the surface as a whole, will also be functions of the same p and q. Hence, in order that no line may be altered in length, the elements of corresponding arcs measured from any point in any direction on S and Σ will be equal.

Now for an element ds of an arc measured from (x, y, z) on the given surface S, $(ds)^2 = E(dp)^2 + 2Fdpdq + G(dq)^2$, where E, F, G are definite functions of p and q, viz. $x_1^2 + y_1^2 + z_1^2$, $x_1x_2 + y_1y_2 + z_1z_2$ and $x_2^2 + y_2^2 + z_2^2$, and since the corresponding value of $(ds)^2$ must be the same for all directions measured on the surface Σ, that is for all values of dp and dq, the coefficients of $(dp)^2$, $dpdq$ and $(dq)^2$ must be the same functions of p and q; therefore the coordinates ξ, η, ζ of the point, corresponding to (x, y, z), in the deformed surface must satisfy the differential equations

$$\xi_1^2 + \eta_1^2 + \zeta_1^2 = E, \quad \xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2 = F, \quad \xi_2^2 + \eta_2^2 + \zeta_2^2 = G. \quad (1)$$

If a general solution of these equations could be obtained we should have a solution of the problem, *to find all the surfaces into which a given surface can be transformed by bending without extension or creasing.*

Since E, F, and G are functions of p and q unaltered by the deformation of the surface, the same is true of any function of these quantities and in particular of the specific curvature, Art. 798.

The invariability of the specific curvature is a necessary but not a sufficient condition of the possibility of a transformation from S to Σ, since we cannot generally deduce from this alone the equality of all corresponding arcs.

It should be observed that the integral curvature of any closed area is the sum of the integral curvatures of all the elements of the area; also the integral curvature of an element is the area of the element multiplied by the specific curvature which is unchanged by deformation, therefore the integral curvature of any closed area, as well as the specific curvature at any point, is unchanged by deformation.

822. The subject of deformation has been variously treated by Liouville*, Jellett†, Minding‡, Bonnet§, Bour‖, and Maxwell¶; but anything like a complete solution has been obtained in only a very few cases.

823. It will give some idea of the form in which the general solution of the equations (1) Art. 821 would appear, if we take a particular case which has considerable generality. We will suppose that the curvilinear coordinates are orthotomic, and that ζ is a function of p only, so that $(ds)^2=(Pdp)^2+(Qdq)^2$ and $\zeta_2=0$; in this case

$$\xi_1^2+\eta_1^2=P^2-\zeta_1^2=U^2, \text{ suppose,}$$

$$\xi_2^2+\eta_2^2=Q^2, \quad \xi_1\xi_2+\eta_1\eta_2=0.$$

$$\text{Let } \xi_1=U\cos v, \quad \eta_1=U\sin v, \quad (1)$$

$$\xi_2=Q\cos v', \quad \eta_2=Q\sin v'; \quad (2)$$

$$\therefore\ UQ\cos(v'-v)=0 \quad\text{and}\quad v'-v=\tfrac{1}{2}\pi,$$

$$\text{and } \xi_2=-Q\sin v, \quad \eta_2=Q\cos v,$$

$$\xi_{12}=U_2\cos v-Uv_2\sin v=-Q_1\sin v-Qv_1\cos v,$$

$$\eta_{12}=U_2\sin v+Uv_2\cos v=\ Q_1\cos v-Qv_1\sin v;$$

$$\therefore\ U_2+Qv_1=0 \quad\text{and}\quad Q_1-Uv_2=0; \quad (3)$$

$$\therefore\ (U^2)_2\,v_2+(Q^2)_1\,v_1=0 \quad\text{or}\quad (P^2)_2\,v_2+(Q^2)_1\,v_1=0,$$

and this being a linear equation, if $f(p, q)=\beta$ be a solution of

$$(Q^2)_1\,dq-(P^2)_2\,dp=0, \quad v=F\{f(p, q)\}, \quad U=Q_1/v_2;$$

$$\therefore\ \zeta_1^2=P^2-(Q_1/v_2)^2,$$

whence ξ, η, and ζ could be determined in terms of p and q.

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* *Monge ed. Liouville*, p. 583.
† *Proc. Ir. Ac.* vol. XXII. p. 343.
‡ *Crelle*, XVIII. pp. 297, 365.
§ *Jour. L'Ec. Polyt.* tome XIX.
‖ *Ibid*, tome XXII.
¶ *Camb. Phil. Tr.* vol. IX.

824. *To find surfaces applicable to each other for which $(ds)^2=(dq)^2+(q^2+a^2)(dp)^2$, and for which the distance of any point from a fixed plane is a function either of p (i) or q (ii) only.*

Take the plane for that of xy.

i. In this case $z_2 = 0$, $x_1^2+y_1^2 = q^2+a^2-z_1^2 = \lambda^2$, (1)

$$x_1x_2+y_1y_2=0, \quad x_2^2+y_2^2=1.$$

$$\text{Let } x_1=\lambda\sin v, \quad y_1=\lambda\cos v,$$

$$x_2=\sin v', \quad y_2=\cos v',$$

$$\cos(v'-v)=0, \quad v'=v+\tfrac{1}{2}\pi,$$

$$x_2=\cos v, \quad y_2=-\sin v, \quad (2)$$

$$x_{12}=\lambda_2\sin v+\lambda v_2\cos v=-v_1\sin v,$$

$$y_{12}=\lambda_2\cos v+\lambda v_2\sin v=-v_1\cos v;$$

$$\therefore\ v_2=0 \quad \text{or} \quad v=f(p),$$

$$\text{and } \lambda_2=-v_1=-f'(p) \quad \text{or} \quad \lambda=-qf'(p)+\phi(p);$$

$$\therefore, \text{ by (1)}, f'(p)=\pm 1, \quad \phi(p)=0, \quad z_1^2=a^2, \quad x_1=-q\sin p, \quad y_1=\mp q\cos p;$$

$$\therefore\ z=ap, \quad x=q\cos p, \quad y_1=\mp q\sin p,$$

since by (2) the arbitrary functions vanish and the origin may be chosen so that any constant will disappear.

ii. In this case $z_1=0$,

$$x_1^2+y_1^2=q^2+a^2=P^2,$$

$$x_2^2+y_2^2=1-z_2^2=\mu^2, \quad (1), \text{ where } \mu_1=0,$$

and, as before, $x_1=P\sin v$, $y_1=P\cos v$, (2), $x_2=\mu\cos v$, $y_2=-\mu\sin v$, (3)

$$x_{12}=P_2\sin v+Pv_2\cos v=-\mu v_1\sin v,$$

$$y_{12}=P_2\cos v-Pv_2\sin v=-\mu v_1\cos v;$$

$$\therefore\ P_2+\mu v_1=0, \quad v_2=0; \quad \therefore\ v=f(p), \quad \text{also } q/\sqrt{(q^2+a^2)}+\mu f'(p)=0,$$

and μ being a function of q only, $f'(p)$ is constant $=-n$, suppose;

$$\therefore\ n\mu=q/\sqrt{(q^2+a^2)},$$

$$\text{by (2) } nx=\sqrt{(q^2+a^2)}\cos np, \quad ny=\sqrt{(q^2+a^2)}\sin np,$$

since, by (3) the arbitrary functions of q disappear.

$$\text{Again, } n^2z_2^2=n^2-q^2/(q^2+a^2)=\{a^2n^2-(1-n^2)q^2\}/(q^2+a^2);$$

$$\therefore\ nz=\int dq\sqrt{\frac{a^2n^2-(1-n^2)q^2}{q^2+a^2}},$$

$$\text{and if } \frac{q\sqrt{(1-n^2)}}{na}=-\cos\theta, \qquad \frac{nz}{a}=\int\frac{n^2\sin^2\theta\, d\theta}{\sqrt{(1-n^2\sin^2\theta)}},$$

$$\text{or } nz/a=F(n,\theta)-E(n,\theta).$$

Case i. gives a helicoid, and case ii. a surface of revolution for which the equation of the meridian is given by $nr=\sqrt{(q^2+a^2)}$,

$$\text{and } z=\int\frac{dr}{\sqrt{(n^2r^2-a^2)}}\sqrt{\{a^2-(1-n^2)n^2r^2\}}.$$

These surfaces are therefore applicable to each other.

If $n<1$, since $q^2<a^2n^2/(1-n^2)$, a portion only of the helix is applicable.

If $n=1$, $z=a\log\{r/a+\sqrt{(r^2/a^2-1)}\}$, and $r=\frac{1}{2}a\,(e^{z/a}+e^{-z/a})$;

therefore the surface is formed by the revolution of a catenary about its directrix.

If $n>1$, and $q\sqrt{(1-n^{-2})}=-a\cot\theta$, $r=a\sqrt{(\operatorname{cosec}^2\theta-n^{-2})/(n^2-1)}$,

$$\frac{z}{a}=\int\frac{d\theta}{\sin^2\theta\sqrt{(1-n^{-2}\sin^2\theta)}}.$$

825. Maxwell* has given a means of forming a conception of what takes place when a surface changes its form by bending. He has also shewn how to construct the lines about which the bending takes place.

826. *General conditions of the applicability of two surfaces.*

i. Every point, line, and angle on one surface has its corresponding point, line, and angle on the second.

ii. If any line whatever be drawn on the first surface the corresponding line on the second will be equal to it in length.

iii. If two curves on the first surface intersect, the corresponding curves on the second will intersect at the same angle; for if on the first a third curve be drawn so as to form a triangle with the first two, and be then moved towards the point of intersection, the triangle becomes ultimately a plane triangle and the corresponding triangle on the second surface will have its sides, and therefore its angles, equal to those of the first triangle.

827. *Polyhedrons can be inscribed in each of two surfaces which are applicable to one another, such that corresponding facets are parallelograms equal in all respects.*

Let one of the surfaces be striated by two systems of curves such that the tangents at each point of their intersection are parallel to conjugate diameters of the indicatrix at that point. If *abcd* be a quadrilateral formed by two consecutive curves of each system, it will ultimately be a parallelogram inscribed in the conic section made by the plane of *abc*, whose sides are necessarily parallel to conjugate diameters of the conic. The two systems of curves are called conjugate systems.

Consider two surfaces applicable one to the other; take two points, one on each surface, corresponding to one another, and suppose the surfaces to be so placed that the tangent planes at the two points are coincident, and then the surfaces to be turned about the common normal, until the tangents to corresponding curves drawn through the points are also coincident, which, by iii. of the last article, is possible. The indicatrices of the two surfaces are conics which intersect in four points, and these points are the angular points of a parallelogram whose sides are parallel to conjugate diameters of both conics.

There are therefore at every point of one of the surfaces two particular directions which are conjugate and remain conjugate when deformation takes place. Hence there are two conjugate systems which remain conjugate systems on deformation.

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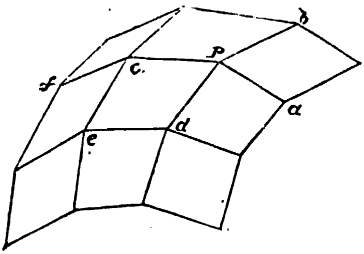
* *Camb. Phil. Tr.* vol. IX.

828. *Deformation of the corresponding polyhedrons inscribed in two applicable surfaces.*

Let P be a solid angle of the polyhedron inscribed in one of the surfaces, the four facets of which are the quadrilaterals, ultimately



parallelograms, ab, bc, cd, da, the deformation about P is effected by changing the angle of inclination of two of the facets having a common edge Pa, in consequence of which there will be a definite change of the angles about the other edges, and these changes define the new positions of the adjacent facets, such as ef.

When we proceed to the limit, the polygons bPd, aPc, &c., become the curves which belong to the conjugate systems common to both surfaces, which are thus *lines of bending.*

829. *Instantaneous and permanent lines of bending.*

In passing from one surface to another by deformation we may suppose the bending to take place continuously. The lines of bending in the passage from any one surface to the next consecutive will generally change their position at every instant, and for every intermediate form of the surface there will be a definite system which are *instantaneous lines of bending.* But, under certain conditions, these lines may retain the same position while a finite amount of bending takes place; they are then *permanent lines of bending.* An example of this will be seen in the case of the scroll described in Art. 833.

830. *Specific curvature at any point is unchanged by deformation.*

The horograph of the portion of the polyhedron including the four facets A, B, C, D forming the solid angle at P, is a quadrilateral $a'b'c'd'$ whose sides are arcs of great circles. The arc $a'b'$ is generated by the extremities of radii corresponding to lines perpendicular to the edge (A, B), which take the place of normals to the surface, hence the inclination of the planes of $a'b'$ and $a'd'$ is equal to the angle of the facet A, which does not change.

The specific curvature of the polyhedron at P is measured by the area $a'b'c'd'$, which depends only on the angles, and therefore remains constant as the polyhedron changes its form.

Hence, proceeding to the limit, we obtain another proof of the proposition, which is due to Gauss, already proved in Art. 821.

Ruled Surfaces.

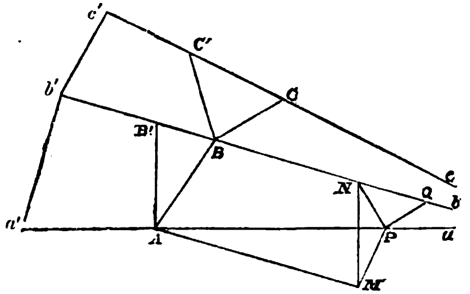
831. *Scroll referred to curvilinear coordinates.*

Let Aa, Bb, Cc... be consecutive generating lines of a scroll, each inclined at the same indefinitely small angle dp to the preceding line. Let AB', BC', &c., be the shortest distances of Bb from Aa, Cc from Bb, &c., so that the polygon ABC... is ultimately the line of striction. The extremities of radii of an unit sphere drawn parallel to Aa, Bb, Cc, &c., will ultimately form a curve, and p will be an arc of this curve measured from a fixed point in it.

AB' depends only on the position of Aa and the inclination of Bb to Aa, and may be represented by βdp, where β is a function of p only; similarly $BB' = \alpha' dp$, and the angle between BC' and $AB' = \gamma dp$, where α' and γ are functions of p, and $\alpha' = d\alpha/dp$.

If the scroll be striated by the generating lines and curves cutting them orthogonally, let any one of these curves be identified by q the distance measured along a generating line from one of them $a'b'c'$... chosen as the curve of reference, and let q_0 be the value of q at the line of striction, so that $dq_0 = \alpha' dp$, and $q_0 = \alpha$.

Let PQ be an element ds of a curve crossing Aa, Bb at P, Q, whose coordinates are p, q and $p + dp$, $q + dq$, and let a plane through P, perpendicular to Aa, meet Bb in N, and draw NM perpendicular to the plane through Aa parallel to Bb, then,



ultimately, $NQ = dq$, $PM = (q - q_0)\,dp$, $MN = \beta dp$, and

$$PQ^2 = NQ^2 + PN^2 = NQ^2 + PM^2 + MN^2,$$

$$\therefore (ds)^2 = (dq)^2 + \{(q - q_0)^2 + \beta^2\}\,(dp)^2.$$

If I be the inclination of the tangent plane at P to that at A,

$$\tan I = PM/MN = (q - q_0)/\beta.$$

The specific curvature at any point (p, q) is

$$-P_{22}/P = -\beta^2/\{(q - q_0)^2 + \beta^2\}^2 = -\cos^4 I/\beta^2,$$

which is negative, the surface being anticlastic.

The three quanti α, β and γ are the functions of p, whic are sufficient to deter the form of the scroll, and it is to b observed that ds is indep ent of γ, which is the only one of th three quantities altered by deformation, Art. 833.

The cone, whose vertex is the centre the unit sphere anc generating lines parallel to those of the scro called by Bou the *director cone* of the scroll.

832. The equation $P^2 = (q-\alpha)^2 + \beta^2$ might have be duced from Bour's equations of Art. 816, for H, being the curv of the generating line, vanishes; $\therefore (TP^2)_2 = 0$ or $TP^2 = f(p)$, $P_{22} = PT^2 = \{f(p)\}^2/P^3$, (1), which is satisfied by $P^2 = Lq^2 + 2Mq$ where L, M, N are functions of p only, and, by a proper choic the parameter p, $L=1$; hence P^2 is of the form $(q-\alpha)^2 + \beta^2$, substituting this value in the equation (1), $\beta = f(p)$.

By the remaining equation $T_1 + (PK)_2 = 0$, we obtain

$$PK = -\frac{d}{dp}\int\frac{\beta dq}{(q-\alpha)^2+\beta^2} = -\frac{d}{dp}\left\{\tan^{-1}\frac{q-\alpha}{\beta} + \phi(p)\right\},$$

$$\therefore K = -d\{I + \phi(p)\}/Pdp,$$

therefore, since K is the curvature of the geodesic tangent to t curve (q), and Pdp an element of the curve, $-d\{I+\phi(p)\}$ is t angle of contingence of the normal section perpendicular to t generating line, and $-d\phi(p)$ is therefore the angle between t consecutive shortest distances; $\therefore \phi'(p) = -\gamma$. It should be notic that I is unchanged on deformation.

Since a radius of the unit sphere which is in the direction of th shortest distance is perpendicular to two consecutive generatin lines, the cone generated by these radii is the cone reciprocal to th director cone, and a small arc of its intersection with the sphere i equal to γdp the corresponding angle of contingence of the directo cone.

833. *Deformation of a scroll when the generating lines are the lines of bending.*

The scroll may be deformed in such a manner that all the shortest distances are parallel, by turning the whole portion of the scroll above Bb about Bb through the angle γdp until BC' is parallel to AB', and then about Cc and so on, and in that case the director cone becomes a plane.

By bending this new scroll back about its generating lines, through proper angles, it may be deformed into another scroll, whose director cone is perfectly arbitrary, which is one of Bour's theorems.

When the scroll is deformed so that the shortest distances are all parallel, the director cone becoming a plane to which all the generating lines are parallel, the projections of the shortest

[illegible]tances upon that plane is a curve of which $a'dp = d\sigma$ is an [illegible]mentary arc; and since p is the angle made by the tangent to [illegible] curve with a fixed line, this is the intrinsic equation of the [illegible]rve.

The deformed scroll can therefore be generated by constructing [illegible] a plane the curve $a'dp = d\sigma$, and a cylinder of which this curve [illegible] the base, take $MP = \int \beta\, dp$ on the generating line of the cylinder [illegible]ssing through a point M [illegible] the base, whose distance from a fixed [illegible]int measured along the base is α; draw a line parallel to the base [illegible]aching the cylinder at P, this will be a generating line of the [illegible]oll. If the cylinder were unwrapped the locus of the points of [illegible]ct would be the curve given by $x = \alpha$, $y = \int \beta dp$, being the [illegible]ned line of striction of the deformed scroll.

We can illustrate the propositions given above by taking the particular [illegible]ich is a hyperboloid of revolution of one sheet, given by the equations [illegible] its generating lines $x/a = \cos\phi - \sin\phi\, z/c$, and $y/a = \sin\phi + \cos\phi\, z/c$, [illegible]ith the notation of the preceding articles

$$[\mathrm{c}]\mathrm{os}\, dp = \{a^2 \sin\phi \sin(\phi + d\phi) + a^2 \cos\phi \cos(\phi + d\phi) + c^2\}/(a^2 + c^2),$$

$$\text{or } 1 - \tfrac{1}{2}(dp)^2 = 1 - \tfrac{1}{2}a^2 (d\phi)^2/(a^2 + c^2), \quad \therefore\ dp = a\,d\phi/\sqrt{(a^2 + c^2)}.$$

[illegible] the line of striction for this surface is, by symmetry, the principal [illegible] section, and $-a'dp$ is therefore the projection of the arc $a\,d\phi$ of this [illegible]ose direction-cosines are $-\sin\phi$, $\cos\phi$, 0, on the generating line;

$$-a'dp = a^2 d\phi/\sqrt{(a^2 + c^2)} = a\,dp, \text{ and } (\beta dp)^2 = (a\,d\phi)^2 - (a\,dp)^2 = c^2 (dp)^2;$$

$$\therefore\ (ds)^2 = (dq)^2 + \{(q + ap)^2 + c^2\}(dp)^2.$$

[illegible] μ, ν) be the direction of the shortest distance of consecutive generating [illegible]nce it is perpendicular to both

$$-\lambda a \sin\phi + \mu a \cos\phi + \nu c = 0 \quad \text{and} \quad \lambda\cos\phi + \mu\sin\phi = 0;$$

$$\therefore\ \lambda/\sin\phi = \mu/-\cos\phi = \nu c/a = c/\sqrt{(a^2 + c^2)},$$

[illegible] is the angle between consecutive shortest distances,

$$\cos(\gamma dp) = \{c^2 \sin\phi \sin(\phi + d\phi) + c^2 \cos\phi \cos(\phi + d\phi) + a^2\}/(a^2 + c^2);$$

$$\therefore\ 1 - \tfrac{1}{2}(\gamma dp)^2 = 1 - \tfrac{1}{2}(d\phi)^2 c^2/(a^2 + c^2);$$

$$\therefore\ \gamma dp = c\,d\phi/\sqrt{(a^2 + c^2)} \text{ and } \gamma = c/a.$$

[illegible]en the scroll is deformed so that all the generating lines are parallel [illegible]ed plane, the cylinder which they all touch has for its intrinsic equation [illegible] $\sigma = 0$, and is therefore circular with radius a, and the locus of the points [illegible]tact cuts the generating lines of the cylinder at the constant angle [illegible] $/c$).

Surfaces of Revolution.

[illegible]ondition of applicability of two surfaces of revolution.

[illegible] the curvilinear coordinates which define the position of a [illegible] P on a surface of revolution be p the longitude, and q the arc [illegible]ured along the meridian from a fixed point in it to P, and let [illegible] the distance of P from the axis, which depends only on q, then [illegible] $= (dq)^2 + r^2 (dp)^2$. If r', p', q' correspond on another surface of [illegible]ution to r, p, q on the first, for this surface $(ds)^2 = (dq')^2 + r'^2 (dp')^2$, [illegible] the surfaces will be applicable to one another, if the two [illegible]essions for $(ds)^2$ are equal for all positions of the element, and

this is clearly the case if $q'=q$, $r'=kr$, $p'=p/k$; in fact, th elements of the arcs of the meridians and parallels of the tw surfaces are respectively $dq'=dq$ and $r'dp'=rdp$.

If ψ, ψ' be the inclination to the axes of revolution of tangent to the meridians at corresponding points,

$$\sin\psi' = -dr'/dq = -kdr/dq = k\sin\psi;$$

hence, taking $k>1$, the only parts of the first surface which ca be applied to the second are those for which $\sin\psi$ is not greate than k^{-1}.

836. Cayley has discussed the forms of the surfaces of revolutio to which portions of a spherical surface included between tw meridians and two parallels can be applied.

Measuring q from the equator which is the great circle of th sphere perpendicular to the axis,

$$(ds)^2 = (dq)^2 + \cos^2 q\,(dp)^2,$$

if the radius of the sphere be taken for the unit; also, for th surface of revolution to which any portion of it is applicable,

$$(ds)^2 = (dq)^2 + k^2\cos^2 q\,(dp')^2.$$

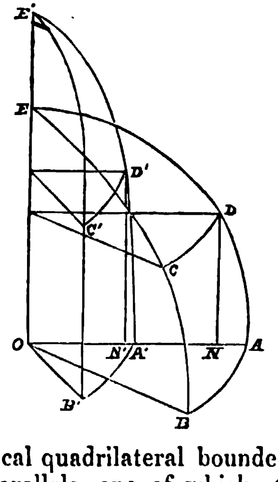
If (z, r) and (z', r') be corresponding points,

$$z=\sin q \quad\text{and}\quad (dz')^2 = (dq)^2 - (dr')^2 = (dq)^2(1-k^2\sin^2 q);$$

$$\therefore z' = \int dq\,\sqrt{(1-k^2\sin^2 q)} = E(k, q),$$

which is an elliptic function of the second kind if $k<1$.

Also, if ψ' be the angle at which the tangent to the curve a (z', r') cuts the axis, $\sin\psi' = -dr'/dq = -kdr/dq = k\sin q$.

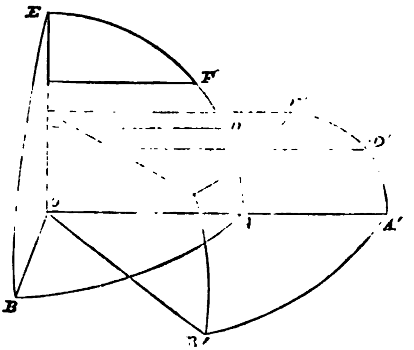


Consider the spherical quadrilateral bounded by two meridian AD, BC, and two parallels, one of which AB is an arc of th equator and the other DC has latitude q. First, let $k<1$, th

face into which the quadrilateral is deformed is $A'B'C'D'$, ere the radius of $A'B'$ is k, and $\angle A'OB' = \angle AOB/k$. Also, DN, $D'N'$ be perpendicular to the equator, $D'N' > DN$, since h term of $\int dq \sqrt{(1 - k^2 \sin^2 q)}$ is greater than the corresponding ns of $\int dq \sqrt{(1 - \sin^2 q)}$. Considering the semilune AEB, deformed $A'E'B'$, at the equator $\sin\psi' = 0$, and at E' for which $q = \frac{1}{2}\pi$, $\psi' = k$; therefore the surface $A'E'B'$ cuts the equator at right les and has a conical point at E', whose semi-vertical angle in^{-1}k.

If $AB > 2\pi k$, $A'OB'$ will be $> 2\pi$, and to effect the deformation hemisphere must be slit along a meridian AE and overlap surface $A'E'B'$.

Next, let $k > 1$, then the latitude of the parallel which bounds portion of the sphere which can be applied to the corresponding ace of revolution cannot be greater than $\sin^{-1}(k^{-1})$. Also



$\psi' = k \sin q$, $\therefore$ the surface meets the equator at right angles. at F', which corresponds to F in the highest latitude on the re, the tangent is parallel to the equator.

37. Def. A helicoidal surface is generated by the revolution given curve about a line, fixed with reference to the curve, h moves in its own direction with a velocity always propor- l to the angular velocity of the curve; so that every point e curve describes a helix of the same pitch.

38. *A helicoidal surface is applicable to a surface of revolution.* et r, θ, ζ be cylindrical coordinates of any point P of the e in an initial position, these will be functions of q, the length e arc measured from a fixed point to the point P; the position in any position of the curve will be given by

$$x = r\cos(p + \theta), \quad y = r\sin(p + \theta), \quad z = ap + \zeta;$$

hence, denoting dr/dq by r', &c.,

$$(dx)^2+(dy)^2=(r'dq)^2+r^2(dp+\theta' dq)^2,\quad dz=adp+\zeta' dq;$$

$$\therefore\ (ds)^2=(dq)^2\{r'^2+r^2\theta'^2+\zeta'^2\}+2dq\,dp\,(r^2\theta'+a\zeta')+(dp)^2(r^2+a^2)$$

$$=(dq)^2\{r'^2+r^2(a\theta'-\zeta')^2/(r^2+a^2)\}+(r^2+a^2)\{dp+dq\,(r^2\theta'+a\zeta')/r^2+a^2)\}^2$$

$=(dq')^2+(r^2+a^2)(dp')^2$, and, since q' is a function of q only, and therefore r^2+a^2 a function of q' only, the helicoidal surface is applicable to a surface of revolution, of which the arc of a meridian is q', and the distance of the point (p', q') from the axis is $\sqrt{(r^2+a^2)}$.

839. If the generating curve be replaced by a straight line cutting the axis at an angle of 45°,

$$x=\tfrac{1}{2}\sqrt{2}\,q\cos p,\quad y=\tfrac{1}{2}\sqrt{2}\,q\sin p,\quad z=ap+\tfrac{1}{2}\sqrt{2}\,q,$$

$$\begin{aligned}\text{and}\quad (ds)^2&=\tfrac{1}{2}\{(dq)^2+q^2(dp)^2\}+(adp+\tfrac{1}{2}\sqrt{2}\,dq)^2\\ &=(dq)^2+\sqrt{2}\,adpdq+(\tfrac{1}{2}q^2+a^2)(dp)^2\\ &=(dq')^2+(\tfrac{1}{2}q^2+a^2)(dp')^2,\end{aligned}$$

$$\begin{aligned}\text{where}\quad dq'^2&=(dq)^2(q^2+a^2)/(q^2+2a^2)\\ dp'&=dp+\sqrt{2}\,adq/(q^2+2a^2)\end{aligned}$$

$$\text{and so }p'=p+\tan^{-1}(q/a\sqrt{2}).$$

Let x', y' be coordinates of (p', q') a point in the curve generating the surface of revolution;

$$y'^2=\tfrac{1}{2}q^2+a^2,\quad \sqrt{2}\,dy'=qdq/\sqrt{(q^2+2a^2)},$$

$$(dx')^2=(dq')^2-(dy')^2=\tfrac{1}{2}(dq)^2;\quad \therefore\ x'^2=\tfrac{1}{2}q^2;$$

hence $y'^2=x'^2+a^2$; therefore the meridian curve is a rectangular hyperbola.

If (x', y', z') be the point in the hyperboloid which corresponds to (p, q), and $q/a\sqrt{2}=\tan\psi$,

$$y'=\sqrt{(\tfrac{1}{2}q^2+a^2)}\cos(p+\psi)=a\cos p-\sqrt{\tfrac{1}{2}}\,q\sin p,$$

$$z'=\sqrt{(\tfrac{1}{2}q^2+a^2)}\sin(p+\psi)=a\sin p+\sqrt{\tfrac{1}{2}}\,q\cos p;$$

$\therefore\ y'\cos p+z'\sin p=a$, the equation of the tangent plane to the hyperboloid $y'^2+z'^2-x'^2=a^2$ at the point $(0, a\cos p, a\sin p)$: hence every generating line of the helicoid has for its corresponding line a generator of the hyperboloid, and since when $q=0$, $y'^2+z'^2=a^2$, the axis is bent into the principal circular section.

840. It has been shewn that, when the form of P in the equation $(ds)^2=(dq)^2+(Pdp)^2$ is given, we have equations sufficient to determine the values of K, H and T for all the surfaces which are applicable to one another and which have the same value of P; and that when these three quantities have been determined, such curves as the lines of curvature and the chief curves can be formed for each surface.

Instead of supposing P given, as by giving one of the above surfaces, we may endeavour to find what must be the form of P in order that the surfaces applicable to each other may all be derived by deformation from a surface possessing a given property which can be represented by a relation between H, K, T; an example of such a property is that the area of the surface included within any closed curve traced on it be less than for any other form of the surface bounded by the same line, in which case the

two principal curvatures at any point must be equal and in opposite directions, the corresponding relation is $H+K=0$; Bour has obtained a differential equation for the value of P in this case.

841. *To find a differential equation for determining P when the corresponding surface is applicable to a surface of minimum area.*

In this case, since $H+K=0$, the fundamental equations become

$$(KP^2)_2=-PT_1,\ (TP^2)_2=PK_1,\ \text{and}\ T^2+K^2=S^2,\ \text{Art. 816},$$

where $-S^2$ is the specific curvature $-P_{22}/P$.

$$\text{Let } K=S\cos\phi,\ T=S\sin\phi,\ \therefore\ KdT-TdK=S^2d\phi;$$

$$\therefore\ S^2\phi_1=KT_1-TK_1=-\{K(KP^2)_2+T(TP^2)_2\}/P=-(S^2P^4)_2/2P^3;$$

$$\therefore\ \phi_1=-\tfrac{1}{2}P\{\log(S^2P^4)\}_2=-\tfrac{1}{2}P\{\log(P^3P_{22})\}_2.$$

$$\text{Similarly } \phi_2=\tfrac{1}{2}P^{-1}\{\log(P^{-1}P_{22})\}_1;\ \text{and}\ \phi_{12}=\phi_{21},$$

$$\therefore\ \frac{d}{dq}\left\{P\frac{d}{dq}\log\left(P^3\frac{d^2P}{dq^2}\right)\right\}+\frac{d}{dp}\left\{\frac{1}{P}\frac{d}{dp}\log\left(\frac{1}{P}\frac{d^2P}{dq^2}\right)\right\}=0,$$

which is the differential equation required.

By Art. 810, if θ be the inclination of a line of curvature to the curve (p), since $Pdp/\sin\theta=dq/\cos\theta$, $T\cos 2\theta=K\sin 2\theta$, so that the subsidiary angle ϕ is 2θ.

COR. If the surfaces which are applicable to surfaces of minimum area be surfaces of revolution, the first term only remains, since P is a function of q only.

LVII.

1. Prove that the curve given by

$$2x^2(a^2-b^2)(a^2-c^2)=a^2\theta^2(2a^2-b^2-c^2),$$
$$2y^2(a^2-b^2)=b^2(a^2-b^2-\theta^2),$$
$$2z^2(a^2-c^2)=c^2(a^2-c^2-\theta^2)$$

is a line of curvature on $x^2/a^2+y^2/b^2+z^2/c^2=1$.

2. If $PMQN$ be an infinitesimal rectangle formed by striating curves of two systems cutting orthogonally, and α, β be the angles made with the tangent plane at P by the tangent at M to MQ and that at N to NQ, prove that $\alpha.MQ=\beta.NQ$ ultimately.

3. Prove that the chief lines on a surface are not generally geodesics.

4. If the line of striction of a scroll cut the generating lines at a constant angle it must be a geodesic line.

5. Shew that every scroll may be deformed into a surface generated by straight lines moving parallel to a fixed plane.

6. When a scroll is deformed so as to become another scroll, the relative position of the tangent planes along any generating line are unaltered by the transformation.

7. The curvilinear coordinates of a point on a sphere are the longitude p and co-latitude q, shew that the differential equation of any geodesic line is

$$\frac{d^2q}{dp^2} - 2\cot q\,\frac{dq}{dp} - \sin q\cos q = 0.$$

Find the complete integral of the equation and determine the constants when the geodesic joins two given points on the sphere.

LVIII.

1. Prove that, if the two sets of curves (α), (β), and (α'), (β') divide a surface into square elements, so that $(ds)^2 = \lambda\{(d\alpha)^2 + (d\beta)^2\} = \lambda'\{(d\alpha')^2 + (d\beta')^2\}$, $\alpha' + i\beta'$ is any arbitrary function of $\alpha + i\beta$.

2. When an ellipsoid is striated by the lines of curvature, corresponding to constant values of p and q, find the principal curvatures at any point in terms of the values of p and q at that point, and shew that they satisfy the equations (7) of Art. 815.

3. Prove that if the curves (p) and (q) on a hyperboloid of one sheet be generating lines the equation of any plane curve will be of the form

$$apq + bp + cq + d = 0.$$

4. Prove that a flexible and inextensible surface in the form of a hyperboloid of revolution of one sheet, whose transverse and conjugate semi-axes are respectively a and c, and the angle of whose conical asymptote is 2β, may be deformed as follows:

i. The principal circle may be bent into a straight line, in which case one system of generating lines will be the generating lines of a conicoid of uniform pitch inclined to the axis at a constant angle, the axis being the straight line into which the circle was bent.

ii. The circle may be bent into a helix of pitch $\frac{1}{2}\pi - \beta$ traced on a cylinder of radius a, in which case the generators become tangent lines to the cylinder perpendicular to the axis.

iii. The surface may be applied to the surface of revolution of which the equation of a meridian curve is $r = c\sec\beta\cosh(x/c)$.

5. Shew that a sphere may be deformed into a surface of revolution, which in its general form resembles that obtained by the revolution of an arc less than a semicircle round its chord, the half chord being greater and the versed sine less than the radius of the sphere. Also that a zone of the sphere contained between two parallels may be deformed into a surface, which in its general form resembles that obtained by the revolution of a half ellipse about a line parallel to and beyond its bounding axis, the semi-axis being less, and the greatest radius of rotation greater, than the radius of the sphere.

6. Find the area and length of the circumference of a small geodesic circle traced on a surface, neglecting higher powers of the radius than the third; and deduce that, on deformation, the specific curvature at any point is unaltered.

7. When a scroll is transformed into another scroll the angles of contingence of the normal sections perpendicular to the same generating line increase or diminish by the same amount as do those of the director cone.

LIX.

1. A hyperboloid of revolution is given by the equations $x = a \sec\phi \cos\theta$, $y = a \sec\phi \sin\theta$, and $z = a \cot\beta \tan\phi$, shew that the curves which cut orthogonally all the generating lines of one system are given by the equation

$$\theta - \phi + \operatorname{cosec}^2\beta \tan\phi = \operatorname{cosec}\beta\, q/a,$$

where q is the constant part of any generating line intercepted by two of the curves, one of which passes through the point $\theta = 0$, $\phi = 0$.

2. Two generating lines of a hyperboloid of revolution intersect the principal circular section at points A, P, the arc AP subtending an angle p at the centre. A curve cuts all the generators of one system orthogonally and meets the generators through A and P in A, A', A''... and Q, Q', Q''..., shew that $AA' = A'A'' = ... = QQ' = Q'Q'' = ...$, and that the length AQ of the curve is $\frac{1}{2}a \cot^2\beta\,(\cot\psi \operatorname{cosec}\psi + \log\cot\frac{1}{2}\psi)$, where 2β is the vertical angle of the asymptotic cone, and $\cot\psi = p \sin\beta \tan\beta$.

3. If a finite quadrilateral be formed on a conicoid by four lines of curvature, the distances of the angular points from the consecutive confocal will form a proportion.

4. Prove that the equation of the wave surface referred to elliptic coordinates is $(q + r - a - b - c)(r + p - a - b - c)(p + q - a - b - c) = 0$.

5. Prove that the complete solution of the equation $\{P(\log P^3 P_{22})_2\}_2 = 0$, Art. 842, Cor. is given by the equations $q + c = a\{t^{n+1}/(n+1) + t^{n-1}/(n-1)\}$ and $P = ma\,(t^{n+1} + t^{n-1})$, the four arbitrary constants being c, a, m and n.

6. Shew that, when a surface of minimum area can be obtained by the deformation of a surface of revolution,

i. Its lines of curvature are cut at a constant angle by a geodesic which is one of the deformed meridians.

ii. The same line of curvature cuts any two deformed meridians at angles whose difference varies as the difference of their longitudes.

7. Shew that the surface formed by the revolution of a hypocycloid of four cusps about the line joining opposite cusps, may, by slitting it along the circular cuspidal edge, be deformed into two parts of another surface generated by a similar hypocycloid, which cut one another at an angle 2α, where $\cos^2\alpha$ is the ratio of the linear magnitudes of the two hypocycloids.

8. A point moves along a generator of a scroll with velocity V, prove that the angular velocity of the tangent plane to the scroll at the point is $V/\sqrt{(R_1 R_2)}$, $R_1 R_2$ being the principal radii of curvature at the point.

LX.

1. A quadrilateral figure $ABCD$ is formed on an ellipsoid by four lines of curvature, prove that if O be the centre of the ellipsoid, $OA^2+OC^2=OB^2+OD^2$, and that the chords AC, BD will be equal.

Prove that when a confocal ellipsoid intersects the hyperboloids, which determined the above four lines of curvature, in another quadrilateral $A'B'C'D'$

$$OA'^2-OA^2=OB'^2-OB^2=OC'^2-OC^2=OD'^2-OD^2$$

2. The principal radii of curvature at the intersection of two of the three confocals $x^2/(a+p)+y^2/(b+p)+z^2(c+p)=1$, (1), $x^2/(a+q)+\ldots=1$, (2), and $x^2(a+r)+\ldots=1$, (3), are, for the confocal (1), ρ_q, ρ_r; for (2), σ_r, σ_p; for (3), τ_p, τ_q, ρ_q corresponding to the normal section touching the line of curvature common to (1) and (2), &c. Prove the equations $\rho_q\sigma_r\tau_p=\rho_r\sigma_p\tau_q$, $\sigma_r/\sigma_p+\tau_q/\tau_p=1$, $\tau_p/\tau_q+\rho_r/\rho_q=1$, and $\rho_q/\rho_r+\sigma_p/\sigma_r=1$.

3. If the polar equation of a surface be expressed in the form

$$d\log r=P\,d\theta+Q\sin\theta\,d\phi,$$

prove that the angle ψ between the radius vector and normal will be given by $\tan^2\psi=P^2+Q^2$; verify this result in the case of the ellipsoid.

4. A hyperboloid of one sheet is determined by the equations

$$x/a=(1+pq)/(p+q),\quad y/b=(p-q)/(p+q),\quad z/c=(1-pq)/(p+q),$$

prove that the lines of curvature are given by the equation

$$dp\,(1-2Ap^2+p^4)^{-\frac{1}{2}}\pm dq\,(1-2Aq^2+q^4)^{-\frac{1}{2}}=0,$$

$$\text{where } A=(a^2-2b^2-c^2)/(a^2+c^2).$$

Shew that the striating curves (p) and (q) are the generating lines.

5. Prove that along certain generating lines of a scroll the specific curvature is zero at every point except where it meets the line of striction. Explain the apparent discontinuity.

6. A scroll is generated by a straight line moving so as to intersect the circle $z=0$, $x^2+y^2=a^2$ and the two straight lines $x=0$, $z=c$ and $y=0$, $z=-c$; prove that, if it be deformed by bending about its generating lines until they are all parallel to the same plane, their projections on that plane will envelope an epicycloid.

Shew that the generating lines will touch a cylinder on the epicycloid as base at points of a curve of constant inclination to the base.

7. If a be the radius of a flexible and inextensible sphere, p, q the longitude and latitude of a point upon it, $a\delta a$, δp, δq the increments of a, p, q when any part of the sphere is bent infinitesimally, prove that

$$\frac{d\delta q}{dq}+\delta a=0,\qquad \frac{d}{dq}(\sec q\,\delta q)-\sec q\,\frac{d\delta p}{dp}=0,$$

$$\text{and } \frac{d}{dp}(\sec q\,\delta q)+\cos q\,\frac{d\delta p}{dq}=0;$$

shew that the general real value of δp is $f(p+iu)+f(p-iu)$, where $u=\log(\sec q+\tan q)$.

Deduce the particular solution

$$\delta p=\cos sp\,\{A\tan^s(\tfrac{1}{4}\pi-\tfrac{1}{2}q)+B\cot^s(\tfrac{1}{4}\pi-\tfrac{1}{2}q)\},$$

$$\sec q\,\delta q=-\sin sp\,\{A\tan^s(\tfrac{1}{4}\pi-\tfrac{1}{2}q)-B\cot^s(\tfrac{1}{4}\pi-\tfrac{1}{2}q)\},$$

$$\delta a=\sin sq\,\{A(s+\sin q)\tan^s(\tfrac{1}{4}\pi-\tfrac{1}{2}q)+B(s-\sin q)\cot^s(\tfrac{1}{4}\pi-\tfrac{1}{2}q)\}.$$

CHAPTER XXV.

FUNCTIONAL AND DIFFERENTIAL EQUATIONS OF FAMILIES OF SURFACES. ENVELOPES.

842. Before discussing the general question, we will consider a simple case, with which the student is already familiar.

He has seen, Art. 209, that if a straight line move so as always to intersect three given lines, which do not themselves intersect, the straight line will generate a definite surface. But consider a straight line which has only the two constraints, that it must intersect two given straight lines which do not themselves intersect, the straight line no longer generates a surface when it assumes all its possible positions, but the freedom which the omission of one of the constraining lines has given to it will enable it to fill all space; any arbitrary additional constraint which still allows it some freedom of motion will cause it to sweep out a definite surface as it assumes all its possible positions. And all such surfaces have a common character due to the two constraints common to them all, although they differ among themselves in consequence of the arbitrary nature of the third constraint.

843. Thus, if $y=0$, $z=c$ and $z=0$, $x=a$ be the equations of the two fixed lines, those of the moving line will be $x=a+\alpha z$, $z=c+\beta y$, intersecting each of the former, and, by a proper choice of α and β, passing through any point in space; let us introduce any other arbitrary constraint, such as that it shall always intersect a curve whose equations are $F(x, y, z)=0$, $G(x, y, z)=0$; since there are values of x, y and z which satisfy these equations and those of the moving line, we can eliminate these three quantities and obtain a relation between α and β, viz. $f(\alpha, \beta)=0$, or $\beta=\phi(\alpha)$, where f or ϕ denote arbitrary functions, and the moving line will then generate any one of the family of surfaces denoted by

$$f\{(x-a)/z,\ (z-c)/y\}=0 \quad \text{or} \quad (z-c)/y=\phi\{(x-a)/z\}.$$

844. If we take away one of the two given constraints, we must, in order to oblige the line to generate a surface, subject it to two arbitrary constraints, and the family of surfaces generated will be more extensive, including the former as a particular species.

Thus, if the only given restriction be that the line shall always intersect the given line whose equations are $z=0$, $x=a$, the equa-

tions of the generating line will be $x=a+\alpha z$, $y=\beta z+\gamma$, and if it be constrained to intersect each of two arbitrary curves we shall obtain two arbitrary relations between α, β and γ by two eliminations, whence $\beta=f(\alpha)$, $\gamma=\phi(\alpha)$, and the equation of the family of surfaces, which the straight line might generate, would be

$$y=zf\{(x-a)/z\}+\phi\{(x-a)/z\},$$

which includes the former if $-cf$ be written for ϕ.

845. If the functions be eliminated in the ordinary way by differentiation we obtain for the first case

$$(x-a)(z-c)p+yzq=z(z-c),$$

and for the second

$$q^2(x-a)^2r+2q(x-a)\{z-p(x-a)\}s+\{z-p(x-a)\}^2t=0.$$

These are the partial differential equations of these families of surfaces, from which the functional equations can be reproduced.

846. *Generation of a definite surface by the motion of a curve.* Consider now the case of a curve of a particular species; such a curve is represented by means of equations, the manner in which the current coordinates enter the equations defining the species, and the constants determining the position and dimensions of the curve; if one of these constants be made to vary it is called a *parameter*, and as this parameter changes continuously the curve changes its position or magnitude or both, and generates a definite surface represented by the equation resulting from the elimination of the parameter from the equations of the curve.

The same thing is obvious if there were n such variable parameters, connected by $n-1$ fixed and independent conditions, the $n-1$ equations representing these conditions, with the equations of the curve, giving $n+1$ independent equations, from which the n parameters could be eliminated and the equation of a definite surface obtained, which would be the locus of the curve.

847. *Families of surfaces generated by curves, on which one or more arbitrary conditions are imposed.*

If one or more of the conditions, which would constrain a curve to generate a definite surface, were removed, there would be no point in space through which, by a proper determination of the parameters, the curve could not be constrained to pass; but if, in place of each removed condition, we were to substitute an arbitrary condition, surfaces would be generated by the curve, as it assumed all its possible states, which would be definite for each arbitrary condition, and they would be represented by an equation involving arbitrary functions, and form a *family* of surfaces having a common

character; and by the elimination of the arbitrary functions we should obtain a partial differential equation, which would be satisfied at every point of each individual of the family.

848. *Functional equation of surfaces generated by curves subject to one arbitrary condition.*

Let the equations of the variable curve contain n parameters, subject to $n-2$ independent conditions; these, with the two equations of the curve, supply n equations, from which $n-1$ of the parameters can be eliminated, and two final equations, $u=\alpha$, $v=\beta$ found, u and v being determinate functions of x, y, z; these equations represent the variable curve, and in order that a surface may be generated an arbitrary relation between α and β must be imposed, let this be $\beta=f(\alpha)$. Hence, the equation of any surface so generated must be of the form $v=f(u)$, where $f(u)$ is an arbitrary function of u.

849. We may notice that, whatever arbitrary condition is imposed which compels the generating curve to describe a surface as it passes through all its states by the variation of the parameters, we may always replace this condition by means of an arbitrary curve which the generating curve must intersect. For whatever surface is described we can trace upon it a curve which will intersect all the curves by which the surface was generated; therefore if the surface be removed with the exception of this curve, it can be again reproduced by constructing the generating curves which intersect this curve.

850. *Partial differential equation of surfaces generated by curves, subject to one arbitrary condition.*

Since each surface of the family is generated by curves for each of which u and v are constant, the tangent plane at any point (x, y, z) must contain an element of the generating curve through that point, whose projections on the axes are dx, dy, dz, which satisfy the equations

$$u_1dx+u_2dy+u_3dz=0, \quad v_1dx+v_2dy+v_3dz=0,$$

where u_1, u_2, u_3, &c. are the partial differential coefficients of u, v with respect to x, y, and z; hence $dx:dy:dz=P:Q:R$, where $P=u_2v_3-u_3v_2$, &c., and, if $z=f(x, y)$ be the equation of the surface, $pdx+qdy=dz$; $\therefore Pp+Qq=R$, which is the partial differential equation of the family of surfaces.

851. Conversely, if we have given the partial differential equations of a surface,

$$PF'(x)+QF'(y)+RF'(z)=0, \text{ or } Pp+Qq=R, \qquad (1)$$

it is evident that every element of any of the curves determined at any point by $dx/P=dy/Q=dz/R$ (1) will lie in the tangent plane to the surface at that point; hence, if we travel along that curve, we shall at no point leave the surface, that is, the curve lies wholly on the surface; and since the solution of the equations (1) involves

two arbitrary constants, or is of the form $u=\alpha$, $v=\beta$; one of the curves represented by these equations may be made to pass through every point of the surface, and, by what has been shewn before will lie entirely on the surface, which will therefore be generated by such curves, and this requires that $\beta=f(\alpha)$; therefore, $v=f(u)$ is the form of the equation of the surface.

This mode of stating the nature of the solution of the partial differential equation was suggested by Moulton.

852. *Functional and differential equations of cylindrical surfaces.*

Let (l, m, n) be the direction of the generating lines. The equations of any generating line may be put in the form $ly-mx=\alpha$, $lz-nx=\beta$, in which α, β are the parameters connected by the arbitrary relation $\beta=f(\alpha)$. The functional equation is

$$lz-nx=f(ly-mx).$$

The differential equation may be found by the usual method of eliminating the function, or immediately from the mode of generation we have

$$dx/l=dy/m=dz/n, \text{ therefore } lp+mq=n,$$

representing that the normals to the surface are all perpendicular to the same direction.

853. *Functional and differential equations of conical surfaces.*

Let (a, b, c) be the vertex, then the equations of a generating line may be written $y-b=\alpha(x-a)$, $z-c=\beta(x-a)$, and $\beta=f(\alpha)$; hence the functional equation is $\dfrac{z-c}{x-a}=f\left(\dfrac{y-b}{x-a}\right)$.

Since $dx : dy : dz = x-a : y-b : z-c$, the differential equation is

$$(x-a)p+(y-b)q=z-c,$$

representing that the tangent planes all pass through the vertex.

854. *Functional and differential equations of conoidal surfaces having a given axis and given directing plane.*

DEF. A conoidal surface, or conoid, is a surface generated by a straight line moving so as to intersect a given straight line, the *axis*, and to remain parallel to a given plane, the *directing plane*. If the axis be perpendicular to the directing plane the surface is called a *right conoid*.

Let $u=0$, $v=0$ be the equations of the given axis, $w=0$ that of the directing plane. The plane containing the axis and a generating line, and a plane parallel to the directing plane, give the equations of the generating line in the form $v=\beta u$, $w=\alpha$; the general functional equation will therefore be $v=uf(w)$. That of the right conoid, whose axis is that of z, is $y=xf(z)$, if the axes be rectangular.

With the notation of Art. 850 the differential equation is

$$\begin{vmatrix} uv_1-vu_1, & uv_2-vu_2, & uv_3-vu_3 \\ w_1\ \ , & w_2\ \ , & w_3 \\ p\ \ , & q\ \ , & -1 \end{vmatrix}=0.$$

For the right conoid it is $xp+yq=0$.

The student should shew that the tangent plane at any point contains the generating line.

855. *Functional and differential equations of surfaces of revolution round a given axis.*

These may be considered as generated by the motion of a circle, whose centre lies on a fixed straight line, to which its plane is perpendicular, and

whose circumference meets a curve in the same plane as the axis, called a meridian.

Let (a, b, c) be a point on the axis, whose direction is (l, m, n). The equations of the generating circle may be written

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2, \quad lx + my + nz = p.$$

The form of the meridian, which is arbitrary, gives a relation between an abscissa $p - la - mb - nc$ and an ordinate $\sqrt{\{r^2 - (p - la - mb - nc)^2\}}$, so that r^2 is an arbitrary function of p, and the required functional equation is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = f(lx + my + nz).$$

The differential equation is

$$\{m(z-c) - n(y-b)\}\,p + \{n(x-a) - l(z-c)\}\,q = l(y-b) - m(x-a).$$

The student should find the differential equation of surfaces of revolution considered as surfaces all the normals to which intersect a fixed straight line.

856. *Families of surfaces generated by curves having more than one arbitrary condition.*

Consider now the general case in which, after determining the species of curves by which a surface is generated, that is, after all the fixed given conditions have been introduced, there remain more than two parameters; these are to be connected by arbitrary conditions, such that, if we name any one of the parameters, each of the remaining parameters will be an arbitrary function of that one.

The equations of the curve will then be of the form

$$F\{x, y, z, \alpha, \phi(\alpha), \psi(\alpha), \ldots\} = 0,$$
$$G\{x, y, z, \alpha, \phi(\alpha), \psi(\alpha), \ldots\} = 0.$$

The general functional equation would be found if we could eliminate α between these equations; but this is obviously not generally possible, since the functions ϕ, ψ, ... are arbitrary for the family, determinate only for a particular surface.

But in certain cases it happens that one of the final equations involves only one of the parameters so that it appears under the form $u = \alpha$, where u is a known function of x, y, z. In this case the general equation of the family of surfaces can be obtained, viz.

$$F\{x, y, z, \phi(u), \psi(u), \ldots\} = 0.$$

857. *Partial differential equations of surfaces generated by curves subject to more than one arbitrary condition.*

Take the case in which one of the final equations resulting from the elimination of parameters is $u = \alpha$, involving only one of the parameters, the general equation of the family of surfaces being

$$F\{x, y, z, \phi(u), \psi(u), \chi(u), \ldots\} = 0,$$

in which there are m arbitrary functions.

Differentiate with respect to x and y as independent variables,

$$F'(x) + F'(z)\,p + F'(u)\,(u_1 + u_3 p) = 0,$$
$$F'(y) + F'(z)\,q + F'(u)\,(u_2 + u_3 q) = 0,$$

where $F'(x)$, $F''(y)$, $F''(z)$ involve the m functions, and $F'(u)$ their differential coefficients as well, which may therefore be eliminated all at once, and an equation of the first order will be obtained, viz. $H_1 = 0$, involving only the m functions. In the same way an equation of the second order $H_2 = 0$ may be obtained, involving the m functions only, and so on.

From m equations $H_1 = 0, \ldots H_m = 0$, and the original equation, the m functions can be eliminated and a final differential equation obtained of the order m, the number of the arbitrary functions.

If from the original equation and the $m-1$ equations $H_1 = 0, \ldots H_{m-1} = 0$, we eliminate all but one of the m arbitrary functions, we shall have m different differential equations, of the $(m-1)^{\text{th}}$ order, each involving an arbitrary function, and these will be m first integrals of the final equation of the m^{th} order.

858. It should be observed that if we were required to eliminate m arbitrary functions of *different* known functions of x, y, z, we should have generally to proceed to a higher order than the number of arbitrary functions.

We should thus have two equations of the first order involving $\phi, \psi, \ldots$ and $\phi', \psi', \ldots$, three of the second order involving $\phi'', \psi'', \ldots$ as well, and so on to the sth order, the number of equations would be

$$1 + 2 + \ldots (s+1) = \tfrac{1}{2}(s+1)(s+2),$$

and the number of functions to be eliminated would be $(s+1)m$; therefore $\frac{1}{2}(s+2)$ must be greater than m, or $s = 2m - 1$.

859. *To shew that the order of the differential equation of families of surfaces generated by curves subjected to more than one arbitrary condition is the number of arbitrary conditions.*

A general method of eliminating the parameters of a curve which generates a surface, so as to obtain the partial differential equation of the family of surfaces, depends upon the consideration that the whole of the curve in each position lies on the surface.

Let the equations of the generating curve in any position be reduced, by the elimination of one of the current coordinates, to the two

$$\left.\begin{aligned} F(x, y, z, \alpha, \beta, \gamma, \ldots) &= 0, \\ f(x, y, \alpha, \beta, \gamma, \ldots) &= 0, \end{aligned}\right\} \qquad (1)$$

in which $\alpha, \beta, \gamma, \ldots$ are n parameters connected by $n-1$ arbitrary relations.

These equations still hold if we write for x, y, z the coordinates $x+dx$, $y+dy$, $z+dz$ of an adjacent point on the same curve; hence dx, dy, dz satisfy the equations

$$F_1 dx + F_2 dy + F_3 (p dx + q dy) = 0,$$
$$\text{and} \quad f_1 dx + f_2 dy = 0,$$

where F_1 is written for $\dfrac{dF}{dx}$, &c.,

$$\therefore f_2 p - f_1 q = G(x, y, z, \alpha, \beta, \gamma \ldots), \qquad (2)$$

nd this equation being true at every point of the curve considered,

$$f_2(rdx+sdy)-f_1(sdx+tdy)+pdf_2-qdf_1$$
$$=G_1dx+G_2dy+G_3(pdx+qdy);$$
$$\therefore f_2^2r-2f_1f_2s+f_1^2t=H(x, y, z, p, q, \alpha, \beta, \gamma\ldots). \qquad (3)$$

If there be only three parameters we can now eliminate them rom the equations (1), (2) and (3), and the result will be the artial differential equation of the family of surfaces.

If there be four parameters we can treat (3) as we treated (2) nd obtain another equation, and as before, by elimination, we shall ave the partial differential equation of the third order, which elongs to the family.

Differential Equations of Ruled Surfaces.

860. In all cases of ruled surfaces the partial differential equations of families satisfying given conditions may be found as follows.

Let $F(x, y, z)=0$ be the equation of any ruled surface, and let the equations of any generating line be

$$(\xi-x)/\lambda=(\eta-y)/\mu=(\zeta-z)/\nu=r.$$

The equation $F(x+\lambda r, y+\mu r, z+\nu r)=0$ must then be identically true, being true for all values of r; therefore, with the notation of Art. 462,

$$F(x, y, z)=0, \quad DF=0, \ldots D^nF=0,$$

if the equation be of the n^{th} degree.

From these equations, if the conditions to which the generating line is subject be expressed in terms of λ, μ, ν, a series of differential equations may be obtained by eliminating λ, μ, ν, any one of which must be true at every point of the surface, and may be considered as a differential equation of the surface.

It should be noticed that the symbol D denotes the rate of change of the function to which it is applied in passing along the generating line through the point (x, y, z), so that if $D^nF=0$ at every point of the generating line $D^{n+1}F$ must also vanish.

861. If the equation of the surface be given in the form $z=f(x, y)$, the equation $z+\nu r=f(x+\lambda r, y+\mu r)$ will be true for all values of r, and we obtain the equations

$$\nu=p\lambda+q\mu, \quad 0=r\lambda^2+2s\lambda\mu+t\mu^2,$$

$$\text{and generally } 0=\left(\lambda\frac{d}{dx}+\mu\frac{d}{dy}\right)^m z,$$

where m may have all integral positive values except unity.

862. *Family of surfaces generated by a straight line moving parallel to a given plane.*

The direction of the plane being (l, m, n), the straight line must move so as to satisfy the condition $l\lambda+m\mu+n\nu=0$, and the partial differential

equation may be found by eliminating λ, μ, ν from this equation, $DF=0$ an $D^2F=0$, or from $\nu=p\lambda+q\mu$ and $r\lambda^2+2s\lambda\mu+t\mu^2=0$, giving in the last form

$$(m+nq)^2 r - 2(m+nq)(l+np)s + (l+np)^2 t = 0.$$

863. *Family of surfaces generated by a straight line which intersects given axis.*

Let the axis be given by $\xi - a = l\rho$, $\eta - b = m\rho$, $\zeta - c = n\rho$, and the generat by $\xi - x = \lambda r$, $\eta - y = \mu r$, $\zeta - z = \nu r$, then, since they intersect,

$$\lambda r - l\rho + x - a = 0, \text{ \&c.,}$$

and $\lambda\{n(y-b) - m(z-c)\} + \mu\{l(z-c) - n(x-a)\} + \nu\{m(x-a) - l(y-b)\} = 0$, or $L\lambda + M\mu + N\nu = 0$; also $DF = U\lambda + V\mu + W\nu = 0$,

$$\text{whence } \lambda/P = \mu/Q = \nu/R, \text{ where } P = MW - NV, \text{ \&c.;}$$

therefore, since $D^2F=0$, the differential equation is

$$P^2u + Q^2v + R^2w + 2QRu' + 2RPv' + 2PQw' = 0.$$

Or, when z is expressed explicitly, by Art. 861,

$$\nu = p\lambda + q\mu, \text{ and } r\lambda^2 + 2s\lambda\mu + t\mu^2 = 0, \text{ and, as before, } L\lambda + M\mu + N\nu = 0;$$

$$\therefore\ (L+Np)\lambda + (M+Nq)\mu = 0,$$

$$\text{whence } (M+Nq)^2 r - 2(M+Nq)(L+Np)s + (L+Np)^2 t = 0.$$

Torses.

864. The condition of a torse is that at every point the intersection with the tangent plane contains two coincident lines, so tha $DF=0$ and $D^2F=0$ give two equal values of $\lambda:\mu:\nu$; therefore if λ, μ, ν were current coordinates of a point, $DF=0$ would be a tangent plane to the cone $D^2F=0$, that is, with the notation of Art. 461, $DU/U = DV/V = DW/W = \rho$, and, eliminating λ, μ, and ρ, the general partial differential equation of a torse is the bordered Hessian

$$\begin{vmatrix} u, & w', & v', & U \\ w', & v, & u', & V \\ v', & u', & w, & W \\ U, & V, & W, & 0 \end{vmatrix} = 0,$$

or, if $z = f(x, y)$, $rt - s^2 = 0$.

865. *General Ruled Surfaces.*

If there be no restriction on the motion of a straight line except that it must generate a surface, the differential equation which will be of the third order, will be found by eliminating λ, μ, from the equations $DF=0$, $D^2F=0$, and $D^3F=0$.

When the equation of the surface is supposed to be in the form $z=f(x,y)$, the corresponding equation is found by eliminating the ratio $\lambda:\mu$ from the quadratic $r\lambda^2+2s\lambda\mu+t\mu^2=0$, and the cubic $\left(\lambda\dfrac{d}{dx}+\mu\dfrac{d}{dy}\right)^3 z = 0$.

866. When the equation of a particular surface is given, it can be shewn whether the surface belongs to a certain family of surfaces, by comparing its equation with either the functional or the differential equation of the family. The method of doing this may be seen by examining which of the conicoids are, for example, a conoid, a surface of revolution, or a torse.

Thus, if we ask when a conicoid is a conoid. Taking the axis of z for the axis of a conoid, and a plane containing the axis of x for the directing plane, the functional equation of the conoid family is $y = xf(my + nz)$, which is an equation of the second degree if $f(u)$ is of the form $(\alpha u + \beta)/(\gamma u + \delta)$, and the equation is

$$y\{\gamma(my + nz) + \delta\} - x\{\alpha(my + nz) + \beta\} = 0, \qquad (1)$$

and, taking the usual form for the conicoid,

$$a = 0, \;\; b = m\gamma, \;\; c = 0, \;\; 2a' = n\gamma, \;\; 2b' = -n\alpha, \;\; 2c' = -m\alpha;$$

$$\therefore\; bb' = 2a'c', \quad \text{hence} \quad aa'^2 + bb'^2 + cc'^2 - abc - 2a'b'c' = 0 \qquad (2)$$

is satisfied when these axes are chosen, and, the left side being an invariant, this is a condition which must hold for any axes.

The equation (1) becomes

$$(by + 2a'z)(a'y + b'x) + 2a'(a''x + b''y) = 0,$$

thus the only conicoid is a hyperbolic paraboloid.

If the conicoid be a right conoid $m = 0$, $\therefore\; c' = 0$ and $b = 0$; hence for all positions of the axes the invariant $a + b + c$ as well as $aa'^2 + bb'^2 + cc'^2 - abc - 2a'b'c'$ will vanish.

Since the equation in this case is

$$2a'yz + 2b'zx + 2a''x + 2b''y = 0,$$

the conicoid is a hyperbolic paraboloid of which the principal sections are equal parabolas.

Envelopes and Families of Envelopes.

867. The problem of finding the envelope of a series of surfaces will separate into two distinct classes, viz. those in which the general equation of the series involves *one* or *two* arbitrary parameters. The following are simple illustrations of the marked distinction between the two cases.

When a sphere of constant radius moves with its centre always in a given straight line, in every position which it assumes it is touched by the cylinder whose radius is that of the sphere, and axis the line described by the centre; and the contact is not in an isolated point, but all along a circle, which circle is the ultimate intersection of two consecutive positions of the sphere.

This is an example of a fixed surface which envelopes a variable surface in every position which it can assume in consequence of the changes of one parameter.

When the sphere moves with its centre confined to a fixed plane instead of a straight line, it is touched in every position which it can assume by each of two planes parallel to the fixed plane and at a distance from it equal to the radius of the sphere, the contact being in this case in isolated points, which are the ultimate intersections of an infinite number of consecutive spheres.

This is an example of a fixed surface enveloping a system of surfaces which can alter their form or position in consequence of the changes of two variable parameters.

868. *Envelope of surfaces whose equations have one variable parameter.*

Let S_1, S_2, S_3 be three surfaces corresponding to three consecutive values of a single parameter, and let C_1 be the curve of intersection of S_1 and S_2, C_2 that of S_2 and S_3. Since C_1, C_2 both lie on S_2, and ultimately coincide when the three values of the parameters are ultimately equal, any surface which passes through the curves C_1, C_2 will touch S_2 all along the curve C_1.

Hence the locus of the curves which are the ultimate intersections of consecutive surfaces of the system is a surface which touches each individual along a curve, and is the envelope of the system.

Any curve C is called the *characteristic* of the envelope.

869. *Envelope of a system of surfaces whose form and position depend upon two independent parameters.*

Let S be any surface of the system corresponding to the values a, b of the parameters, in this case there is no definite sequence of surfaces which enables us to speak of the next consecutive surface, but there will be an infinite number of surfaces for which the parameters differ from a and b by quantities which may be taken as small as we please; let S_λ be a surface corresponding to the values $a + \delta a$, $b + \delta b$ of the parameters where $\delta b = \lambda \delta a$, S_λ and S will intersect in a curve C_λ, which is the curve of ultimate intersection when δa becomes indefinitely small, and, since the parameters are independent, λ may have any value and there will be an infinite number of such curves as C_λ which will cover the whole surface S. These curves, it will be seen, intersect in a finite number of points, at each of which the tangent plane to S contains the tangent lines to all the curves.

870. *Equation of the envelope of a system of surfaces whose form and position depend upon one parameter.*

Let $F(x, y, z, a) = 0$ be the equation of any surface S of the system, corresponding to the value a of the parameter, the equation of a surface S' corresponding to a consecutive value $a + \delta a$ of the parameter is $F(x, y, z, a + \delta a) = 0$.

At every point of the curve of intersection of S and S'

$$F(x, y, z, a+\delta a) - F(x, y, z) = 0;$$

if now δa becomes indefinitely small, this equation will be replaced by $F'(a)=0$; thus $F(x, y, z, a)=0$, and $F'(a)=0$ are the equations of the curve of ultimate intersection of S with the next consecutive surface of the system.

By the elimination of a from these equations we obtain an equation $\phi(x, y, z)=0$, which is the locus of all the curves of ultimate intersection and which envelopes all the surfaces, touching each all along the curve, which is called the *characteristic* of the envelope.

871. The property proved in Art. 868 that the envelope of a series of surfaces having one parameter touches each individual of the series at every point of the characteristic can be shewn by means of the equations in the last article.

$F(x, y, z, a)=0$ is the equation of any one of the series of surfaces, and also of the envelope, a being constant in the first case, and in the second a function of x, y, and z, derived from $F'(a)=0$.

At any point (x, y, z) of the characteristic, the direction-cosines of a tangent line are in the ratio $dx : dy : dz$.

For the particular surface, since a is constant,

$$F'(x)\,dx + F'(y)\,dy + F'(z)\,dz = 0,$$

and for the envelope at the same point

$$F'(x)\,dx + F'(y)\,dy + F'(z)\,dz + F'(a)\,da = 0;$$

hence, since $F'(a)=0$, the tangent lines coincide.

872. *Equation of the edge of the envelope.*

Two consecutive characteristics will generally intersect in a finite number of points, and the locus of these points of ultimate intersection is called the *edge* of the envelope.

The equations of two consecutive characteristics are $F=0$, $F'(a)=0$, and $F+F'(a)\,da=0$, $F'(a)+F''(a)\,da=0$; hence, where they ultimately intersect, $F=0$, $F'(a)=0$, $F''(a)=0$; and the locus of such points will be found by eliminating a, which will give the two equations of the edge of the envelope.

Thus, in the case of a developable surface, derived from the motion of a plane whose equation involves one parameter, the characteristic is a straight line and the edge is the edge of regression.

The following problem serves to illustrate the principal points connected with surfaces having one parameter.

873. *Envelope of spheres having for diametral planes one system of circular sections of an ellipsoid.*

Let $ax^2+by^2+cz^2=1$ be the equation of the ellipsoid,

$x\sqrt{(b-a)}+z\sqrt{(c-b)}=\alpha$ (1) that of a circular section of the given system,

$x\sqrt{(b-a)}-z\sqrt{(c-b)}=\beta$, of the opposite system,

$$b(x^2+y^2+z^2)-1-(\alpha+\beta)\sqrt{(b-a)}\,x+(\alpha-\beta)\sqrt{(c-b)}\,z+\alpha\beta=0$$

is the equation of a sphere containing the two circles, where, if the centre lie in (1), $\beta(c-a)=\alpha(c+a)$; and the equation becomes

$$(c-a)\{b(x^2+y^2+z^2)-1\}-2\alpha\{c\sqrt{(b-a)}\,x+a\sqrt{(c-b)}\,z\}+\alpha^2(c+a)=0.$$

The characteristic of the envelope of these spheres is

$$c\sqrt{(b-a)}\,x+a\sqrt{(c-b)}\,z=\alpha(c+a),$$

and the envelope either of the two forms

$$(c^2-a^2)\{b(x^2+y^2+z^2)-1\}=\{c\sqrt{(b-a)}\,x+a\sqrt{(c-b)}\,z\}^2$$

or $$(c^2-a^2)(ax^2+by^2+cz^2-1)=\{a\sqrt{(b-a)}\,x+c\sqrt{(c-b)}\,z\}^2.$$

Turning the axes of x, z through an angle $\tan^{-1}\frac{a}{c}\sqrt{\frac{c-b}{b-a}}$, the envelope may be shewn to be a prolate spheroid, whose axis passes through the two umbilics which are the foci.

The characteristics will be real only when

$$\alpha^2(c+a)\,ac = \text{ or } < (c-a)\{(c+a)b-ac\}.$$

The last real characteristic is at the umbilic.

The edge of the envelope becomes the two circular points at infinity, which lie in the plane $cx\sqrt{(b-a)}+az\sqrt{(c-b)}=0$.

874. *Equation of the envelope of a series of surfaces depending on two arbitrary parameters.*

Let the equation of a surface of the system be

$$F\equiv F(x, y, z, a, b)=0,$$

that of any neighbouring surface

$$F(x, y, z, a+\delta a, b+\delta b)\equiv F+F'(a)\,\delta a+F'(b)\,\delta b+\ldots=0.$$

The curve in which they ultimately intersect is given by the equations $F=0$ and $F'(a)\,da+F'(b)\,db=0$, and, since a and b are independent, and therefore $da:db$ may have any value, there are an infinite number of such curves which intersect in a definite number of points given by the equations $F=0$, $F'(a)=0$, $F'(b)=0$. The equation required is found by eliminating a and b from these three equations.

875. That the locus of ultimate intersections of the series of surfaces is an envelope, that is, touches each individual of the series, as shewn in Art. 869, appears also from the equations $F=0$, $F'(a)=0$, $F'(b)=0$ (1).

For the relations between dx, dy and dz in the locus, at the point (x, y, z), one of the points of ultimate intersection may be found from $F=0$, by considering a and b as functions of x, y, z, given by $F'(a)=0$ and $F'(b)=0$, and the result is

$$F'(x)\,dx+F'(y)\,dy+F'(z)\,dz=0,$$

in which a and b are to be expressed in terms of x, y and z. But this is the same equation for any surface of the series which passes through (x, y, z) since a and b are constant. Hence, all the tangent lines at the points of ultimate intersection are common to the locus and the surfaces of the series, that is, they have a common tangent plane.

The points of contact may be real or imaginary.

876. *Envelope of a series of spheres, having for diameters a series of parallel chords of an ellipsoid.*

Take the diametral plane of the ellipsoid bisecting the chords for the plane of xy, the principal axes $2a$, $2b$, of the section for the axes of x and y, and the axis of z perpendicular to this plane. Let $(\alpha, \beta, 0)$ be the centre of one of the spheres, the radius r is given by $\alpha^2/a^2 + \beta^2/b^2 + r^2/c^2 = 1$, if $2c$ be the diameter of the ellipsoid parallel to the chords; the equation of the sphere is therefore

$$(x-\alpha)^2 + (y-\beta)^2 + z^2 + c^2\alpha^2/a^2 + c^2\beta^2/b^2 = c^2;$$

for the envelope, $x = \alpha\,(1 + c^2/a^2)$, $y = \beta\,(1 + c^2/b^2)$, whence the equation of the envelope may be found, and reduced to $x^2/(a^2+c^2) + y^2/(b^2+c^2) + z^2/c^2 = 1$, an ellipsoid whose focal ellipse is the section of the original ellipsoid by the diametral plane bisecting the chords.

It may be shewn that this ellipsoid has a double contact with the given ellipsoid at points where a normal to the envelope coincides with one of the given system of chords; also, that the contact of the spheres with the envelope will be real only when the centre of the sphere lies within the ellipse

$$(a^2+c^2)\,x^2/a^4 + (b^2+c^2)\,y^2/b^4 = 1.$$

877. *Envelope of a series of surfaces whose equations involve n parameters, connected by $n-1$ or $n-2$ equations.*

This is the general statement of the problem in which the series of surfaces depend upon *one* or *two* independent parameters. It should be observed that there could be no *locus* of ultimate intersections if more than two were involved, for in that case, by making small variations in each of the parameters, we should obtain more than three equations for determining the points in which a given surface ultimately intersects the neighbouring surfaces, from which equations the current coordinates could be eliminated, and a relation be found between the parameters, which would be contrary to the supposition of independence.

The problem might be solved by eliminating all but one or two parameters, and then proceeding as in Arts. 870 and 874; but we may consider that, when the equations of the series of surfaces in the two cases are $F(x, y, z, a) = 0$ and $F(x, y, z, a, b) = 0$, the equation of the envelope being $\phi(x, y, z) = 0$, $\phi(x, y, z)$ is the maximum or minimum value of $F(x, y, z, \ldots)$; obtained by variation of a or of a and b.

The envelope of the surfaces whose general equation is

$$F \equiv F(x, y, z, a_1, a_2 \ldots a_n) = 0,$$

in which $a_1, a_2, \ldots$ are connected by $n-1$ or $n-2$ equations,

$\phi_1 = 0$, $\phi_2 = 0$, &c., will be $\phi(x, y, z) = 0$, where $\phi(x, y, z)$ is th maximum or minimum value of F due to variations of $a_1, a_2, \ldots$.

The ordinary method of proceeding by undetermined multiplier is explained in all treatises on the Differential Calculus.

The following problem exemplifies the manner of work.

878. *Envelope of a series of planes passing through the centre of an ellipsoi and intersecting it in sections of constant area.*

Let the equation of one of the planes be $lx + my + nz = 0$, the parameter l, m, n being connected by the equations

$$l^2 + m^2 + n^2 = 1 \quad \text{and} \quad l^2a^2 + m^2b^2 + n^2c^2 = d^2;$$

differentiate and use undetermined multipliers λ, μ,

$$\text{then } \lambda x + \mu l + a^2 l = 0,$$
$$\lambda y + \mu m + b^2 m = 0,$$
$$\lambda z + \mu n + c^2 n = 0;$$

multiply by l, m, n, and add, then $\mu + d^2 = 0$, and $\lambda x + l(a^2 - d^2) = 0$, &c.;

$$\therefore \quad \frac{x^2}{a^2 - d^2} + \frac{y^2}{b^2 - d^2} + \frac{z^2}{c^2 - d^2} = 0$$

is the equation of the envelope, being a cone whose focal lines are the asymptotes of the focal hyperbola of the ellipsoid.

Differential Equations of Envelopes.

879. When a series of surfaces depends upon the variation of two parameters, one of which is an arbitrary function of the other, there will be a family of envelopes, each individual of which is determined by a special form of the arbitrary function; this family will have a common differential equation.

The only additional consideration is that the differential equation obtained is true for any one of the surfaces as well as for each envelope, since the elimination takes place between equations of the same form, whether the parameters are considered constant or as functions of x, y, z derived from $F'(a) = 0$.

It is unnecessary to state the case when there are n parameters connected by $n - 1$ arbitrary relations.

880. *Differential equation of the family of envelopes of surfaces depending upon two parameters connected by an arbitrary relation.*

Let the equation of one of the surfaces be

$$z = F\{x, y, a, \phi(a)\}, \quad (1)$$

then, since $F'(a) = 0$, both for the envelope and for the surface,

$$p = F'(x), \qquad q = F'(y), \quad (2)$$

and since these equations involve the parameters but not their differential coefficients they can be eliminated from (1) and (2), and the differential equation formed.

881. *Differential equation of the family of envelopes of surfaces depending upon three parameters, two of which are arbitrary functions of the third.*

Let the equation of the surfaces be

$$z = F\{x,\ y,\ a,\ \phi(a),\ \psi(a)\}, \quad (1)$$

both for the envelope and the particular surface, since $F'(a) = 0$,

$$\text{we have } p = F'(x), \quad q = F'(y), \quad (2)$$

and if r, s, t be the differential coefficients for the envelope, r', s', t' for the surface,

$$r - r' = \frac{dp}{da}\frac{da}{dx}, \qquad t - t' = \frac{dq}{da}\frac{da}{dy},$$

$$\text{and} \quad s - s' = \frac{dp}{da}\frac{da}{dy} = \frac{dq}{da}\frac{da}{dx};$$

$$\therefore\ (r - r')(t - t') = (s - s')^2; \quad (3)$$

and since r', s', t' involve the parameters and not their differential coefficients, they can be eliminated from the equations (1), (2), and (3), and an equation obtained of the form

$$rt - s^2 + Rr - 2Ss + Tt + RT - S^2 = 0.$$

882. *Differential equation of a family of tubular surfaces.*

DEF. A *tubular* surface is the envelope of a series of spheres of equal radii, whose centres describe a given curve.

Any sphere of the series is given by the equation

$$(x - a)^2 + (y - \beta)^2 + (z - \gamma)^2 = c^2,$$

where $\beta = \phi(a)$ and $\gamma = \psi(a)$, and $x - a + p(z - \gamma) = 0$, $y - \beta + q(z - \gamma) = 0$, both in the sphere and envelope, $\therefore\ (1 + p^2 + q^2)(z - \gamma)^2 = c^2$.

$$\text{For the sphere} \quad 1 + p^2 + r'(z - \gamma) = 0,$$

$$pq + s'(z - \gamma) = 0,$$

$$1 + q^2 + t'(z - \gamma) = 0,$$

$$\text{and for the envelope} \quad (r - r')(t - t') = (s - s')^2;$$

$$\therefore\ \{(z - \gamma)r + (1 + p^2)\}\{(z - \gamma)t + 1 + q^2\} - \{(z - \gamma)s + pq\}^2 = 0;$$

$$\therefore\ (z - \gamma)^2(rt - s^2) + (z - \gamma)\{r(1 + q^2) - 2spq + t(1 + p^2)\} + 1 + p^2 + q^2 = 0,$$

$$\text{or } c^2(rt - s^2) + c\sqrt{(1 + p^2 + q^2)}\{r(1 + q^2) - 2spq + t(1 + p^2)\} + (1 + p^2 + q^2)^2 = 0,$$

which, by Art. 718, implies that c is one of the principal radii of curvature.

883. *When a family of surfaces depends on two parameters, one of which is an arbitrary function of the other, to find the form of the function in order that the envelope may* (i) *contain a given directing curve,* (ii) *touch a given surface.*

Let $u = 0$ be the equation of one of the family of surfaces.

i. If $v = 0$, $w = 0$ be the equations of the given curve, and u_1, u_2, u_3, v_1... be the partial differential coefficients of u, v... with respect to x, y, z; since the curve must be a tangent to each of the

enveloped surfaces, the same values of $dx : dy : dz$ satisfy th equations derived from $u = 0$, $v = 0$, and $w = 0$;

$$\therefore \begin{vmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \\ w_1, & w_2, & w_3 \end{vmatrix} = 0,$$

and eliminating x, y, z from these four equations, a final r between the two parameters is found.

ii. If $v = 0$ be the equation of the given surface, since th surfaces touch one another, they must have a common normal besides the two equations, we have $u_1/v_1 = u_2/v_2 = u_3/v_3$, w x, y, z can be eliminated, and the parameters connected.

LXI.

(1) Shew that the differential equation of all surfaces which are gen by a circle, whose plane is parallel to the plane of yz, and which passes th the axis of x, is $(y^2 + z^2)\,t + 2(z - yq)\{1 + q^2\} = 0$.

(2) The functional equation of surfaces generated by a straight line secting the axis of z, and meeting the plane of xy in the circle $x^2 + y$ is $\sqrt{(x^2 + y^2)} - a = zf(y/x)$. Find also the differential equation.

(3) In a right conoid of the third degree, in which only one gener line passes through any point of the axis, shew that the section made b plane through the axis will consist of the axis and two generating lines.

(4) Shew that the following equation represents a conoidal surface

$$z^2(2z - x - y)^2 + 2z(a - z)(x - y)^2 - 2a^2(z - x)(z - y) = 0.$$

(5) Spheres are described upon the chords of the circle $x^2 + y^2 = 2ax$, which pass through the origin, as diameters, shew that their envelope has its equation $(x^2 + y^2 + z^2 - ax)^2 = a^2(x^2 + y^2)$.

(6) The envelope of the plane $lx + my + nz = a$, l, m, n being connecte the equations $l^2 + m^2 + n^2 = 1$, $\lambda l + \mu m + \nu n = 0$, is a right circular cylinder, wh equation is $(x^2 + y^2 + z^2 - a^2)(\lambda^2 + \mu^2 + \nu^2) = (\lambda x + \mu y + \nu z)^2$.

(7) A series of similar ellipsoids are described, having a series of sect of a paraboloid, perpendicular to the axis, as principal sections; prove t their envelope will be a paraboloid, similar to the former.

(8) Shew that the envelope of planes cutting off a constant volume fr the cone $ax^2 + by^2 + cz^2 = 0$ is a hyperboloid of which the cone is the asympto

LXII.

(1) A surface is generated by a straight line always passing through t two fixed lines $y = mx$, $z = c$ and $y = -mx$, $z = -c$; prove that the equation the surface generated is of the form

$$\frac{mcx - yz}{c^2 - z^2} = f\left(\frac{mzx - cy}{c^2 - z^2}\right);$$

also that its differential equation is

$$(cy - mzx)\,U + m(mcx - yz)\,V + m^2(c^2 - z^2)\,W = 0.$$

(2) Find the general functional equation of a family of surfaces such that the tangent plane at any point (x, y, z) of one of them intercepts on the axis of z a length k^{n+1}/z^n.

Determine the arbitrary function so that the intercepts on the axes of x and y may be in the ratio $x : y$.

(3) Find the functional equation of a family of surfaces generated by a straight line of constant length c sliding between the coordinate planes of yz, zx, and remaining parallel to the plane of xy.

Shew that the differential equation is $(xp + yq)^2 (p^2 + q^2) = c^2p^2q^2$.

(4) Find the family of surfaces which cut orthogonally a series of spheres which pass through a given point, and have their centres in a given straight line.

(5) Surfaces are built up of sphero-conics on ellipsoids similar to

$$x^2/a + y^2/b + z^2/c = 1;$$

prove that the differential equation of the family is

$$(b - c)\, ap/x + (c - a)\, bq/y = (a - b)\, c/z.$$

(6) If an enveloping cone of an ellipsoid be a cone of revolution, the plane of contact will touch a hyperbolic cylinder.

(7) The equation of a surface is $(x - a)(y - a)(z - a) + a^3 = 0$; shew that if a be a variable parameter the equation of the envelope of such surfaces is $x^{-\frac{1}{2}} + y^{-\frac{1}{2}} + z^{-\frac{1}{2}} = 0$, and that of the projection of the characteristic on the plane of xy is $xy = a\{x + y + (xy)^{\frac{1}{2}}\}$.

(8) If a cone be described with any point of a central conicoid as vertex and the conjugate central section as base, this cone will envelope a similar, concentric, and similarly situated conicoid.

(9) Find the envelopes of the surfaces

$$(1)\quad \begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ x, & y, & z \end{vmatrix} \times \begin{vmatrix} a', & b', & c' \\ \alpha, & \beta, & \gamma \\ x, & y, & z \end{vmatrix} = m, \qquad (2)\quad (\alpha x + \beta y + \gamma z) \times \begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ x, & y, & z \end{vmatrix} = m,$$

α, β, γ in each case satisfying the condition $\alpha^2 + \beta^2 + \gamma^2 = 1$.

LXIII.

(1) The differential equation of surfaces generated by the intersection of two spheres, one through each of the circles, $y^2 + z^2 = a^2$, $x = 0$, and $x^2 + z^2 = c^2$, $y = 0$, is $2x(x + pz)/(r^2 - a^2) + 2y(y + qz)/(r^2 - c^2) = 1$; where $r^2 = x^2 + y^2 + z^2$.

Find the general cubic surface satisfying this equation.

(2) Shew that every right conoid of the n^{th} degree will be cut by any plane perpendicular to the axis in a number of straight lines not exceeding $n - 1$.

(3) The generating lines of a ruled surface pass through a given straight line $(x - a)/l = (y - b)/m = (z - c)/n$; shew that the general functional equation is $z = xf(u) + y\phi(u)$, where $u = \{n(y - b) - m(z - c)\}/\{n(x - a) - l(z - c)\}$.

(4) Shew that the differential equation of a family of surfaces which intersect the tangent planes in curves, the two branches of which through the point of contact are at right angles, is

$$(V^2 + W^2)\, u + (W^2 + U^2)\, v + (U^2 + V^2)\, w - 2VWu' - 2WUv' - 2UVw' = 0,$$

with the notation of Art. 461.

(5) Shew that the only conoid possessing the property of (4) is a right conoid, and that its equation may be reduced to the form $y = x \tan(z/c)$.

(6) A straight line moves so as always to intersect a fixed circle and a straight line through the centre perpendicular to its plane, find the functional and differential equations of the family of surfaces thus generated.

If the equations of the circle be $x^2 + y^2 = c^2$ and $z = 0$, shew that the functional equation will be $z = \{c - \sqrt{(x^2 + y^2)}\} f(y/x)$, and that the equation of the osculating plane of the geodesic which touches the circle at the point $(x', y', 0)$ will be $xx' + yy' - c^2 = czf(y'/x')$.

(7) Prove that the edge of regression of the torse, whose generating lines intersect the two curves, whose equations are $y^2 = 4ax$, $z = 0$, and $x^2 = 4ay$, $z = c$, is given by the equations $cx^2 = 3ayz$ and $cy^2 = 3ax(c - z)$.

(8) Along a normal at a point P of an ellipsoid is measured PQ of a length inversely proportional to the perpendicular from the centre on the tangent plane at P; prove that the locus of Q is another ellipsoid, and that the envelope of all such ellipsoids is the surface of centres of the original ellipsoid.

LXIV.

(1) Surfaces are generated by an ellipse moving so that its plane is always parallel to a fixed plane and its axes unchanged in direction; shew that the differential equation of such surfaces is $q^2r - 2pqs + p^2t + a^{-2}b^{-2}(a^2p^2 + b^2q^2)^{\frac{3}{2}} = 0$, the semi-axes being a, b, and the fixed plane that of xy.

Find also the two first integrals of this equation by the method of Art 857.

(2) A thin ellipsoidal shell is bounded by similar and similarly situated concentric ellipsoids, and planes are drawn through the centre parallel to the tangent planes at points where the thickness of the shell is constant. Find the cone which is the envelope of all these planes.

(3) Find the envelope of a series of spheres described on parallel chords of a hyperbolic paraboloid as diameters.

(4) Apply the differential equation of LXIII. (4), to determine at what points of the surface

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + 2a''x + 2b''y + 2c''z + d = 0$$

the generating lines are at right angles to each other.

(5) Shew that the only surface of revolution, in which the two branches of the curve of intersection with the tangent plane are at right angles to each other at every point, is the surface generated by the revolution of a catenary about its directrix.

(6) A cone is described with its vertex at a fixed point, and one axis parallel to an axis of a given central conicoid, and the cone cuts the conicoid in plane curves; shew that these planes envelope a parabolic cylinder whose directrix-plane passes through the fixed point.

(7) A torse touches a sphere along a sphero-conic, prove that the projection of the edge of regression on a principal plane of the cone determining the sphero-conic is the evolute of a conic.

(8) Find the differential equation of the projection on the plane of xy of each family of lines of curvature of the surface which is the envelope of spheres which pass through the origin, and whose centres lie on a parabola $x^2 + 4ay = 0$, $z = 0$.

(9) All cubic torses are cones, including cylinders.

CHAPTER XXVI.

ENERAL THEORY OF POLARS AND TANGENTS. DEGREE OF RECIPROCAL OF A SURFACE.

84. THE method adopted in the examination of the properties olars of Surfaces, represented by rational algebraical equations, ıat employed by Joachimsthal,* by Cayley in a discussion con- ing tangents to curves in *Crelle's Journal*,† and by Salmon ıe *Quarterly Journal of Mathematics*,‡ in which he gives an unt of the whole theory of the contact of right lines with ıces. In this method the position of a point in a straight line etermined by the ratio of its distances from two other points he same line. Nearly all that is good in this chapter is due ʃalmon.

385. *The position of any point in a straight line determined with rence to two other given points in the line.*

Let P, P' be two points (x, y, z, w) and (x', y', z', w'), and R other point in the line joining them, whose algebraical distances n P and P', estimated in the directions which correspond to a ition between P and P', are in the ratio $\lambda' : \lambda$, so that R would the centre of gravity of masses in the ratio $\lambda : \lambda'$ placed at and P'.

The coordinates of R are $(\lambda x + \lambda' x')/(\lambda + \lambda')$, ..., the position of being in PP' produced, when the ratio $\lambda' : \lambda$ is negative and merically greater than unity, and in $P'P$ produced if that ratio negative and less than unity.

The coordinates of R are of the same form whether quadriplanar tetrahedral coordinates be used, or when x/w, y/w, z/w are ırtesian coordinates, w being written for unity, so as to make ıy Cartesian equation homogeneous.

886. *When a straight line is drawn through any two points, to nd the points of intersection with a given surface.*

Let $F \equiv F(x, y, z, w) = 0$ be the equation of a given surface of ıe n^{th} degree when expressed in a rational homogeneous form, and ɜt P, P' be two points (x, y, z, w), (x', y', z', w'), R any point in he line drawn through P, P' determined by the ratio $\lambda' : \lambda$ as in ‌he preceding article.

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* *Crelle's Journ.* Vol. XXXIII. † Vol. XXXIV. ‡ Vol. I. p. 329.

The points of intersection with the surface are given by
n values of the ratio which satisfy the equations

$$F(\lambda x + \lambda' x',\ \lambda y + \lambda' y',\ \lambda z + \lambda' z',\ \lambda w + \lambda' w') = 0. \qquad ($$

If we expand the function by the ordinary methods, wri
for the operations

$$D = x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz} + w'\frac{d}{dw}$$

$$\text{and } D' = x\frac{d}{dx'} + y\frac{d}{dy'} + z\frac{d}{dz'} + w\frac{d}{dw'},$$

the equation (1) may be written in either of the symbolical fo
$\lambda^n e^{\lambda'/\lambda\, D} F = 0$ or $\lambda'^n e^{\lambda/\lambda'\, D'} F' = 0$, each expansion terminating a
the $(n+1)^{\text{th}}$ term, since $D^{n+1}F = 0$; and it should be observed, fr
the identity of the two forms, that $r!\, D'^{n-r} = (n-r)!\, D^r$.

887. *Poles and Polars.* The surfaces which are represented
the equations

$$DF = 0,\quad D^2F = 0,\quad \ldots,\quad D^{n-1}F = 0,$$

or by the equivalent equations

$$D'^{n-1}F' = 0,\quad D'^{n-2}F' = 0,\quad \ldots,\quad D'F' = 0,$$

are called the 1st, 2nd, ..., $(n-1)^{\text{th}}$ *Polars* of the surface $F=$
with respect to the point (x', y', z', w'), which is called the *Pole.*

The particular polars $D^{n-1}F = 0$ and $D^{n-2}F = 0$, or $D'F' = 0$ a
$D'^2F' = 0$, which are of the first and second degrees respectivel
are called the *polar plane* and *polar conicoid* of the surface.

COR. If $f(x, y, z) = 0$, the equation of a surface expressed
Cartesian coordinates, be arranged in homogeneous functions

$$u_0 + u_1 + u_2 + \ldots + u_n = 0,$$

this will become in the homogeneous form

$$u_0 w^n + u_1 w^{n-1} + \ldots + u_n = 0,$$

and the equation of the r^{th} polar will be $D^rF = 0$, whence t
equation of the r^{th} polar of the origin is, since $D = w'd/dw$,

$$n(n-1)\ldots(n-r+1)\,u_0 + (n-1)\ldots(n-r)\,u_1 + \ldots + r(r-1)\ldots2.1\,u_{n-r} =$$

The polar plane is $nu_0 + u_1 = 0$, and the polar conicoid

$$\tfrac{1}{2}n(n-1)\,u_0 + (n-1)\,u_1 + u_2 = 0.$$

888. *Geometrical properties of polars.*

Let R_m be one of the points of intersection of PP' with
surface $F = 0$, and let ρ, ρ_m be the distances of P and R_m from I
then $\lambda'_m : \lambda_m = \rho - \rho_m : \rho_m$ will be the corresponding value of λ' :
the general values of which are given by the equation

$$\left(\frac{\lambda'}{\lambda}\right)^n F' + \frac{1}{2}\left(\frac{\lambda'}{\lambda}\right)^{n-1} D'F' + \ldots + \frac{1}{r!}\left(\frac{\lambda'}{\lambda}\right)^{n-r} D'^rF' + \ldots = 0,$$

;iven by the . $D^rF'=0$ is the locus of a point P, such that

$$\Sigma\left(\frac{\lambda'_1\lambda'_2\ldots\lambda'_r}{\lambda_1\lambda_2\ldots\lambda_r}\right)=0;$$

:0. (1)

ods, writing e $(n-r)^{\text{th}}$ polar with respect to P' is the locus of a point ıat $\Sigma\{(\rho^{-1}-\rho_1^{-1})(\rho^{-1}-\rho_2^{-1})\ldots(\rho^{-1}-\rho_r^{-1})\}=0$.

ıolar plane is the locus of a point for which

$$\Sigma(\rho^{-1}-\rho_1^{-1})=0,\ \text{ or }\ n\rho^{-1}=\rho_1^{-1}+\rho_2^{-1}+\ldots+\rho_n^{-1},$$

ves the harmonic property of the polar of a conicoid with › a point, as in Art. 280.

Connection between diametral and polar surfaces.

ılical forms ı the pole is at an infinite distance, since $\rho_1, \rho_2\ldots$ are infinite ıating after $\cdot\rho_1$ &c. remain finite, $\therefore \rho/\rho_1=\rho/\rho_2=\ldots=1$, ultimately, and ərved, from itions for the polar plane and polar conicoid become respec- $(\rho-\rho_1)=0$ and $\Sigma(\rho-\rho_1)(\rho-\rho_2)=0$, or $\Sigma(PR)=0$ and esented by $'R_2)=0$; these surfaces are the *polar diametral plane* and *ımetral conicoid* for a system of parallel chords.

Properties of Polars.

Every polar of a surface, with respect to a given pole, is a ith respect to the same pole, of every polar of a higher degree 'ace $F=0$ *own.*

Pole. For, $D^{k+l}F=D^k(D^lF)$.

$''=0$ and pectively, *Every line, drawn through a pole to a point in the curve ection of the first polar with the surface, meets the surface in cident points.*

ressed in the equation $(\lambda^n+\lambda^{n-1}\lambda'D+\frac{1}{2}\lambda^{n-2}\lambda'^2D^2+\ldots)F=0$ will have ns ıes of λ' equal to zero, if $F=0$ and $DF=0$ simultaneously.

If a surface have a multiple point of the m^{th} degree, that ıll be a multiple point of the $(m-1)^{\text{th}}$ degree on the first ıith respect to any point not on the surface.

P' be the pole, R the multiple point, P any point on $P'R$, ›nce the io $\lambda:\lambda'$, given by the equation $(\lambda^n+\lambda^{n-1}\lambda'D+\ldots)F=0$, re m equal values determining R, therefore

$u_{n-r}=0.$

$$(\lambda^{n-1}+\lambda^{n-2}\lambda'D+\ldots)DF=0$$

-1 values also determining R, where PP' meets the first ihewing that R is a multiple point of the $(m-1)^{\text{th}}$ degree first polar of every point.

s appears also from the geometrical property given in with a .8, which may be written, if ρ_1 be the distance of the multiple ›om P', rom the pole,

$'\lambda':\lambda,$

$$\rho_1^{-1})^m(\rho^{-1}-\rho_{m+1}^{-1})\ldots\{m(\rho^{-1}-\rho_1^{-1})^{-1}+(\rho^{-1}-\rho_{m+1}^{-1})^{-1}+\ldots\}=0.$$

ıilarly the r^{th} polar has a multiple point at R of the $(m-r)^{\text{th}}$

893. *If a tangent cone at a double point of a surface become two non-coincident planes, the first polar of any pole will touch the line in which the planes intersect.*

Let P' be the pole, and R, R' the two consecutive points on the surface in the line of intersection of the two planes. The straight line $P'RP$ meets the surface in two points coincident in R, hence the equation $\lambda^n e^{\lambda'/\lambda D} F = 0$, which gives the ratio $\lambda' : \lambda$ determining the position of R, has two equal roots, and the same value satisfies $\lambda^{n-1} e^{\lambda'/\lambda D} DF = 0$, therefore R is a point in the first polar; similarly, R' is also a point in the first polar, that is, the line of intersection is a tangent to the first polar.

894. *If a tangent cone at a double point become two coincident planes, the first polar will have a tangent plane coincident with them.*

For a line from P' will intersect the surface in two coincident points for *any* direction indefinitely near to $P'RP$, hence the first polar will have a point in the plane coincident with these points, not only at the multiple point, but at the adjacent point also, the plane will therefore be a tangent plane to the first polar.

895. *If r generating lines of a conical tangent coincide, $r-1$ generating lines of the conical tangent of the first polar will also coincide.*

For, let P' be the pole, and RQ the multiple generating line, and take Q indefinitely near R; a line through P', passing near Q and moving up to it, will meet the surface in r points, which will ultimately coincide, and, as before, $r-1$ points on the first polar will coincide in Q; hence RQ will be a multiple generating line of the $(r-1)^{\text{th}}$ degree of the conical tangent of the first polar.

896. *If a surface have a multiple line of the m^{th} degree, the first polar will contain the same line as a multiple line of the $(m-1)^{\text{th}}$ degree.*

For, if P' be the pole, R any point on the multiple line, $P'R$ will have m equal values for the surface and $m-1$ for the first polar.

897. The propositions relating to multiple points may be shewn readily by employing, as a fundamental tetrahedron, one in which the multiple point is one angular point and the pole another.

Thus, using tetrahedral coordinates, let the pole be $A\,(1, 0, 0, 0)$, and the multiple point $D\,(0, 0, 0, 1)$, the equation of the surface may then be written $F \equiv \phi_m w^{n-m} + \phi_{m+1} w^{n-m-1} + \ldots + \phi_n$, ϕ_m representing a homogeneous function of x, y, z of the m^{th} degree; also $\phi_m = 0$ is the equation of the conical tangent at the multiple point.

Since in this case $D = d/dx$, the equation of the first polar is

$$\frac{d\phi_m}{dx} w^{n-m} + \frac{d\phi_{m+1}}{dx} w^{n-m-1} + \ldots = 0, \quad (1)$$

and $d\phi_m/dx = 0$ is the conical tangent, which shews that the multiple point of the surface is a multiple point of the $(m-1)^{\text{th}}$ degree on the first polar.

If the conical tangent reduce to two non-coincident planes, $m = 2$ and $\phi_2 = \psi\chi$, $\psi = 0$ and $\chi = 0$ being the equations of the tangent planes, then, by the equation (1), in which $m = 2$,

$$\frac{d\phi_2}{dx} \equiv \psi \frac{d\chi}{dx} + \chi \frac{d\psi}{dx} = 0$$

will be the equation of the tangent plane of the first polar, and this is satisfied by $\psi = 0$, $\chi = 0$, that is, the line of intersection of the tangent planes is a tangent to the first polar.

If the conical tangent become two coincident planes, $\phi_2 = \psi^2$ and $d\phi_2/dx = 2\psi\, d\psi/dx = 0$, that is, $\psi = 0$ will be the equation of the tangent plane to the first polar.

898. *When a point on the surface is taken for the pole, the polar plane is a tangent plane, at the pole, to the surface, and to all the polars.*

For $D'F' = 0$ is the equation of the polar plane of the surface $F = 0$, and is also that of the tangent plane at P', if P' be a point on the surface.

Again, since

$$\left(x' \frac{d}{dx'} + y' \frac{d}{dy'} + z' \frac{d}{dz'} + w' \frac{d}{dw'}\right)^{n-r} F' \equiv n(n-1)\ldots(r+1)\, F',$$

P' is a point on the r^{th} polar $D'^{n-r}F' = 0$, and its polar plane, which is the same as that of the surface, Art. 890, is a tangent plane at P' to it as well as to the surface.

899. *The locus of poles, whose polar planes pass through a given point, is the first polar with respect to that point.*

Let (f, g, h, k) be the given point Q, which is a point in the polar plane of $P'(x', y', z', w')$, if $(f\, d/dx' + g\, d/dy' + \ldots)F' = 0$, but the equation of the first polar of Q is $(f\, d/dx + g\, d/dy + \ldots)F = 0$, hence P' lies in the first polar of Q, which is therefore the locus of P'.

900. *Every plane is a polar plane corresponding to $(n-1)^3$ poles.*

Take three arbitrary points P_1, P_2, P_3 in the plane, the first polars of these points are of the $(n-1)^{\text{th}}$ degree. The first polar of P_1 is the locus of all points which are poles of planes through P_1,

and therefore contains all poles of the given plane; the three surfaces which are first polars of P_1, P_2, P_3 each contain the pole of the given plane, and therefore, since every common pole is the pole of the plane containing P_1, P_2, P_3, there are $(n-1)^3$ such poles.

901. *The first polars of all points in a straight line have a common curve of intersection.*

The $(n-1)^3$ poles of any plane through two of the points lie on the curve of intersection of the polars of the two points, and this curve must therefore be the locus of the poles of all such planes: any point in the line of intersection of the planes must therefore have its first polar passing through the curve of intersection of the first polars of the two points taken.

Such a curve is a *Polar Curve* corresponding to the line.

COR. 1 If two lines intersect, their polar curves lie on the first polar of the point of intersection.

COR. 2. If any number of planes pass through a point, their poles lie on the first polar of the point.

COR. 3. A tangent line to the surface touches its polar curve at the point of contact with the surface.

Relation of Straight Lines to Surfaces.

902. *To find the condition that a straight line may touch a surface at a given point.*

Let P' (x', y', z', w') be the given point, $P(x, y, z, w)$ any point in a line through P', then n values of the ratio $\lambda : \lambda'$, given by the equation $(\lambda'^n + \lambda'^{n-1}\lambda D' + \frac{1}{2}\lambda'^{n-2}\lambda^2 D'^2 + \ldots) F' = 0$, determine the position of the points in which $P'P$ intersects the surface; and if $P'P$ be a tangent to the surface, two values of the ratio vanish: hence $F' = 0$ and $D'F' = 0$, which are *necessary* conditions for tangency at P', and they shew that generally the locus of $P'P$ is the polar plane of P'.

$D'F' = 0$ is not a *sufficient* condition if the differential coefficients are all zero, for in that case $P'P$ meets the surface in two coincident points for all positions of P, or P' is a multiple point; if D'^2F' do not also vanish identically, the point P' will be a double point: and if the coordinates of P satisfy the equation $D'^2F' = 0$, $P'P$, which then intersects the surface in three points at P', will be a tangent to the surface, and the polar conicoid will be the locus of all the tangent lines, that is, it will be the conical tangent at P'.

The argument is easily continued in the case of triple...r-ple singular points.

903. *Inflexional tangents at an ordinary point.*

At an ordinary point the tangent $P'P$ will meet the surface in three coincident points if $D'F' = 0$ and $D'^2F' = 0$, the inflexional tangents are therefore the lines of intersection of the polar plane

and polar conicoid, which are two straight lines, real or imaginary, since the polar plane is a tangent plane to the conicoid as well as to the surface, Art. 898.

904. If the surface be of a higher degree than the second, and if $D'^2F' = 0$ be identically satisfied at P' for all values of x, y, z, w, we can obtain three straight lines which meet the surface in four coincident points, viz. the intersection of the surface $D'^3F' = 0$ with the tangent plane $D'F' = 0$. And so on to the general case.

905. If the equation $D'F' = 0$ be satisfied identically, or the point P' be a multiple point, the tangent lines which meet the surface in four coincident points will be given by equations $D'^2F' = 0$ and $D'^3F' = 0$, and these are two conical surfaces if they be not also identically satisfied, the first being the conical tangent to the double point and the second determining the six particular generating lines of the tangent cone which satisfy the required condition.

If the singular point have a conical tangent of the r^{th} degree, the number of tangent lines meeting the curve in $r + 2$ coincident points will be $r(r+1)$.

906. *The polar conicoid of a parabolic point on a surface is a cone.*

The inflexional tangents at a parabolic point coincide, therefore the surfaces $D'F' = 0$ and $D'^2F' = 0$ intersect in two coincident lines, and the only conicoid which can be cut by a plane in two coincident lines is a cone; hence $D'^2F' = 0$, the polar conicoid, is a cone.

Or, since one principal radius of curvature is infinite, the equation of the surface may be written

$$z - ay^2 - z(bx + cy + dz) + u_3 + \ldots = 0,$$

the origin being the parabolic point.

The polar conicoid with respect to the origin is

$$(n-1)z - z(bx + cy + dz) - ay^2 = 0 \quad \text{or} \quad zv = ay^2,$$

which represents a cone, of which $z = 0$ and $v = 0$ are tangent planes, and $y = 0$ the plane of contact.

907. *To find the locus of the parabolic points on a surface.*

Since $D'^2F' = 0$, the polar conicoid with respect to a point (x', y', z', w'), is a cone, if Δ be the determinant obtained by eliminating x, y, z, w from the four equations

$$\frac{d}{dx}D'^2F' = 0, \quad \frac{d}{dy}D'^2F' = 0, \quad \frac{d}{dz}D'^2F' = 0, \quad \frac{d}{dw}D'^2F' = 0,$$

$F' = 0$ and $\Delta = 0$ will be the equations of the required locus.

The degree of the surface $\Delta = 0$, called by Salmon the Hessian of the surface, is the same as that of the term

$$\frac{d^2F'}{dx'^2} \cdot \frac{d^2F'}{dy'^2} \cdot \frac{d^2F'}{dz'^2} \cdot \frac{d^2F'}{dw'^2};$$

and is therefore $4(n-2)$.

The degree of the curve containing the parabolic points of the surface is $4n(n-2)$.

908. *To find the equation of the conical surface which envelopes a given surface, and has its vertex at a given point.*

Let $P'(x', y', z', w')$ be the given point, $P(x, y, z, w)$ any point in any tangent line drawn from P'. The equation $\lambda^n e^{\lambda'/\lambda\ D} F = 0$ must give equal values of $\lambda : \lambda'$, hence the equations $\lambda^n e^{\lambda'/\lambda\ D} F = 0$ and $\lambda^{n-1} e^{\lambda'/\lambda\ D} DF = 0$ must have a common root. The eliminant of these equations equated to zero is the equation of the locus of P, and is the equation of the envelope.

If the surface be of the second degree,

$$\lambda^2 F + \lambda\lambda' DF + \tfrac{1}{2}\lambda'^2 D^2 F = 0 \quad \text{and} \quad \lambda DF + \lambda' D^2 F = 0$$

will have their roots equal, or the equation of the envelope will be

$$2FD^2F = (DF)^2 \quad \text{or} \quad 4FF' = (DF)^2.$$

909. *To find the number of tangents which can be drawn from a given point not on the surface to meet it in three consecutive points.*

Let a line from P' meet the surface in three points coinciding in P, the equation $\lambda^n e^{\lambda'/\lambda D} F = 0$ must have three values of $\lambda' : \lambda = 0$, $\therefore F = 0$, $DF = 0$, and $D^2F = 0$ give, by the intersection of the surfaces which they represent, the number of positions of $P'P$ satisfying the conditions; the required number of tangents is therefore $n(n-1)(n-2)$; a point of contact of this kind corresponds to a cuspidal edge of the conical envelope.

If the given point P' be on the surface, through each of the three coincident points on an inflexional tangent can be drawn a line satisfying the conditions; hence the number of the tangent lines meeting the surface in three points distinct from P' will be

$$n(n-1)(n-2) - 6 = (n-3)(n^2+2).$$

910. *To find the number of tangent lines which can be drawn to a surface at a given point so as to touch at another point as well.*

Let P' be the given point, R the second point of contact. For contact at P' two values of $\lambda : \lambda'$ in $\lambda'^n e^{\lambda/\lambda'\ D'} F' = 0$ are zero; hence $F' = 0$ and $D'F' = 0$; the equation which gives the remaining values of $\lambda : \lambda'$ is $u \equiv \tfrac{1}{2}\lambda'^{n-2} D'^2 F' + \ldots + \lambda^{n-2} F = 0$, two of these values are equal, viz. those which determine the position of R; $\therefore du/d\lambda = 0$ and $du/d\lambda' = 0$ have a common root, the eliminant v of these equations will be of the same degree as $F^{n-3}(D'^2F')^{n-2}$,

$\therefore v=0$ represents a surface meeting the tangent plane in $(n+2)(n-3)$ straight lines, and this will be the number of double tangents touching at two different points, one of which is given. The equation $v=0$ represents a surface which intersects the tangent plane at P' in the tangent lines having double contact.

911. The number of the double tangents might be obtained by taking P as the second point of contact, so that for contact at P', $F'=0$, $D'F'=0$, and for contact at P, $F=0$, $DF=0$; the position of P would then be given by the intersection of the two surfaces $F=0$, $DF=0$ with the plane $D'F'=0$, which would give $n(n-1)$ positions of P, but, since all three equations are satisfied by the coordinates of P', $D'F=0$ being a tangent plane to both surfaces $F=0$ and $DF=0$, its curve of intersection with each has two branches which touch the inflexional tangents, so that there are six of the $n(n-1)$ points which coincide with P' and are to be rejected, and the remainder is $(n+2)(n-3)$, as before.

912. If P' be a multiple point of the r^{th} degree, in order that $P'P$ may be a proper tangent at P' and some point R in $P'P$, $r+1$ values of $\lambda:\lambda'$ must be zero, and the remaining values, two of which must be equal, will be given by

$$u \equiv \lambda'^{n-r-1} D'^{r+1} F' / (r+1)\,! + \ldots + \lambda^{n-r-1} F = 0,$$

and the eliminant v of $du/d\lambda' = 0$ and $du/d\lambda = 0$ will be of the degree of $(D'^{r+1}F')^{n-r-2} F^{n-r-2}$, that is $(n+r+1)(n-r-2)$, thus the number of double tangents will be the number of tangents which are the intersection of $v=0$ with the tangent cone $D'^{r}F'=0$, that is $r\{n(n-1)-(r+1)(r+2)\}$.

913. By the method of Art. 911, $D'^{r}F'=0$ is the conical tangent of both $F=0$ and $DF=0$, and the curve of intersection with each of these touches the tangents of closest contact, $r(r+1)$ in number, hence among the $rn(n-1)$ points of intersection of the three surfaces $r(r+1)(r+2)$ coincide with P', and the number of double tangents is, as before, $r\{n(n-1)-(r+1)(r+2)\}$.

914. *To find the locus of tangents which can be drawn from a multiple point of a surface to the surface.*

Let P' be the multiple point of the r^{th} degree, so that $F'=0$, $D'F'=0$, $\ldots D'^{r-1}F'=0$, the equation which gives the remaining points of intersection of a line $P'P$ with the surface is

$$u \equiv \lambda'^{n-r} D'^{r} F' / r\,! + \ldots + \lambda^{n-r} F = 0,$$

and if this equation have equal roots, $P'P$ is a generating line of $v=0$, where v is the eliminant of $du/d\lambda' = 0$ and $du/d\lambda = 0$, whose degree is that of $(D'^{r}F')^{n-r-1} F^{n-r-1}$ or $(n+r)(n-r-1)$.

915. *To find the number of double tangent lines which can be drawn from a fixed point not on the surface.*

Take P' for the fixed point, P for one point of contact; then two positions of R coincide in P, hence, from the equation $\lambda^{n} e^{\lambda'/\lambda\, D} F=0$, $F=0$ and $DF=0$, and, for the second point of contact, two roots of $u \equiv \frac{1}{2}\lambda^{n-2} D^{2} F + \ldots + \lambda'^{n-2} F' = 0$ are equal, the degree of v the

eliminant of $du/d\lambda = 0$ and $du/d\lambda' = 0$ is that of $(D^2F)^{n-3}$ or $(n-2)(n-3)$, and the number of solutions of $v=0$, $F=0$ and $DF=0$ is double the number required, since each double tangent corresponds to two solutions, hence the number required is

$$\tfrac{1}{2}n(n-1)(n-2)(n-3);$$

this is the number of double sides of the conical envelope whose vertex is P'.

On the degree of a reciprocal surface.

916. The theory of reciprocal surfaces, as far as it has been completed, has been elaborated by Salmon and Cayley; but the whole subject of the reduction of the class of a surface or the degree of its reciprocal is so complicated that we shall only give explanations of the effect which some of the simpler singularities of a surface have upon the degree of its reciprocal, with the object of introducing the student to the subject.

917. *On the class of a surface.*

The class of a surface represented by an algebraical equation of the n^{th} degree is the number of proper tangent planes to the surface which pass through an arbitrary straight line. Let A, B be any two points in an arbitrary straight line, P the point of contact of a tangent plane passing through AB, then AP, BP are tangents to the surface at P, each containing two coincident points at P. But the converse is not true that APB will be a tangent plane when AP, BP each contain two coincident points, for if P be a multiple point or a point in a multiple line on the surface, every line through P will pass through two or more points coincident in P, and in order that AP should be a proper tangent the number of points coincident in P must be one more than the degree of the multiple point or line.

When straight lines drawn through A meet the surface in two coincident points, these points lie in the curve of intersection of the surface and A's first polar, similarly for straight lines drawn through B, hence the number of positions of P such that AP and BP both pass through two coincident points on the surface, is the number of points of intersection of the surfaces of degree n and the two first polars each of degree $n-1$, and the degree of the reciprocal surface is $n(n-1)^2$ when ABP is a proper tangent plane for every position of P.

If the surface have singularities such as multiple points and multiple lines, the effect of these singularities in reducing the degree $n(n-1)^2$ has to be estimated.

918. It should be observed that by Art. 903 the first polars of every point in the line AB have a common curve of degree $(n-1)^2$, called the polar curve, whose intersections with the surface are the $n(n-1)^2$ points given above.

919. *To estimate the effect on the class when the surface has a double point.*

Since, Art. 892, where there is an ordinary double point on the surface, the first polars of any point pass through the double point, therefore the polar curve of any given straight line passes through that point. Hence the lines drawn from the double point in the plane containing it and the given line, although they meet the surface in two coincident points, are not generally tangent lines.

Therefore, since the polar curve meets the surface in two points at the double point, two of the planes passing through the given line are not tangent planes.

The number of proper tangent planes is therefore less by two than the number found above.

920. *To estimate the effect on the class when the tangent cone at a double point becomes two planes.*

i. When the two planes are not coincident, the first polars touch their line of intersection, Art. 893; hence, besides the two points coincident in the double point, there is added one more point, namely on the line of intersection of the two planes, which is common to the three surfaces. On this account the apparent number of tangents is diminished by three.

ii. When the two planes are coincident, the first polars touch one of the planes, hence each plane contains three coincident points in each of the three surfaces, and the reduction in this case is six.

921. The surface of the third degree, whose equation is

$$ax^{-1} + by^{-1} + cz^{-1} + dw^{-1} = 0,$$

has four double points one at each angle of the fundamental tetrahedron. Hence, the class of the surface is 3.2^2 diminished by 2 for each double point, and is therefore $3.2^2 - 2.4 \equiv 4$.

Reciprocating the surface with respect to the auxiliary conicoid

$$x^2 + y^2 + z^2 + w^2 = 0,$$

let $(\xi, \eta, \zeta, \omega)$ be a point on the reciprocal surface, $\xi x + \eta y + \zeta z + \omega w = 0$ is a tangent plane to the surface at some point (x', y', z', w');

$$\therefore\ ax'^{-2}/\xi = by'^{-2}/\eta = cz'^{-2}/\zeta = dw'^{-2}/\omega,$$

$$\text{and}\ \ \xi x' + \eta y' + \zeta z' + \omega w' = 0;$$

$$\therefore\ (a\xi)^{\frac{1}{2}} + (b\eta)^{\frac{1}{2}} + (c\zeta)^{\frac{1}{2}} + (d\omega)^{\frac{1}{2}} = 0,$$

which, when rationalized, is of the fourth degree.

922. The surface $xyz = w^3$ has double points at A, B, C of the fundamental tetrahedron, and at each the tangent cone is two non-coincident planes, for each double point there is therefore a reduction of 3, and the degree of the reciprocal surface is $3.2^2 - 3.3 \equiv 3$. The reciprocal surface has an equation of the same form as that of the given surface.

923. If the equation of the wave surface be put into the homogeneous form

$$(a^2x^2 + b^2y^2 + c^2z^2)(x^2 + y^2 + z^2) - \{a^2(b^2+c^2)x^2 + b^2(c^2+a^2)y^2 + c^2(a^2+b^2)z^2\}w^2 + a^2b^2c^2w^4 = 0,$$

it is easily shewn that there are four double points in each of the principal

planes, real or imaginary, and four in the plane at infinity, viz. the four points of intersection of that plane, and the two cones $a^2x^2+b^2y^2+c^2z^2=0$ and $x^2+y^2+z^2=0$.

The degree of the reciprocal is therefore $4.3^2-4.4.2\equiv 4$.

In Art. 532 it is shewn that if $lx+my+nz=p$ be a tangent plane to the wave surface, (l, m, n) being the direction of the perpendicular p,

$$l^2/(p^2-a^2)+m^2/(p^2-b^2)+n^2/(p^2-c^2)=0;$$

hence, taking the origin for the reciprocating point, if $pr=aa'=bb'=cc'$, and (x, y, z) be the point corresponding to the tangent plane

$$a'^2x^2/(r^2-a^2)+b'^2y^2/(r^2-b^2)+c'^2z^2/(r^2-c^2)=0,$$

therefore the reciprocal surface is of the fourth degree as stated.

924. *To estimate the effect of a double straight line on the class of a surface.*

The first polars for any two points in an arbitrary straight line contain the line singly which is double on the surface, and, by Art. 451, the number of points which correspond to the line which is common to the three surfaces is $5n-8$. But for each of the points of the multiple line at which the two tangent planes coincide there will be an additional point common to the three surfaces, and the number of such points must be deducted.

Let CD of the fundamental tetrahedron be a double line, the equation of the surface will be of the form $Px^2+2Qxy+Ry^2=0$, where P, Q, and R are of the $(n-2)^{\text{th}}$ degree; the tangent planes will be coincident if $PR-Q^2=0$, which represents a surface whose degree is $2(n-2)$, which is the number of the points in which it meets the double line.

Hence, for a double line the class is lowered by $7n-12$.

925. *To estimate the effect of a multiple straight line of the r^{th} degree of multiplicity.*

The polars contain the line, each in the $(r-1)^{\text{th}}$ degree of multiplicity, Art. 896, and the number of points which correspond to the common multiple lines is, Art. 451,

$$(r-1)^2n+2r(r-1)(n-1)-2r(r-1)^2\equiv(r-1)\{(3r-1)n-2r^2\}.$$

Taking, as before, CD for the multiple line, the equation of the surface may be written in the form

$$F_r(x, y)\equiv Px^r+Qx^{r-1}y+\ldots+Ty^r=0,$$

in which the coefficients are of the $(n-r)^{\text{th}}$ degree.

The equation of the tangent planes at any point $(0, 0, z', w')$ in CD is $F_r'(x, y)\equiv P'x^r+Q'x^{r-1}y+\ldots=0$, where P', Q', ... are the values of P, Q, ... when $0, 0, z', w'$ are substituted for x, y, z, w.

Now the points in CD at which there are two coincident tangent planes may be obtained by eliminating x, y from the equations

$$dF_r'(x, y)/dx=0 \text{ and } dF_r'(x, y)/dy=0,$$

and the eliminant is of the $(r-1)^{\text{th}}$ degree in the coefficients of each equation; therefore the degree of the resulting equation in

z', w' is $2(r-1)(n-r)$, and, since the polars touch the line at each of these points, this is the number of additional points which lie on the multiple line.

Hence the total number of points corresponding to the multiple line, each of which is a point which gives an improper tangent plane, is

$$(r-1)\{(3r-1)n-2r^2+2(n-r)\} \equiv (r-1)\{(3r+1)n-2r(r+1)\};$$

this is therefore the number by which the degree $n(n-1)^2$ is reduced in the reciprocal surface.

926. If a surface of the n^{th} degree contain a multiple line of the $(n-1)^{\text{th}}$ degree which must be a straight line, the degree of the reciprocal will be

$$n(n-1)^2-(n-2)\{(3n-2)n-2(n-1)n\} \equiv n,$$

so that the reciprocal surface will be of the same degree as the original surface.

927. If a surface of the n^{th} degree contain two multiple lines which do not intersect, which are respectively of the degrees $n-r$ and r, the number of points which give improper tangents will be

$$(n-r-1)[\{3(n-r)+1\}n-2(n-r)(n-r+1)]+(r-1)\{(3r+1)n-2r(r+1)\}$$
$$\equiv n^2(n-r-1)+(r-1)\{n^2-(3r+1)n+2r(r+1)\}+(r-1)\{(3r+1)n-2r(r+1)\}$$
$$\equiv n^2(n-2),$$

and the degree of the reciprocal will be as before the same as that of the original surface.

928. *To estimate the effect af a curve of the r^{th} degree of multiplicity on a given surface, the curve being the complete intersection of two surfaces of the k^{th} and l^{th} degrees.*

The number of points which correspond to the multiple curve is, by Art. 452,

$$kl\{n(r-1)^2+2(n-1)r(r-1)-(k+l)r(r-1)^2\}.$$

Let $K=0$ and $L=0$ be the equations of the two surfaces, that of the given surface will be

$$F(K, L) \equiv PK \ + QK^{r-1}L+\ldots=0,$$

where the coefficients P, Q, ... are of degree sufficient to make each term of the n^{th} degree.

At any point in the multiple curve for which $P=P'$, $Q=Q'$, &c. the approximate form of the surface in the neighbourhood of that point will be given by the r values of the ratio $K:L$, obtained from the equation

$$F'(K, L) \equiv P'K_r+ \ Q'K^{r-1}L+\ldots=0,$$

and two tangent planes will be coincident for all points in the curve for which the eliminant of $dF'/dK=0$ and $dF'/dL=0$ vanishes; this eliminant is of the degree

$$(n-kr)(r-1)+(n-lr)(r-1) \equiv (r-1)\{2n-(k+l)r\}.$$

The class of the surface is therefore reduced in consequence of the multiple curve by the number

$$kl(r-1)\{n(r-1)+2r(n-1)-(k+l)r(r-1)\}+kl(r-1)\{2n-(k+l)r\}$$
$$\equiv kl(r-1)\{(3r+1)n-2r-(k+l)r^2\},$$

which agrees with Art. 925, when $k=1$ and $l=1$.

LXV.

(1) The polars of any order, the same for all surfaces of the n^{th} degree which pass through $\phi(n)-1$ given arbitrary points, have a common curve of intersection, $\phi(n)$ being the number of conditions which a surface of the n^{th} degree may be constructed to satisfy.

(2) The r^{th} polars of all surfaces of the n^{th} degree passing through $\phi(n)-2$ arbitrary points have $(n-r)^3$ common points.

(3) U, V are the first polars of two points P, Q with respect to a surface of the n^{th} degree; prove that the first polars of P with respect to V and of Q with respect to U are the same surface.

(4) If U, V be respectively the p^{th} polar of P and the q^{th} polar of Q with respect to any surface, prove that the p^{th} polar of P with respect to V will be the q^{th} polar of Q with respect to U.

(5) When the p^{th} polar of P with respect to a surface of the n^{th} degree has a double point Q, prove that the $(n-p-1)^{\text{th}}$ polar of Q with respect to the same surface has a double point at P.

(6) If a cubic surface have a double straight line, the number of cuspidal edges on the proper enveloping cone will be diminished by 3.

(7) If the polar conicoid of $l/x+m/y+n/z+r/w=0$, whose pole is P, resolve into two planes, there will be four positions of P given by equations similar to $-x/l=y/m=z/n=w/r$.

The corresponding conicoid is the two planes $x=0$ and $y/m+z/n+w/r=0$, and the plane polar is $3x/l-y/m-z/n-w/r=0$.

(8) Prove that the inflexional tangents at any point on a surface are generators of the polar conicoid of that point.

LXVI.

(1) The points on a surface of the n^{th} degree, for which the perpendiculars upon the tangent planes from a given point are equal, lie on a surface whose degree is $2(n-1)$.

(2) The envelope of the first polars of all points on the surface

$$x^m/a^m+y^m/b^m+z^m/c^m=1,$$

with respect to the surface

$$U\equiv u_n+u_{n-1}+\ldots+u_1+u_0=0,$$

$$\text{is } (a\,dU/dx)^{\frac{m}{m-1}}+(b\,dU/dy)^{\frac{m}{m-1}}+(c\,dU/dz)^{\frac{m}{m-1}}=P^{\frac{m}{m-1}},$$

$$\text{where } P\equiv-u_{n-1}-2u_{n-2}-\ldots-nu_0.$$

(3) The condition that the first polar of (x', y', z', w'), with respect to the surface $l/x + m/y + n/z + r/w = 0$, may be a sphere is, when tetrahedral coordinates are employed,

$$a^2mn + a'^2lr = b^2ln + b'^2mr = c^2lm + c'^2nr \equiv \rho.$$

The pole is given by the ratios

$$x' : y' : z' : w' = c'^2/m + b'^2/n + a'^2/r - \rho/mnr : \ldots : \ldots.$$

Also, the locus of such poles, corresponding to all surfaces of this form, is the curve

$$a^2yz + a'^2xw = b^2zx + b'^2yw = c^2xy + c'^2zw.$$

(4) When a cone circumscribes a surface of the n^{th} degree, prove that the number of points of inflexion in a plane section of the cone is $4n(n-1)(n-2)$.

(5) Prove that if a straight line be wholly on a surface of the n^{th} degree, it will touch the parabolic curve of that surface in $n-2$ points.

(6) The equation of a surface is $azx^2 + 2bwxy + czy^2 = 0$, shew that the equation of the reciprocal surface with respect to the auxiliary conicoid $x^2 + y^2 + z^2 + w^2 = 0$ is $w(ay^2 + cx^2) + 2bxyz = 0$, and explain the reduction of the degree.

(7) Prove that the degree of the locus of points on a surface of the n^{th} degree, at which lines can be drawn to meet the surface in four coincident points, is $11n - 24$.

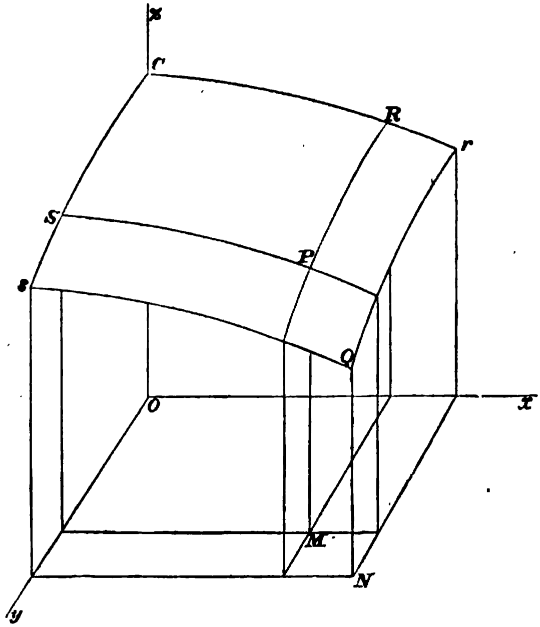
(8) Prove that the degree of the locus of points, at which tangents meet a surface of the n^{th} degree in three coincident points, the tangents touching the surface at some other point as well, is $(n-4)(3n^2 + 5n - 24)$.

CHAPTER XXVII.

VOLUMES, AREAS OF SURFACES, &c. LINE SURFACE AND VOLUME-INTEGRALS.

929. *To find the differential coefficients of the solid contained between a surface, given in rectangular coordinates, the coordinate planes, and planes parallel to two of them drawn through any point of the surface.*

Let x, y, z and $x+\Delta x$, $y+\Delta y$, $z+\Delta z$ be the coordinates of two points P and Q upon the surface.



Draw planes through P and Q parallel to the planes of yz, zx, and let V be the volume $CRPSOM$ cut off by these planes from the given solid. If $\Delta_x V$ be the increment of V, when x is changed to $x+\Delta x$, while y remains constant, and a similar interpretation be given to the operation Δ_y, the volume $PrM = \Delta_x V$; also the volume $PQNM$, which is the increment of $\Delta_x V$ when y changes to $y+\Delta y$, $=\Delta_y(\Delta_x V)$, which is easily seen to be the same as $\Delta_x(\Delta_y V)$.

Let z_1, z_2 be the least and greatest values of z within the ›rtion of the surface PQ, then $PQNM$ lies between $z_1\Delta x\Delta y$ ıd $z_2\Delta x\Delta y$;

$$\therefore \frac{\Delta_y\left(\frac{\Delta_x V}{\Delta x}\right)}{\Delta y} \text{ or } \frac{\Delta_x\left(\frac{\Delta_y V}{\Delta y}\right)}{\Delta x} \text{ lies between } z_1 \text{ and } z_2.$$

Proceeding to the limit, in which $z_1 = z_2 = z$, we obtain

$$\frac{d^2V}{dy\,dx} \text{ or } \frac{d^2V}{dx\,dy} = z.$$

We may observe that, since the volume PrM is ultimately equal the area $RM \times \Delta x$, the partial differential coefficient dV/dx presents the area RM, and similarly dV/dy the area SM.

930. The differential coefficient of the volume of a wedge of the solid ntained between the planes of zx, xy, a plane through the axis of z, and plane parallel to yOz may be obtained as follows.

Let V be the volume included between the planes zOx, xOy, the surface, plane whose equation is $y = tx$, and a plane parallel to yOz through any int (x, y, z), then $\Delta_t V$ is the increment of V when t changes to $t + \Delta t$, emaining constant, and is the volume which stands on a base whose area $\frac{1}{2}x\Delta t.x$; $\Delta_x(\Delta_t V)$ is the increment of $\Delta_t V$ when x changes to $x + \Delta x$, and he volume which stands on a base whose area is

$$\tfrac{1}{2}(x + \Delta x)^2\Delta t - \tfrac{1}{2}x^2\Delta t = (x + \tfrac{1}{2}\Delta x)\,\Delta x\,\Delta t;$$

ıce, as before, $\dfrac{\Delta_x(\Delta_t V)}{\Delta x\,\Delta t}$ is between $z_1(x + \frac{1}{2}\Delta x)$ and $z_2(x + \frac{1}{2}\Delta x)$, and, proding to the limit, $\dfrac{d^2V}{dx\,dt} = zx$.

931. *To find the differential coefficient of the portion of a surface en in rectangular coordinates, cut off by two coordinate planes, and nes parallel to them drawn through any point of the surface.*

Let P, Q be the points (x, y, z) and $(x + \Delta x, y + \Delta y, z + \Delta z)$, he surface $PRCS$, cut off by the planes through P. $\Delta_x S$ is surface Pr, which is the increment of S when x is changed $x + \Delta x$.

$\Delta_y(\Delta_x S)$ is the surface PQ, which is the increment of $\Delta_x S$ when changed to $y + \Delta y$, and is evidently the same as $\Delta_x(\Delta_y S)$.

Let γ_1, γ_2 be the greatest and least inclinations of the tangent ne to the plane of xy for any point within the surface PQ.

Therefore PQ is intermediate between $\Delta x\Delta y\sec\gamma_1$ and $\Delta y\sec\gamma_2$.

Hence $\dfrac{\Delta_y\left(\frac{\Delta_x S}{\Delta x}\right)}{\Delta y}$ or $\dfrac{\Delta_x\left(\frac{\Delta_y S}{\Delta y}\right)}{\Delta x}$ is intermediate between $\sec\gamma_1$ $\sec\gamma_2$, which are, in the limit, each equal to $\sec\gamma$.

Therefore, $\dfrac{d^2S}{dy\,dx}$ or $\dfrac{d^2S}{dx\,dy} = \sec\gamma = \sqrt{\left\{1 + \left(\dfrac{dz}{dx}\right)^2 + \left(\dfrac{dz}{dy}\right)^2\right\}}$.

932. If S be the surface contained between the plane zOx, and a pl
whose equation is $y = tx$, we can shew, by proceeding as in Art. 930, that

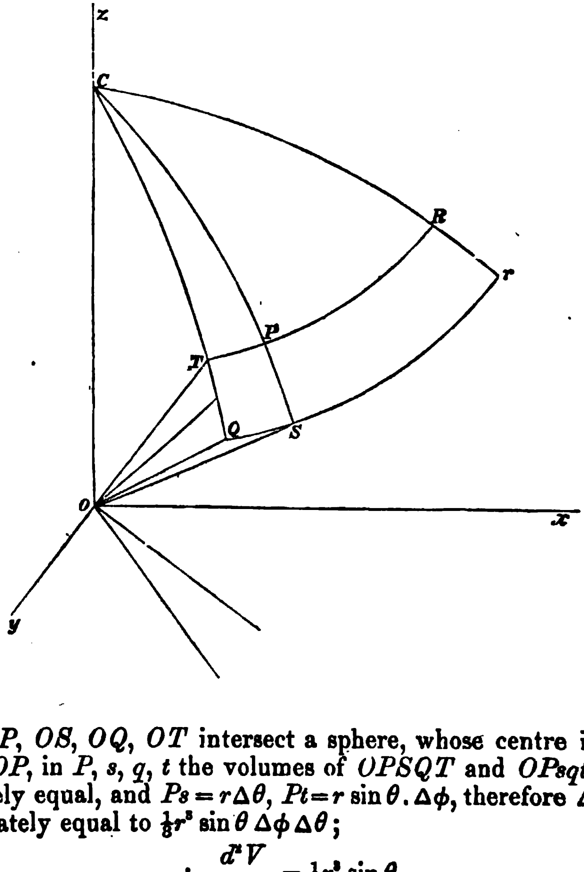
$$\frac{d^2S}{dx\,dt} = \sqrt{\left[\left\{1+\left(\frac{dz}{dx}\right)^2\right\}x^2+\left(\frac{dz}{dt}\right)^2\right]}.$$

933. *To find the differential coefficients of the volume of a sur referred to polar coordinates.*

Let r, θ, ϕ be the polar coordinates of a point P in the surfa θ being measured from Oz, and ϕ from the plane zOx, and let be the volume of the wedge of a cone contained between the pla zOx and zOP, and the given surface, the axis of the cone bei Oz, and θ the semi-vertical angle.

$OPRrS$ is the increase of the volume when θ increases by Δ ϕ remaining constant, therefore $OPRrS = \Delta_\theta V$.

$OPSQT$ is the increase of $\Delta_\theta V$ when ϕ becomes $\phi + \Delta\phi$, a therefore $= \Delta_\phi(\Delta_\theta V)$, and similarly $= \Delta_\theta(\Delta_\phi V)$.



If OP, OS, OQ, OT intersect a sphere, whose centre is O a radius OP, in P, s, q, t the volumes of $OPSQT$ and $OPsqt$ will ultimately equal, and $Ps = r\Delta\theta$, $Pt = r\sin\theta\,.\,\Delta\phi$, therefore $\Delta_\phi(\Delta_\theta$ is ultimately equal to $\frac{1}{3}r^3\sin\theta\,\Delta\phi\,\Delta\theta$;

$$\therefore\ \frac{d^2V}{d\phi\,d\theta} = \tfrac{1}{3}r^3\sin\theta.$$

934. *To find the differential coefficient of a surface referred to polar coordinates.*

Let r, θ, ϕ be the polar coordinates of P, and let S be the surface CPR,

$\Delta_\theta S$ is the increment Pr when θ changes to $\theta + \Delta\theta$,

$\Delta_\phi(\Delta_\theta S)$ is the increment PQ when ϕ changes to $\phi + \Delta\phi$.

Let ψ_1, ψ_2 be the least and greatest inclinations of tangent planes at points taken on the surface $PSQT$, to tangent planes at the corresponding points of the surface $Psqt$ in the construction of the last article, then the ratio of the surfaces $PSQT$ and $Psqt$ lies between $1 : \cos\psi_1$, and $1 : \cos\psi_2$, each of which becomes ultimately $r : p$, where p is the perpendicular from O on the tangent plane at P; $\therefore \Delta_\phi(\Delta_\theta S) : r^2 \sin\theta\, \Delta\phi\Delta\theta :: r : p$, ultimately;

$$\therefore \frac{d^2S}{d\phi\, d\theta} = \frac{r^3}{p}\sin\theta,$$

$$\text{and } \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4\sin^2\theta}\left(\frac{dr}{d\phi}\right)^2, \qquad \text{Art. 502;}$$

$$\therefore \frac{d^2S}{d\phi\, d\theta} = r\left[\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}\sin^2\theta + \left(\frac{dr}{d\phi}\right)^2\right]^{\frac{1}{2}}.$$

935. *To find the volume of a closed surface, the boundaries of which are portions of known surfaces, given by equations in Cartesian coordinates.*

Let (x, y, z) be a point P within the closed surface, and let Δx, Δy, Δz be the lengths of the edges of a small parallelepiped, whose faces are parallel to the coordinate planes, the volume of this elementary parallelepiped will be $\Delta x\Delta y\Delta z$, if the axes be rectangular.

We imagine the volume to be made up of an infinite number of such elements, each of which is supposed indefinitely small, and in order to obtain the volume we have to sum these elements, and we must be guided by the form of the surfaces, in our choice of the order in which we propose to effect the summation. We can give general directions only, leaving to the student's ingenuity the task of adapting them to particular cases.

If we commence by summing the elements, for which x, y have constant values, we shall obtain the parallelepiped $(z_2 - z_1)\Delta x\Delta y$, since the incomplete elements near the boundaries of the surface vanish compared with the parallelepiped upon $\Delta x\Delta y$, when Δx and Δy are indefinitely diminished; $z_2 - z_1$ can be expressed in terms of x and y by means of the equations of the bounding surfaces. This supposes the closed surface to be pierced by the ordinate through $(x, y, 0)$ in only two points; if it were pierced $2n$ times, the first summation would give $\Sigma_1^n (z_{2r} - z_{2r-1})\, dxdy$; we shall not further consider such cases.

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If we next sum the parallelepipeds for all values of y, keep[ing] x constant, we shall obtain the sum of all the elements which [lie] between two planes at distances x and $x+\Delta x$ from the pla[ne] of yz. The first and last of the parallelepipeds must vanis[h]; therefore the summation must generally be made between val[ues] of y obtained from the equation $z_2 - z_1 = 0$, x being constant; [let] y_1, y_2 be those values of y, supposing only two to exist; the wh[ole] sum will then be obtained by summing these sheets of eleme[nts] between values of x obtained from the equation $y_2 - y_1 = 0$.

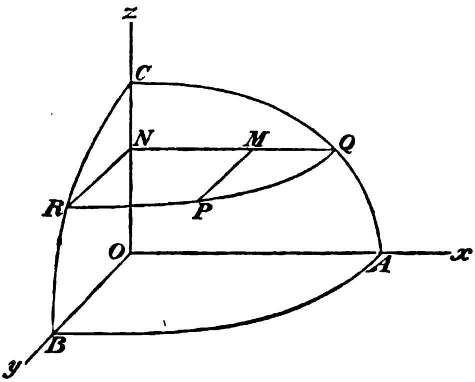
In the case of a closed surface, which is pierced by no straig[ht] line in more than two points, the process for finding the volume is expressed as follows:

$$V = \int_{x_1}^{x_2} dx \left\{\int_{y_1}^{y_2} dy \left(\int_{z_1}^{z_2} dz\right)\right\} = \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} \{f_2(x, y) - f_1(x, y)\}\, dy$$

$$= \int_{x_1}^{x_2} dx \{\phi(x, y_2) - \phi(x, y_1)\} = \psi(x_2) - \psi(x_1).$$

936. The student will have to determine in every particular case the b[est] order in which to make the summation of the elements; in some cases it w[ill] be advisable to take elementary slices of the surface, instead of the elementa[ry] parallelepipeds, as when the area of a plane section is known.

Thus, in the case of an ellipsoid, the area of a section RPQ is $\pi QN.R[N]$ and a slice of the thickness $dz = \pi abc^{-2}(c^2 - z^2)\, dz$, whence the volum[e] $\dfrac{\pi ab}{c^2}\displaystyle\int_{-c}^{+c}(c^2 - z^2)\, dz = \tfrac{4}{3}\pi abc.$



937. He must also judge whether it is advisable to use other coordin[ates] than those in which the equation of the surface is given.

Thus, the equation of an anchor-ring being

$$(x^2 + y^2 + z^2 + c^2 - a^2)^2 - 4c^2(x^2 + y^2) = 0,$$

if we make $x^2 + y^2 = r^2$, and so $z^2 = a^2 - (r-c)^2$, we can sum the elements wh[ich] have their projections on the circular ring $2\pi r dr$, and the volume is

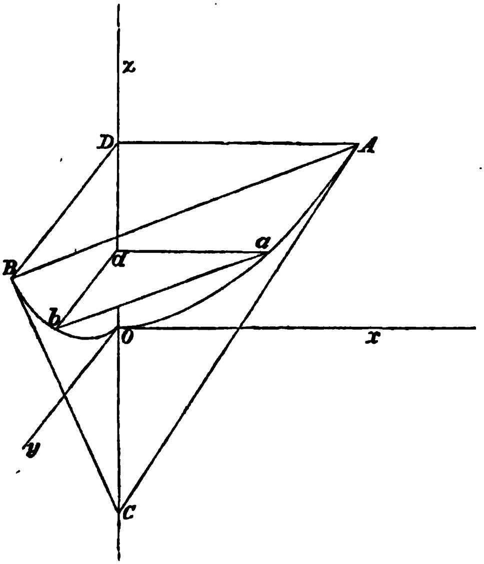
$$\int_{c-a}^{c+a} 4\pi r dr \sqrt{\{a^2 - (r-c)^2\}} = \int_{-a}^{+a} 4\pi (r' + c)\, dr' \sqrt{(a^2 - r'^2)} = 2\pi c \,.\, \pi a^2.$$

938. *To find the volume contained between the surface whose equation is* $c + y)^2 = 4az$, *the tangent plane at a given point, and the planes of* zx *and* yz.

Let the given point be (f, g, h), the equation of the tangent plane is $+ y = \sqrt{(a/h)}\,(z + h)$; the volume required is $\iiint dx\,dy\,dz$, the limits being from $= \sqrt{(h/a)}\,(x + y) - h$ to $(x + y)^2/4a$, then from $y = 0$ to $y = 2\sqrt{(ah)} - x$, since ie tangent plane meets the surface where $(x + y)^2 - 4\sqrt{(ah)}\,(x + y) + 4ah = 0$, stly from $x = 0$ to $x = 2\sqrt{(ah)}$. The volume required is

$$\iint \frac{1}{4a}\,\{x + y - 2\sqrt{(ah)}\}^2\,dy\,dx = \int -\frac{1}{12a}\,\{x - 2\sqrt{(ah)}\}^3\,dx = \tfrac{1}{3}ah^2.$$

This result may be verified thus. Let AOB be the surface, ACB the tangent lane along the line AB, ADB a plane parallel to xOy, adb any section of ie surface parallel to xOy.



$$\text{Then area } adb : \text{area } ADB :: ad^2 : AD^2 :: Od : OD;$$

$$\text{therefore volume } AOBD = \int_0^h 2ah.\frac{z}{h}\,dz = ah^2;$$

$$\text{also volume } ACBD = \tfrac{1}{3}2ah.2h = \tfrac{4}{3}ah^2;$$

$$\text{hence the volume required is } \tfrac{1}{3}ah^2.$$

939. *To find the volume of the elliptic paraboloid* $y^2/b + z^2/c = 2x$, *cut off by* ie *plane* $lx + my + nz = p$.

Perform the integration in the order x, y, z,

$$x_1 = \tfrac{1}{2}\,(y^2/b + z^2/c), \qquad x_2 = (p - my - nz)/l.$$

'or a given value of z, the values of y at the curve of intersection are given y the equation $x_1 = x_2$, or $y^2 + 2ybm/l + z^2b/c - 2b\,(p - nz)/l = 0$, of which $_1$, y_2 are the roots, and z must be taken between the limits which correspond) $y_1 = y_2$, that is z_1, z_2 are the roots of the equation

$$z^2b/c - 2b\,(p - nz)/l = b^2m^2/l^2. \quad (1)$$

$$\text{The volume} = \int_{z_1}^{z_2}\int_{y_1}^{y_2}\left(\frac{p-my-nz}{l}-\frac{y^2}{2b}-\frac{z^2}{2c}\right)dy\,dz$$

$$=\frac{1}{2b}\int_{z_1}^{z_2}\int_{y_1}^{y_2}(y-y_1)(y_2-y)\,dy\,dz \text{ by (1)},$$

$$=\frac{1}{2b}\int_{z_1}^{z_2}\int_{y_1}^{y_2}\{(y-y_1)(y_2-y_1)-(y-y_1)^2\}\,dy\,dz$$

$$=\frac{1}{2b}\int_{z_1}^{z_2}\tfrac{1}{6}(y_2-y_1)^3\,dz,$$

but $(y_2-y_1)^2=(y_2-y_1)^2-4y_2y_1=4(z-z_1)(z_2-z)\,b/c$ by (1),

therefore the volume

$$=\frac{1}{12b}\int_{z_1}^{z_2}8\left(\frac{b}{c}\right)^{\frac{3}{2}}\{(z_2-z_1)(z-z_1)-(z-z_1)^2\}^{\frac{3}{2}}\,dz$$

$$=\tfrac{2}{3}\frac{b^{\frac{1}{2}}}{c^{\frac{3}{2}}}\int_{-\gamma}^{\gamma}(\gamma^2-u^2)^{\frac{3}{2}}\,du, \text{ where } 2\gamma=z_2-z_1,\ u=z-\tfrac{1}{2}(z_1+z_2)$$

$$=\tfrac{4}{3}\frac{b^{\frac{1}{2}}}{c^{\frac{3}{2}}}\gamma^4\int_0^{\frac{1}{2}\pi}\cos^4\theta\,d\theta, \text{ putting } u=\gamma\sin\theta,\ =\tfrac{4}{3}\frac{b^{\frac{1}{2}}}{c^{\frac{3}{2}}}\gamma^4\cdot\frac{3.1}{4.2}\cdot\frac{\pi}{2},$$

$$\text{and } \gamma^2=\tfrac{1}{4}(z_2-z_1)^2=c^2n^2/l^2+2cp/l+bcm^2/l^2;$$

therefore the volume is $\frac{1}{4}\pi\sqrt{(bc)}\,(2pl+bm^2+cn^2)^2/l^4$.

The student may verify this result by the summation of elementary s[…] bounded by planes parallel to the given plane.

9·10. *To find the volume contained between surfaces given […] polar coordinates.*

The volume of an elementary parallelepiped is

$$r^2\sin\theta\,dr\,d\theta\,d\phi.$$

If we integrate this expression from $r=r_1$ to $r=r_2$, r_1, r_2 be[…] the radii of the bounding surfaces corresponding to θ, ϕ, we ob[…] a frustrum of a pyramid, the angular breadths of whose faces […] $d\theta$, $d\phi$, intercepted between the two surfaces or the two sh[…] of the same surface; its volume is $\frac{1}{3}\sin\theta\,d\theta\,d\phi\,(r_2^3-r_1^3)$, the r[…] being given in terms of θ and ϕ.

If now we integrate, considering ϕ and $\phi+d\phi$ constant, f[…] $\theta=\theta_1$ to $\theta=\theta_2$, θ_1, θ_2 being given in terms of ϕ by the boundar[…] of the volume considered, we obtain the portion included betw[…] the planes inclined to zOx at angles ϕ and $\phi+d\phi$,

$$=d\phi\int_{\theta_1}^{\theta_2}\tfrac{1}{3}(r_2^3-r_1^3)\sin\theta\,d\theta.$$

The whole volume is found by integrating from $\phi=\phi_1$ to $\phi=$[…] the extreme planes between which the volume is included.

$$\text{The volume is therefore } \int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2}\tfrac{1}{3}(r_2^3-r_1^3)\sin\theta\,d\theta\,d\phi.$$

941. *To find the volume of a sphere cut off by three planes through the centre.*
Let the radius of the sphere be a, O its centre, and let ABC be the spherical triangle cut off. Take OC for the axis of z, and a plane perpendicular to AOB for that of zx; and let α be the inclination of OC to the plane AOB, $-\beta$ that of the planes AOC and zx, then the equation of the plane AOB will be $\cos\phi = \tan\alpha \cot\theta$, and the limits of integration will be

$$r=0 \quad \text{to} \quad r=a,$$
$$\theta=0 \quad \text{to} \quad \theta=\cot^{-1}(\cot\alpha\cos\phi),$$
$$\phi=-\beta \quad \text{to} \quad \phi=C-\beta;$$

$\therefore$ the volume $= \frac{1}{3}a^3 \iint \sin\theta \, d\theta \, d\phi$

$$= \tfrac{1}{3}a^3 \int_{-\beta}^{C-\beta} \left\{1 - \frac{\cos\alpha\cos\phi}{\sqrt{(1-\cos^2\alpha\sin^2\phi)}}\right\} d\phi$$
$$= \tfrac{1}{3}a^3 \left[C - \sin^{-1}\{\cos\alpha\sin(C-\beta)\} - \sin^{-1}(\cos\alpha\sin\beta)\right]$$
$$= \tfrac{1}{3}a^3 (A+B+C-\pi),$$

since $\cos B = \cos\alpha \sin(C-\beta)$ and $\cos A = \cos\alpha\sin\beta$.

We have given this as an example of the determination of the limits in the case of polar coordinates, but the result is obtained immediately from the area of the spherical triangle, the volume required being the sum of an infinite number of pyramids whose vertices are in the centre, the volume of any one of which is $\frac{1}{3}a\,dS$, and the whole volume $= \frac{1}{3}a \times$ area of the spherical triangle.

942. *To find the volume of a wedge of a sphere cut off by a right circular cylinder, a diameter of whose base is a radius of the sphere.*
Let the equation of the sphere be $\rho^2 + z^2 = a^2$, and that of the cylinder $\rho = a\cos\phi$, and let α be the inclination of the planes of the wedge.

The volume is $$\int_0^\alpha \int_0^{a\cos\phi} 2\rho\sqrt{(a^2-\rho^2)}\, d\rho\, d\phi = \int_0^\alpha \tfrac{2}{3}(a^3 - a^3\sin^3\phi)\, d\phi$$
$$= \tfrac{2}{3}a^3\{\alpha - \tfrac{1}{4}\textstyle\int(3\sin\phi - \sin 3\phi)\, d\phi\} \, d\phi \text{ from } 0 \text{ to } \alpha.$$
$$= \tfrac{2}{3}a^3\{\alpha - \tfrac{3}{4}(1-\cos\alpha) + \tfrac{1}{12}(1-\cos 3\alpha)\}.$$

The surface $$= \iint \sqrt{\left\{\rho^2 + \left(\frac{dz}{d\phi}\right)^2 + \rho^2\left(\frac{dz}{d\rho}\right)^2\right\}}\, d\rho\, d\phi, \text{ between the same limits,}$$
$$= \iint \rho\sqrt{\left(1 + \frac{\rho^2}{a^2-\rho^2}\right)}\, d\rho\, d\phi = \iint \frac{a\rho\, d\rho}{\sqrt{(a^2-\rho^2)}}\, d\phi$$
$$= a^2 \textstyle\int(1-\sin\phi)\, d\phi = a^2(\alpha - 1 + \cos\alpha).$$

943. *To find the volume of a solid whose bounding surfaces are given by tetrahedral coordinates.*
Let ξ, η, ζ be coordinates referred to rectangular axes of a point whose tetrahedral coordinates are x, y, z, w.
Since x, y, z are linear functions of ξ, η, ζ,

$$\iiint d\xi\, d\eta\, d\zeta = C \iiint dx\, dy\, dz,$$

and if V be the volume of the tetrahedron of reference,

$$\iiint d\xi\, d\eta\, d\zeta = V;$$

but the limits for the tetrahedron are, since $x+y+z+w=1$,

$$z=0 \text{ to } w=0 \text{ or } z=1-x-y,$$
$$y=0 \text{ to } y=1-x, \text{ and } x=0 \text{ to } x=1,$$

and with these limits $\iiint dx\, dy\, dz = \frac{1}{6}$; therefore $6V = C$.

Hence, if $F(x, y, z, w)=0$ be the equation of any closed surf[ace] the volume will be $6V\iiint dx\,dy\,dz$, the limits of the integration be[ing] obtained from $F(x, y, z, 1-x-y-z)=0$. This method is d[ue] to Slesser.*

944. *To find the element of a surface employing curvili[near] coordinates.*

It has been explained in Chapter XXIV. how, when each of [the] coordinates x, y, z of any point in the surface is regarded as [a] function of p and q, the surface may be striated by two ser[ies] of curves corresponding to different constant values of p and [q] respectively; and it has been shewn that if $PMQN$ be an elementa[ry] quadrilateral of which the sides PM, NQ are elements of curv[es] traced on the surface for which q and $q+dq$ respectively a[re] constant, PN, MQ those for which p and $p+dp$ are constant, t[he] lengths of the elements PM, PN will be respectively $dp\sqrt{E}$ a[nd] $dq\sqrt{G}$; also that, if ω be the angle NPM, $\cos\omega=F/\sqrt{(EG)}$, [so] that the area of an element dS of the surface S is

$$dp\,dq\sqrt{(EG-F^2)}=dp\,dq\,(A^2+B^2+C^2)^{\frac{1}{2}}.$$

945. *To find the surface of an ellipsoid expressed by elliptic coordinates.*

Let the equation of the ellipsoid be $x^2/a+y^2/b+z^2/c=1$, and let μ, [ν be] the elliptic coordinates of the point (x, y, z), so that

$$x^2/(a+\mu)+y^2/(b+\mu)+z^2/(c+\mu)=1 \text{ and } x^2/(a+\nu)+y^2/(b+\nu)+z^2/(c+\nu)=1$$

Now, with the notation of Art. 287,

$$-\beta\gamma x^2=a(a+\mu)(a+\nu),\quad -\gamma\alpha y^2=b(b+\mu)(b+\nu),\ \text{and}\ -\alpha\beta z^2=c(c+\mu)(c+\nu);$$

$$\therefore\ -\gamma\alpha\, dy^2/d\mu=b(b+\nu),\qquad -\alpha\beta\, dz^2/d\mu=c(c+\nu),$$

$$-\gamma\alpha\, dy^2/d\nu=b(b+\mu),\qquad -\alpha\beta\, dz^2/d\nu=c(c+\mu);$$

$$\therefore\ \alpha^2\beta\gamma\left(\frac{dy^2}{d\mu}\frac{dz^2}{d\nu}-\frac{dy^2}{d\nu}\frac{dz^2}{d\mu}\right)=bc(b-c)(\mu-\nu);$$

$$\therefore\ 4\alpha\beta\gamma yzA=bc(\mu-\nu).$$

Now, if p be the perpendicular on the tangent plane at (x, y, z), $p^2=abc$ . . Art. 291, and the cosine of its inclination to the axis of x is

$$A/\sqrt{(A^2+B^2+C^2)}=xp/a;$$

$$\therefore\ \sqrt{(A^2+B^2+C^2)}=Aa/xp=abc(\mu-\nu)/4\alpha\beta\gamma xyzp,$$

but, by (1), $-\alpha^2\beta^2\gamma^2x^2y^2z^2p^2=a^2b^2c^2(a+\mu)(b+\mu)(c+\mu)(a+\nu)(b+\nu)(c+\nu)$ a[nd]

$$\therefore\ dS=\tfrac{1}{4}d\mu\,d\nu\,(\mu-\nu)\sqrt{(\mu\nu)}/\sqrt{\{-(a+\mu)(b+\mu)(c+\mu)(a+\nu)(b+\nu)(c+\nu)\}},$$

whence the surface can be expressed in the form of a double integral.

If $M=\sqrt{\mu}/\sqrt{\{-(a+\mu)(b+\mu)(c+\mu)\}}$ and $N=\sqrt{\nu}/\sqrt{\{(a+\nu)(b+\nu)(c+\nu)\}}$, th[e] area of the surface cut off by two confocal hyperboloids of one sheet for w[hich] $\mu=\mu_1$ and μ_2, and two of two sheets for which $\nu=\nu_1$ and $\nu=\nu_2$ can be [ex]p[ressed] in the form

$$\tfrac{1}{4}\int_{\mu_1}^{\mu_2}\mu M\,d\mu\times\int_{\nu_1}^{\nu_2}N\,d\nu-\tfrac{1}{4}\int_{\mu_1}^{\mu_2}M\,d\mu\times\int_{\nu_1}^{\nu_2}\nu N\,d\nu.$$

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* *Quar. Jour. of Math.*, vol. II.

946. If the elliptic coordinates be the primary semi-axes a', a'' of the confocal hyperboloids, as in Art. 286, the equation of the ellipsoid being $x^2/a^2+y^2/(a^2-\beta^2)+z^2/(a^2-\gamma^2)=1$, we must write a^2 for a, a'^2-a^2 for μ, and a''^2-a^2 for ν, and the double integral which represents the surface becomes

$$\iint \frac{(a'^2-a''^2)\sqrt{\{(a^2-a'^2)(a^2-a''^2)\}}}{\sqrt{\{(a'^2-\beta^2)(\gamma^2-a'^2)(\beta^2-a''^2)(\gamma^2-a''^2)\}}}\,da'\,da''.$$

947. *To find the volume bounded by surfaces defined by curvilinear coordinates.*

Let the position of a point be given by the intersection of three surfaces whose equations are $F(x, y, z)=p$, $G(x, y, z)=q$, and $H(x, y, z)=r$ (1); when p is constant, the variation of q and r gives rise to two series of curves striating the surface $F(x, y, z)=p$, an element of this surface is therefore $(A^2+B^2+C^2)^{\frac{1}{2}}\,dr\,dq$, where $A=\frac{dy}{dq}\frac{dz}{dr}-\frac{dy}{dr}\frac{dz}{dq}$, &c., A, B, C being the minors of the Jacobian of equations $x=f(p, q, r)=0$, $y=g(p, q, r)=0$, $z=h(p, q, r)=0$, (2) derived from (1).

The equation of the tangent plane at the element considered is $A(\xi-x)+B(\eta-y)+C(\zeta-z)=0$, the perpendicular upon which from a point whose curvilinear coordinates are $p+dp$, q, r is

$$(A\,dx/dp+B\,dy/dp+C\,dz/dp)\,dp/\sqrt{(A^2+B^2+C^2)},$$

hence the volume of the elementary parallelepiped, whose opposite faces correspond to p and $p+dp$ constant, is

$$(A\,dx/dp+B\,dy/dp+C\,dz/dp)\,dp\,dq\,dr,$$

and the volume is $\iiint J\,dp\,dq\,dr$ or $\iiint J'^{-1}\,dp\,dq\,dr$ taken between limits corresponding to the boundaries, where J, J' are the Jacobians of the two systems (2) and (1) respectively.

Line, Surface, and Volume-Integral.

948. We give here two theorems relating to line, surface, and volume-integrals, which are of great importance in certain problems in Electricity, Hydrodynamics, and Conduction of Heat, and which serve as illustrations of the subjects of this chapter.

949. *Definitions of line-integral and surface-integral.*

i. If R be any quantity having direction, called a vector quantity, and ε be the angle between its direction and that of the tangent to a curve at any point (x, y, z) estimated in a definite direction, the integral $\int R\cos\varepsilon\,ds$ is called the *line-integral* of R along the line s, supposed measured from a fixed point. This integral may be written $\int\left(u\frac{dx}{ds}+v\frac{dy}{ds}+w\frac{dz}{ds}\right)ds$, u, v, w being the components of R in the direction of the axes of x, y, and z.

ii. If η be the angle between the direction of R and a n... to a surface at any point (x, y, z), the integral $\iint R \cos\eta$ ... called the *surface-integral*, the summation being taken over ... whole of a surface S. The integral may be written

$$\iint(Ul + Vm + Wn)\, dS,$$

U, V, W being components of R parallel to the axes of x, ... and l, m, n the direction-cosines of the normal to the surfa... (x, y, z) estimated in a definite direction.

950. *To shew that a line-integral taken round a given ... curve can be represented as a surface-integral over a surface bo... by the given curve.*

Suppose the closed curve L to be filled up by any surfac... and suppose S to be divided into an infinite number of s... elements, one of which is σ bounded by the line λ. If we t... the sum of the line-integrals for two of these lines which h... a common part μ, both estimated in the same direction, the ... portions of the sums taken over μ will be taken in opposite d... tions, and being of the same magnitude will vanish; those li... which abut upon L are the only portions which will not be trave... twice; hence the sum of all the line-integrals for the eleme... will be that of the line L.

The proposition will, therefore, be proved if we shew that i... true for any elementary line λ and corresponding surface σ.

Let (x, y, z) be any point on σ, and $(x + \xi, y + \eta, z + \zeta)$ ... point on λ; the line-integral for λ, u, v, w being given at $(x, y, ...$

$$= \int\left\{\left(u + \frac{du}{dx}\xi + \frac{du}{dy}\eta + \frac{du}{dz}\zeta\right) d\xi + \ldots\right\} \text{ ultimately.}$$

Since λ is a closed curve, $\int d\xi = 0$, $\int \xi d\xi = 0$, and, if we sup... the summation taken in the direction from x to y,

$$-\int \eta d\xi = \int \xi d\eta = n\sigma;$$

hence the line-integral for λ is $\left\{\left(\dfrac{dv}{dx} - \dfrac{du}{dy}\right) n + \ldots\right\} \sigma$.

The line-integral of L is, therefore, equal to the surface-inte... $\iint(Ul + Vm + Wn)\, dS$, when $U = \dfrac{dw}{dy} - \dfrac{dv}{dz}$, &c.

951. *The surface-integral of a directed quantity or vector, ... over a closed surface, may be expressed as a volume-integral ... certain function.*

We observe that if the theorem can be proved for an elemen... portion of the volume enclosed within the surface, within w... the directed quantity is supposed to be continuous, the gene... theorem will follow, as well as its modification, when the enc... volume is intersected by surfaces across which the directed quanti... changes discontinuously.

'or, if v_1, v_2 be two elementary volumes enclosed by the ices σ_1, σ_2 to which a portion of σ' is common, the normal ponents along σ', which belong respectively to σ_1 and σ_2, being pposite directions and of the same magnitude, will disappear ie summation.

f, therefore, we sum for all elements within a volume V, ughout which the value of the vector changes continuously, only sets of the resolved vectors which are not destroyed will be e which belong to the points of those elements which abut on the osing surface S.

f the vector change discontinuously in passing surfaces Σ_1, Σ_2, &c., theorem will hold for the portions V_1, V_2... into which they le V, and the volume-integral over V would be equal to the ice-integral over S, together with the surface-integrals over Σ_2; the differences of the vectors on opposite sides of these ices replacing the vectors in the first integral.

.et an elementary volume v be inclosed by the surface σ, $y, z)$ being any point within v, and let $(x+\xi, y+\eta, z+\zeta)$ be point upon σ, and u, v, w the components of the vector at $y, z)$ parallel to the axes.

'he surface-integral for σ is

$$\iint\left\{\left(u+\frac{du}{dx}\xi+\ldots\right)l+\left(v+\frac{dv}{dy}\eta+\ldots\right)m+\ldots\right\}d\sigma,$$

$$\text{and } \iint l\,d\sigma=0, \quad \iint\xi l\,d\sigma=\iint\xi\,d\eta\,d\zeta=v, \text{ \&c.},$$

e the surface-integral for $\sigma=\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)v$ ultimately;

efore, if u_1, u_1' be the values of u on opposite sides of Σ_1,

$$\iint(ul+vm+wn)\,dS=\iiint\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)dV$$
$$+\iint\{(u_1-u_1')\,l'+(v_1-v_1')\,m'+(w_1-w_1')\,n'\}\,d\Sigma_1$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

h represents the theorem in its most general form.

52. Some elegant applications of the theorem which expresses urface-integral as a volume-integral have been made by Webb.* Ie supposes the interior of a closed surface to be filled up with ices constructed according to some definite law such that the ding surface is one of the set.

'or the components of the vector he chooses the direction-ies l, m, n of the normal at a point P of one of the interior ices measured outwards, so that at the bounding surface

* *Mess. of Math.*, vol. IX., p. 170.

$lu + mv + nw = 1$, and if ρ_1, ρ_2 be the principal radii of curv[...] of the interior surface of which P is a point,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

Hence, if there be an interior as well as an exterior surfac[...] difference of these surfaces $= \iiint (\rho_1^{-1} + \rho_2^{-1})\, dx\, dy\, dz$; and in [...] to calculate the surface from this expression, $\rho_1^{-1} + \rho_2^{-1}$ mu[...] expressed in terms of x, y, z by eliminating the parameter w[...] defines the particular surface passing through (x, y, z), using [...] this purpose the equation of that surface.

953. An example is given of a sphere filled with sphere[...] touching at a given point; also the surface of an ellipsoid is f[...] in the form of a triple integral by filling it with similar conc[...] ellipsoids; and it is observed that if the ellipsoid be filled w[...] confocal ellipsoids the interior surface will be twice the area of [...] focal ellipse which is the limit of a very flat ellipsoid.

LXVII.

(1) Find the portion of the cylinder $x^2 + y^2 - 2rx = 0$, intercepted be[...] the planes $ax + by + cz = 0$ and $a'x + by + cz = 0$.

(2) Prove that the volume of the surface $xy + yz + zx - a^2 = 0$, cut [...] the plane $x + y + z = c\sqrt{3}$, is $\frac{2}{3}\pi (c + 2a)(c - a)^2$.

(3) Shew that the surfaces $y^2 + z^2 = 4ax$, and $x - z = a$, include a volum[...]

(4) Prove that the volume included between the surfaces $r = a$, $z = 0$, $z = mr\cos\theta$, is $\frac{4}{3}ma^3$, r and θ being polar coordinates in the plane xy.

(5) Shew that the volume enclosed by the surfaces $x^2 + y^2 = az$, $x^2 + y^2$[...] and $z = 0$ is $\frac{3}{16}\pi a^3$, and draw a figure representing the progress of summa[...]

(6) Prove that the volume included between a cylinder $y^2 = 2cx - x^2$, a[...] boloid $ax^2 + by^2 = 2z$, and the plane of xy, is $\frac{1}{8}\pi c^4 (5a^{-1} + b^{-1})$.

(7) Two cones have a common vertex in the centre of an ellips[...] their bases are curves in which the surface is intersected by planes par[...] the same principal plane, prove that the volume of the ellipsoid con[...] between the cones varies as the distance between the planes.

(8) If ΔS be an element of the surface of an ellipsoid at any point, [...] the area of a section by a plane drawn through the centre parallel [...] tangent plane at that point, prove that the limit of $\Sigma(\Delta S/A) = 4$, the sum[...] being taken over the whole surface.

Find ΔS in terms of α, β, if $x = a\cos\alpha$, $y = b\sin\alpha\cos\beta$, and $z = c\sin\alpha$[...]

LXVIII.

(1) Shew that the whole volume of the surface whose equation is $+ y^2 + z^2)^2 = cxyz$ is equal to $c^3/360$.

(2) Prove that the volume cut off by the planes $y = \pm k$ from the surface $+ b^2z^2 = 2(ax + bz)y^2$ is $4\pi k^5/5ab$.

(3) A cavity is just large enough to allow of the complete revolution of cular disc of radius c, whose centre describes a circle of the same radius c, e the plane of the disc is constantly parallel to a fixed plane, and perpenlar to that of the circle in which the centre moves. Shew that the volume he cavity is $\frac{2}{3}c^3(3\pi + 8)$.

(4) State limits which can be used to find the volume of a closed conicoid se equation is $ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 1$. Shew that the volume $\pi(abc + 2a'b'c' - aa'^2 - bb'^2 - cc'^2)^{-\frac{1}{2}}$.

(5) Find the form of the surface whose equation is

$$(x^2/a^2 + y^2/b^2 + z^2/c^2)^2 = x^2/a^2 + y^2/b^2 - z^2/c^2,$$

shew that the volume is $\pi^2 abc/4\sqrt{2}$.

(6) P is a point in a fixed circle, radius a, whose centre is C; in a plane sing through the radius CP, and perpendicular to the plane of the circle, rcle is described whose centre is P and radius equal to the distance from P fixed diameter of the given circle. Shew that the volume of the surface erated by the variable circles is $\pi^2 a^3$.

(7) Prove that the area of the portion of the surface $z = f(ax + by)$ cut off the planes $x = 0$, $y = 0$, and $ax + by = c$ is $(ab)^{-1}\int_0^c \xi d\xi \sqrt{\{1 + (a^2 + b^2)[f'(\xi)]^2\}}$.

(8) If S be a closed surface, dS an element about P, at a distance r from ed point O, ϕ the angle which the normal drawn inwards makes with OP, w that the volume contained by the surface $= \frac{1}{3}\iint r \cos\phi \, dS$, the summation ig extended over the whole surface.
O being the centre of an ellipsoid, apply the formula to find its volume, rpreting geometrically the steps of the integration.

LXIX.

(1) Prove that the volume cut off from the cone

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0$$

the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is $\frac{4}{3}\pi abc(1 - k)$, the curves of intersection he cone and ellipsoid being ellipses, and k given by the equation

$$\frac{gh}{gh - uf} + \frac{hf}{hf - vg} + \frac{fg}{fg - wh} = \frac{1}{k^2}.$$

(2) Investigate the form of the surface whose equation is

$$\{(x^2 + z^2)^{\frac{1}{2}} - a\}^2 + y^2 = a^2\{\tan^{-1}(z/x)\}^2/4\pi^2;$$

shew that its volume between values of $\tan^{-1}(z/x)$ from 0 to 2π is $\frac{2}{3}\pi^2 a^3$.

(3) Shew that $\iint x^2 dS/p$ extended over the surface of an ellipsoid is
to $\frac{1}{5}(3 + a^2/b^2 + a^2/c^2) \times$ volume of the ellipsoid.

(4) If each element of a closed surface be multiplied by $\mu r^{-2} \cos\psi$,
r is the distance of the element from a point O, and ψ is the angle
the direction of r and the normal to the surface measured outwards,
the sum of all such products is 0 or $4\pi\mu$, according as O is without or
the surface.

(5) Illustrate the method given in Art. 952 by finding the surface
anchor ring from the volume-integral, obtained by supposing a series of
rings to fill the given ring, all the rings having the same circular axis.

(6) Shew that the value of the integral $\iiint e^{2(x+y+z)/\sqrt{(a^2+b^2+c^2)}}\, dx\, dy$
over the volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is $\frac{1}{4}\pi abc\, (e^2 +$

(7) If r be the distance from a point O of any element dS of a
surface, determine the form of the function $f(r)$ when $\iint f(r)\, dS$, the
being effected over the whole surface of the sphere, is constant for all
of O within the sphere.

(8) The shortest distances between generating lines of the same
drawn at the extremities of diameters of the principal elliptic section
hyperboloid, whose equation is $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$, lie on the
whose equations are $\dfrac{cxy}{x^2 + y^2} = \pm \dfrac{abz}{a^2 - b^2}$. Prove that the volume included
these surfaces and the hyperboloid is $\dfrac{abc}{3}\left(\dfrac{a^4 - b^4}{a^2 b^2} + 8 \log \dfrac{a}{b}\right)$.

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THE END.

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